ATING CONVERGENCE

not assume that the terms a_n

erges absolutely if the series

s a_n converges. This is now a
pply the tests for convergence
s important, because we have:

sequence, and assume that the

qual to a_n itself if $a_n \geqq 0$. Let
$-a_n$ if $a_n \leqq 0$. Then both a_n^+
arison with $\sum |a_n|$, we see that

$\sum_{n=1}^{\infty} a_n^-$

,

es our theorem.

gence of a series which may

Theorem 5.2. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series such that*

$$\lim_{n\to\infty} a_n = 0,$$

such that the terms a_n *are alternately positive and negative, and such that* $|a_{n+1}| \leqq |a_n|$ *for* $n \geqq 1$. *Then the series is convergent.*

Proof. Let us write the series in the form

$$b_1 - c_1 + b_2 - c_2 + b_3 - c_3 + \cdots,$$

with $b_n, c_n \geqq 0$. Let

$$s_n = b_1 - c_1 + b_2 - c_2 + \cdots + b_n,$$
$$t_n = b_1 - c_1 + b_2 - c_2 + \cdots + b_n - c_n.$$

Since the absolute values of the terms decrease, it follows that

$$s_1 \geqq s_2 \geqq s_3 \geqq \cdots \quad \text{and} \quad t_1 \leqq t_2 \leqq t_3 \leqq \cdots,$$

i.e. that the s_n are decreasing and the t_n are increasing. Indeed,

$$s_{n+1} = s_n - c_n + b_{n+1} \quad \text{and} \quad 0 \leqq b_{n+1} \leqq c_n.$$

Thus we subtract more from s_n by c_n than we add afterwards by b_{n+1}. Hence $s_n \geqq s_{n+1}$. Furthermore, $s_n \geqq t_n$. Hence we may visualize our sequences as follows:

$$s_n \geqq s_{n+1} \geqq \cdots \geqq t_{n+1} \geqq t_n.$$

Note that $s_n - t_n = c_n$, and that c_n approaches 0 as n becomes large. If we let L be the greatest lower bound for the sequence $\{s_n\}$, and M be the least upper bound for the sequence $\{t_n\}$, then

$$s_n \geqq L \geqq M \geqq t_n$$

for all n. Since the difference $s_n - t_n$ becomes arbitrarily small, it follows that $L - M$ is arbitrarily small, and hence equal to 0. Thus $L = M$, and this proves that s_n and t_n approach L as a limit, whence our series $\sum a_n$ converges to L.