Nagoya University, G30 program Autumn 2014

Mathematics Tutorial I (linear algebra part) Instructor : Serge Richard

Quiz III

Name : ______________________

Exercise 1 *Find a row equivalent matrix in the standard form for the following matrix :*

$$\begin{pmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 & -2 & 3/2 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -4 \end{pmatrix} \sim \underline{\underline{\begin{pmatrix} 1 & 0 & -7/2 & 5/2 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}}}.$$

Exercise 2 *By using elementary row operations, find the inverse for the following matrix :*

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{pmatrix}$$

One has

$$\left(\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array}\right) \sim \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right).$$

The inverse is $\underline{\underline{\begin{pmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{pmatrix}}}$.

Exercise 3 *In terms of the coefficients a_{ij} of a matrix $\mathcal{A} \in M_n(\mathbb{R})$, write down the explicit conditions for $\mathcal{A}$ to be*

1. *diagonal :* $a_{ij} = 0$ whenever $i \neq j$
2. *upper-triangular :* $a_{ij} = 0$ whenever $i > j$
3. *skew-symmetric :* $a_{ij} = -a_{ji}$ for any i, j.

Exercise 4 *Consider the following matrices in $M_2(\mathbb{R})$:* $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, *and briefly answer the following questions:*

1. *Are these matrices linearly independent ?* Yes, one has

$$\lambda_1 \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \iff \begin{cases} \lambda_1 = -\lambda_3 \\ \lambda_1 = -\lambda_3 \\ \lambda_2 = 0 \\ \lambda_2 = -\lambda_3 \end{cases}$$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$ is the only solution.

2. *What is the dimension of* Vect $\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right)$ *?* 3

3. *What is the dimension of $M_2(\mathbb{R})$?* 4

Exercise 5 *For which values of the constant k is the following matrix invertible (don't compute explicitly the inverse but justify your answer) ?*

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 0 & 1 & k^2+1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & k-1 \\ 0 & 1 & k^2+1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & k-1 \\ 0 & 0 & k^2-k+2 \end{pmatrix}$$

$\Rightarrow$ this upper-triangular matrix is always invertible since $k^2-k+2 = (k-\frac{1}{2})^2 + \frac{7}{4} > 0$,

Indeed an upper-triangular matrix with no 0 on its diagonal is invertible, and thus the original matrix is now equivalent to an invertible matrix.

Exercise 6 *We consider the set $C(\mathbb{R})$ of all continuous real functions on $\mathbb{R}$. Let $P : C(\mathbb{R}) \to C(\mathbb{R})$ be the map sending any $f \in C(\mathbb{R})$ to $Pf \in C(\mathbb{R})$ with*

$$[Pf](x) = \frac{1}{2}(f(x) + f(-x)) \qquad \textit{for all } x \in \mathbb{R}.$$

1. *Is P a linear map ? (Justify your answer)* Yes, because

$[P(f+g)](x) = \frac{1}{2}(f(x)+g(x)) + \frac{1}{2}(f(-x)+g(-x)) = \frac{1}{2}(f(x)+f(-x)) + \frac{1}{2}(g(x)+g(-x))$
$= (Pf)(x) + (Pg)(x)$. Since x is arbitrary, $P(f+g) = Pf + Pg$.

Similarly, $(P(\lambda f))(x) = \frac{1}{2}((\lambda f)(x) + (\lambda f)(-x)) = \lambda \frac{1}{2}(f(x) + f(-x)) = \lambda (Pf)(x)$.
Since x is arbitrary, one gets $P(\lambda f) = \lambda Pf$.

2. *What is* Ker(P) *?*

$f \in \mathrm{Ker}(P) \iff (Pf)(x) = 0 \ \forall x \in \mathbb{R} \iff f(x) = -f(-x) \ \forall x \in \mathbb{R}$.

Thus $\mathrm{Ker}(P) = \{ f \in C(\mathbb{R}) \mid f(x) = -f(-x) \ \forall x \in \mathbb{R} \}$
$= \{ f \in C(\mathbb{R}) \mid f \text{ is an odd function} \}$

3. *What is* Ran(P) *?*

Observe that $(Pf)(x) = \frac{1}{2}(f(x) + f(-x)) = (Pf)(-x)$.
Thus Pf is an even function, which means that $\mathrm{Ran}(P) \subset \{$even functions$\}$. To show that $\mathrm{Ran}(P) = \{ g \in C(\mathbb{R}) \mid g \text{ is even} \}$, it is sufficient to observe that if g is even, then $Pg = g$.
Thus, any even function is in the range of P.

4. *What can you say about P^2, with $P^2 f = P(Pf)$*

By the previous observation, one gets

$$P^2 f = P(\underbrace{Pf}_{\text{even}}) = Pf \implies P^2 = P.$$