Nagoya University, G30 program Autumn 2014

Linear algebra I Instructor : Serge Richard

Quiz III

Name : ______________________________

Exercise 1 *Let V be a vector space over a field $\mathbb{F}$, and let X_1, X_2 be elements of V. Recall the definition:*

$\mathrm{Vect}(X_1, X_2) =$ $\{ \lambda_1 X_1 + \lambda_2 X_2 \mid \lambda_1, \lambda_2 \in \mathbb{F} \}$

Exercise 2 *What is the dimension of the vector space of all 4×4 diagonal matrices ?* 4

Exercise 3 *Provide a basis for the set of 2×2 symmetric matrices.*

One basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. they are symmetric and linearly independent.

Exercise 4 *For which value(s) of $k \in \mathbb{R}$ are the following vectors of $\mathbb{R}^3$ linearly independent ?*

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ k \end{pmatrix}.$$

Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, then $\lambda_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} -1 \\ 0 \\ k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Leftrightarrow \begin{cases} \lambda_1 = \lambda_3 \\ \lambda_1 = \lambda_2 \\ \lambda_2 = -k\lambda_3 \end{cases} \Leftrightarrow \begin{cases} \lambda_2 = \lambda_1 \\ \lambda_3 = \lambda_1 \\ \lambda_1 = -k\lambda_1 \end{cases}$. This has a non-trivial solution if $k = -1$ (for example $\lambda_1 = \lambda_2 = \lambda_3$) and no non-trivial solution if $k \neq -1$.

For $k \neq -1$, the 3 vectors are linearly independent.

Exercise 5 *For a matrix $\mathcal{A} = (a_{ij}) \in M_2(\mathbb{R})$ one writes $\mathrm{Tr}(\mathcal{A}) := a_{11} + a_{22}$ for the sum of the elements of its diagonal. We then define*

$$S := \{\mathcal{A} \in M_2(\mathbb{R}) \mid \mathrm{Tr}(\mathcal{A}) = 0\}.$$

Show that S is a subspace of $M_2(\mathbb{R})$. Let $\mathcal{A}, \mathcal{B} \in S \Leftrightarrow a_{11} + a_{22} = 0$ and $b_{11} + b_{22} = 0$

Then $\mathcal{A} + \mathcal{B} \in S$ since $(a_{11} + b_{11}) + (a_{22} + b_{22}) = (a_{11} + a_{22}) + (b_{11} + b_{22})$
$= 0 + 0 = 0.$

Also $\lambda \mathcal{A} \in S$ since $\lambda a_{11} + \lambda a_{22} = \lambda (a_{11} + a_{22}) = \lambda 0 = 0$.

Thus, the 2 conditions for a subspace of $M_2(\mathbb{R})$ are satisfied $\Rightarrow S$ is a subspace of $M_2(\mathbb{R})$.