Nagoya University, G30 program Autumn 2014

Linear algebra I Instructor : Serge Richard

Midterm exam

Exercise 1 *Let* $A = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ *and* $C = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ *be elements of* $\mathbb{R}^4$. *Compute the following expressions : (i)* $A+B$, *(ii)* $A+C-B$, *(iii)* $2A+3B$, *(iv)* $\|\overrightarrow{AB}\|$, *(v)* $\overrightarrow{AB}\cdot\overrightarrow{AC}$, *(vi) the cosine between the located vectors* $\overrightarrow{AB}$ *and* $\overrightarrow{AC}$.

Exercise 2 *In* $\mathbb{R}^3$, *consider the four points* $P_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $P_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ *and* $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

(i) Find the equation of the plane passing through the three points P_1, P_2 *and* P_3,

(ii) Determine the equation of the line passing through $\mathbf{0}$ *and perpendicular to the plane defined in* (i),

(iii) If $Q = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ *and* $N = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$, *determine the intersection of the line passing through* Q *and having the direction* N *with the triangle defined by the three points* $\mathbf{0}$, P_1 *and* P_2.

Exercise 3 *Let* $\mathcal{A} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in M_{31}(\mathbb{R})$, $\mathcal{B} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \in M_3(\mathbb{R})$ *and* $\mathcal{C} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \in M_{13}(\mathbb{R})$. *Compute the products (i)* $\mathcal{BA}$, *(ii)* $\mathcal{AC}$, *(iii)* $\mathcal{CA}$.

Exercise 4 *Let* $\mathcal{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{R})$. *Determine all matrices* $\mathcal{B} \in M_2(\mathbb{R})$ *such that* $\mathcal{AB} = \mathcal{BA}$.

Exercise 5 *For* $c \in \mathbb{R}$, *consider the matrix* $\mathcal{U} \in M_4(\mathbb{R})$ *given by*

$$\mathcal{U} = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

1. *Compute* ${}^t\mathcal{U}$,

2. *Determine* $\mathcal{U}^{-1}$,

3. *For which value(s) of* c *does the equality* ${}^t\mathcal{U} = \mathcal{U}^{-1}$ *hold ?*

4. *For* $c = 1$ *and any* $X \in \mathbb{R}^4$, *show that* $\|\mathcal{U}X\| = \|X\|$.

Exercise 6 *Find all solutions for the following system of equations:*

$$\begin{cases} 2x + y + 4z + w = 1 \\ -x + 3y + z + 2w = 2 \\ x + y + z = 0 \end{cases} .$$

Midterm

Exercise 1

i) $A+B=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\2\\1\\0\end{pmatrix}$. ii) $A+C-B=\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$

iii) $2A+3B=\begin{pmatrix}2\\5\\3\\0\end{pmatrix}$ iv) $\|\overrightarrow{AB}\|=\|B-A\|=\left\|\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}\right\|=\sqrt{2}$

v) $\overrightarrow{AB}\cdot\overrightarrow{AC}=(B-A)\cdot(C-A)=\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}=2$.

vi) $\|\overrightarrow{AC}\|=2$ $\Rightarrow$ the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is given by $\cos(\Theta)=\dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\|\overrightarrow{AB}\|\,\|\overrightarrow{AC}\|}=\dfrac{2}{2\sqrt{2}}=\dfrac{1}{\sqrt{2}}=\dfrac{\sqrt{2}}{2}$.

Exercise 2

i) One looks to $N=\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}$ such that $\overrightarrow{ON}\perp\overrightarrow{P_1P_2}$ and $\overrightarrow{ON}\perp\overrightarrow{P_1P_3}$

$\Leftrightarrow \begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix}=0$ and $\begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix}=0$

$\Rightarrow n_1=n_2=n_3$ and one can choose $N=\begin{pmatrix}1\\1\\1\end{pmatrix}$

Thus the plane is given by $H_{P_1,N}=H_{P_2,N}=H_{P_3,N}$

$=\left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; X\cdot N=P_1\cdot N\right\}$

$=\left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; x_1+x_2+x_3=1\right\}$.

ii) $L_{O,N}=\{O+tN \mid t\in\mathbb{R}\}=\left\{\begin{pmatrix}t\\t\\t\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; t\in\mathbb{R}\right\}$.

iii) $L_{Q,N}=\{Q+tN \mid t\in\mathbb{R}\}=\left\{\begin{pmatrix}1+2t\\1+2t\\1+3t\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; t\in\mathbb{R}\right\}$

Since the triangle defined by O, P_1 and P_2 lies on the horizontal plane $\left\{\begin{pmatrix}x\\y\\0\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; x,y \text{ arbitrary}\right\}$, the line intersects this plane for $t=-\frac{1}{3}$. Thus, the intersection is given by the point $\begin{pmatrix}1-2/3\\1-2/3\\0\end{pmatrix}=\begin{pmatrix}1/3\\1/3\\0\end{pmatrix}$.

Exercise 3

i) $\mathcal{B}\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$,

ii) $\mathcal{A}C = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}$,

iii) $C\mathcal{A} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 4$.

Exercise 4

Let $\mathcal{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$

while $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$ $\Rightarrow$ $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}$ if and only if $\begin{cases} c = b \\ a = d \end{cases}$. Thus $\mathcal{B} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ for any $a, b \in \mathbb{R}$.

Exercise 5

1) ${}^t\mathcal{U} = \mathcal{U}$,

2) $\mathcal{U}^{-1} = \begin{pmatrix} 1/c & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ as it can easily be checked,

3) One has $\mathcal{U}^{-1} = {}^t\mathcal{U} = \mathcal{U}$ if and only if $c = 1$ or $c = -1$.

4) If $c = 1$, then $\mathcal{U}$ is orthogonal, which implies $\|\mathcal{U}X\| = \|X\|$

cf homework 6, ex 3.

Exercise 6

The corresponding augmented matrix is

$$\begin{pmatrix} 2 & 1 & 4 & 1 & 1 \\ -1 & 3 & 1 & 2 & 2 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 2 & 2 & 2 \\ 0 & -1 & 2 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & -1 \\ 0 & 0 & 10 & 6 & 6 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -4/5 & -4/5 \\ 0 & 1 & 0 & 1/5 & 1/5 \\ 0 & 0 & 1 & 3/5 & 3/5 \end{pmatrix}$$

The system is thus equivalent to

$$\begin{cases} x = +4/5\, w - 4/5 \\ y = -1/5\, w + 1/5 \\ z = -3/5\, w + 3/5 \\ w \text{ arbitrary} \end{cases}$$