Homework 8

Exercise 1

1) If $AX = 0$ and $c \in \mathbb{R}$, then $A(cX) = cAX = c0 = 0$.

2) If $AX = 0$ and $AX' = 0$, then

$$A(X+X') = AX + AX' = 0 + 0 = 0.$$

3) If $AY = B$ and $AX = 0$, then

$$A(Y+X) = AY + AX = B + 0 = B.$$

Exercise 2

$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ is perpendicular to these vectors if $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 9 & 9 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

The corresponding augmented matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 & 0 \\ 1 & 9 & 9 & 7 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 8 & 8 & 6 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & -8 & -18 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 9/4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1/4 & 0 \\ 0 & 1 & 0 & -3/2 & 0 \\ 0 & 0 & 1 & 9/4 & 0 \end{pmatrix}.$$

Thus, one gets the system : $\begin{cases} x_1 + 1/4\, x_4 = 0 \\ x_2 - 3/2\, x_4 = 0 \\ x_3 + 9/4\, x_4 = 0 \\ x_4 \text{ arbitrary} \end{cases}$

$$\Leftrightarrow \begin{cases} x_1 = -1/4\, x_4 \\ x_2 = 3/2\, x_4 \\ x_3 = -9/4\, x_4 \\ x_4 \text{ arbitrary} \end{cases}.$$

Note that adding the column $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ was not really necessary.

Exercise 3

One has $(I_{rs} I_{r's'})_{ij} = \sum_{k=1}^{m} (I_{rs})_{ik} (I_{r's'})_{kj}$

$$= \sum_{k=1}^{m} \begin{cases} 1 & \text{if } i=r,\ j=s',\ k=s=r' \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } i=r,\ j=s',\ s=r' \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (I_{rs'})_{ij} & \text{if } s=r' \\ \mathbb{O}_{ij} & \text{otherwise} \end{cases}$$, from which we infer the statement.

1) Observe that $(\mathbb{1}_m - I_{rr} + c I_{rr})(\mathbb{1}_m - I_{rr} + \frac{1}{c} I_{rr})$

$= \mathbb{1}_m - I_{rr} + \frac{1}{c} I_{rr} - I_{rr} + I_{rr} - \frac{1}{c} I_{rr} + c I_{rr} - c I_{rr} + c \frac{1}{c} I_{rr}$

$= \mathbb{1}_m \implies (\mathbb{1}_m - I_{rr} + c I_{rr})^{-1} = (\mathbb{1}_m - I_{rr} + \frac{1}{c} I_{rr})$.

2) For $r \neq s$, observe that

$(\mathbb{1}_m + I_{rs} + I_{sr} - I_{rr} - I_{ss})(\mathbb{1}_m + I_{rs} + I_{sr} - I_{rr} - I_{ss})$

$= \mathbb{1}_m + I_{rs} + I_{sr} - I_{rr} - I_{ss} + I_{rs} + 0 + I_{rr} - 0 - I_{rs}$

$+ I_{sr} + I_{ss} + 0 - I_{sr} - 0 - I_{rr} - I_{rs} + 0 + I_{rr} + 0$

$- I_{ss} - 0 - I_{sr} + 0 + I_{ss} = \mathbb{1}_m$.

Thus $(\mathbb{1}_m + I_{rs} + I_{sr} - I_{rr} - I_{ss})^{-1} = (\mathbb{1}_m + I_{rs} + I_{sr} - I_{rr} - I_{ss})$.

3) $(\mathbb{1}_m + c I_{rs})(\mathbb{1}_m - c I_{rs}) = \mathbb{1}_m - c I_{rs} + c I_{rs} + 0 = \mathbb{1}_m$

$\implies (\mathbb{1}_m + c I_{rs})^{-1} = (\mathbb{1}_m - c I_{rs})$.

Since we have exhibited an inverse for these matrices, they are invertible.