Exercise 1

1) Observe that if $\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ or $\mathcal{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,
then $\mathcal{A}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbb{1}_2$.

2) Let $\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and let us compute

$\mathcal{A}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{pmatrix}$. This is equal to $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if

$$\begin{cases} a^2+bc = 0 \\ b(a+d) = 0 \\ c(a+d) = 0 \\ bc+d^2 = 0 \end{cases}$$

1°: if $b=0$, then $a^2 = 0 = d^2$

2°: if $c=0$, then $a^2 = 0 = d^2$

3°: if $d = -a$, then $bc = -a^2$

Thus, the solutions are

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ -\frac{a^2}{b} & -a \end{pmatrix}$$

with $a, b, c \in \mathbb{R}^*$.

Exercise 3

1) $\|\mathcal{A}X\|^2 = (\mathcal{A}X)\underset{\text{scalar product}}{\cdot}(\mathcal{A}X) \underset{\text{product of matrices}}{=} {}^t(\mathcal{A}X)(\mathcal{A}X) = {}^tX\,{}^t\mathcal{A}\,\mathcal{A}X \underset{\text{assumption on } \mathcal{A}}{=} {}^tX\,\mathbb{1}_n X$
$= {}^tXX = X\cdot X = \|X\|^2$, $\Longrightarrow \|\mathcal{A}X\| = \|X\|$.

2) $(\mathcal{A}X)\cdot(\mathcal{A}Y) = {}^t(\mathcal{A}X)(\mathcal{A}Y) = {}^tX\,{}^t\mathcal{A}\mathcal{A}Y = {}^tX\,\mathbb{1}_n Y = {}^tXY$
$= X\cdot Y$.

Exercise 2

Let us compute $AB = \begin{pmatrix}1 & a\\ 0 & b\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & a+b\\ 0 & 1\end{pmatrix}$.

It follows that $A^2 = \begin{pmatrix}1 & 2a\\ 0 & 1\end{pmatrix}$ and $A^3 = \begin{pmatrix}1 & 3a\\ 0 & 1\end{pmatrix}$.

One can guess that $A^m = \begin{pmatrix}1 & ma\\ 0 & 1\end{pmatrix}$ but one needs to prove it.

The following proof is a proof by induction:

1) Statement: $A^n = \begin{pmatrix}1 & na\\ 0 & 1\end{pmatrix}$ for any $n \in \mathbb{N}$.

2) The statement is verified for $n = 1$, or 2, or 3...

3) Let us assume that the statement is true for a certain m, and let us show that it is true for $m+1$. Indeed, one has

$$A^{m+1} = A^m A \underset{\substack{\uparrow\\ \text{by assumption}}}{=} \begin{pmatrix}1 & ma\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix} \underset{\substack{\uparrow\\ \text{computation}}}{=} \begin{pmatrix}1 & ma+a\\ 0 & 1\end{pmatrix} \underset{\substack{\uparrow\\ \text{computation}}}{=} \begin{pmatrix}1 & (m+1)a\\ 0 & 1\end{pmatrix}.$$

Thus the statement is verified for $m+1$.

4) Since m was arbitrary, the statement is proved.

Exercise 4

1) One has $\begin{pmatrix}\cos(\pi/4) & -\sin(\pi/4)\\ \sin(\pi/4) & \cos(\pi/4)\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix} = \begin{pmatrix}\sqrt{2}/2 & -\sqrt{2}/2\\ \sqrt{2}/2 & \sqrt{2}/2\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix} = \begin{pmatrix}-\sqrt{2}/2\\ 3\sqrt{2}/2\end{pmatrix}$.

2) Similarly $\begin{pmatrix}\cos(\pi/2) & -\sin(\pi/2)\\ \sin(\pi/2) & \cos(\pi/2)\end{pmatrix}\begin{pmatrix}-1\\ 3\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-1\\ 3\end{pmatrix} = \begin{pmatrix}-3\\ -1\end{pmatrix}$.

4) $R(\theta_1)R(\theta_2) = \begin{pmatrix}\cos(\theta_1) & -\sin(\theta_1)\\ \sin(\theta_1) & \cos(\theta_1)\end{pmatrix}\begin{pmatrix}\cos(\theta_2) & -\sin(\theta_2)\\ \sin(\theta_2) & \cos(\theta_2)\end{pmatrix} =$

$$= \begin{pmatrix}\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & -\cos(\theta_1)\sin(\theta_2) - \sin(\theta_1)\cos(\theta_2)\\ \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) & -\sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2)\end{pmatrix}$$

$$= \begin{pmatrix}\cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2)\\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2)\end{pmatrix} = R(\theta_1+\theta_2).$$

3) One has $R(\theta_1)R(\theta_2) = R(\theta_1+\theta_2) = R(\theta_2+\theta_1) = R(\theta_2)R(\theta_1)$.

5) The inverse of $R(\theta)$ is $R(-\theta)$ since $R(\theta)R(-\theta) = R(0) = \mathbb{1}_2 = R(-\theta)R(\theta)$.

Exercise 5

a) $\begin{pmatrix} 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ c) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ d) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 5 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

e) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & 0 & 6 & 0 \end{pmatrix}$ f) $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Exercise 6

a) $\begin{pmatrix} 0 & 3 & 0 & 4 \\ 1 & 0 & 2 & 0 \\ 5 & 0 & 6 & 0 \\ 0 & 7 & 0 & 8 \end{pmatrix}$ b) $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 5 & 0 & 6 & 0 \\ 0 & 3 & 0 & 4 \\ 0 & 7 & 0 & 8 \end{pmatrix}$ c) $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 5 & 0 & 6 & 0 \\ 0 & 22 & 0 & 28 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 5 & -6 & 6 & -8 \\ 0 & 7 & 0 & 8 \end{pmatrix}$.