Exercise 1

One just has to check the two conditions : stability for addition and for multiplication by a scalar.

Let $X, X' \in \mathbb{R}^3$, with $X = {}^t(x,y,z)$, $X' = {}^t(x',y',z')$, and $\lambda \in \mathbb{R}$.

Recall that $X + X' = {}^t(x+x', y+y', z+z')$ and $\lambda X = {}^t(\lambda x, \lambda y, \lambda z)$.

i) Since $X, X' \in S_1$, one has $x+y+z = 0 = x'+y'+z'$

$\Rightarrow (x+x') + (y+y') + (z+z') = (x+y+z) + (x'+y'+z') = 0+0 = 0$

$\Rightarrow X + X' \in S_1$.

Similarly, $\lambda x + \lambda y + \lambda z = \lambda(x+y+z) = \lambda 0 = 0$

$\Rightarrow \lambda X \in S_1$. Thus S_1 is a subspace of $\mathbb{R}^3$.

ii) Since $X, X' \in S_2$, one has $x = y$, $2y = z$, $x' = y'$, $2y' = z'$

$\Rightarrow x + x' = y + y'$ and $y + y' = 2z + 2z' = 2(z+z')$.

$\Rightarrow X + X' \in S_2$.

Similarly, $\lambda x = \lambda y$ and $\lambda y = \lambda(2z) = 2(\lambda z)$

$\Rightarrow \lambda X \in S_2$. Thus, S_2 is a subspace of $\mathbb{R}^3$.

iii) Since $X, X' \in S_3$, $x + y = 3z$ and $x' + y' = 3z'$

$\Rightarrow (x+x') + (y+y') = (x+y) + (x'+y') = 3z + 3z' = 3(z+z')$

$\Rightarrow X + X' \in S_3$.

Similarly, $\lambda x + \lambda y = \lambda(x+y) = \lambda(3z) = 3(\lambda z)$

$\Rightarrow \lambda X \in S_3$. Thus S_3 is a subspace of $\mathbb{R}^3$.

Exercise 2

Let $W = \{X \in \mathbb{R}^n \mid X \cdot Y = 0 \ \forall Y \in V\}$, here $\cdot$ is the scalar product.

Thus, if $X, X' \in W$ one has $(X+X') \cdot Y = X \cdot Y + X' \cdot Y = 0 + 0$ for any $Y \in V \Rightarrow X + X' \in W$.

Similarly, for any $\lambda \in \mathbb{R}$ one has $(\lambda X) \cdot Y = \lambda (X \cdot Y) = \lambda 0 = 0$ for any $Y \in V \Rightarrow \lambda X \in W$.

Thus, W is a subspace of $\mathbb{R}^n$, often denoted by $V^\perp$.

Exercise 3

Any $Y \in V$ is of the form $Y = \lambda_1 X_1 + \lambda_2 X_2 + \dots + \lambda_r X_r$ for some $\lambda_1, \dots, \lambda_r \in \mathbb{R}$. Thus, if $X \in \mathbb{R}^n$ satisfies $X \cdot X_j = 0$ for $j = 1, \dots, r$, then for any $Y \in V$ one has

$$X \cdot Y = X \cdot (\lambda_1 X_1 + \dots + \lambda_r X_r) = \lambda_1 X \cdot X_1 + \dots + \lambda_r X \cdot X_r = 0$$

which means that X is perpendicular to all elements of V. As a consequence, $X \in V^\perp$, as introduced in Exercise 2.

On the other hand, if $X \cdot Y = 0$ for any $Y \in V$, one has in particular that $X \cdot X_j = 0$ for $j = 1, 2, \dots, r$.

Conclusion: $X \cdot X_j = 0$ for $j = 1, 2 \dots, r$

$$\Longleftrightarrow \quad X \in \left(\mathrm{Vect}\{X_1, \dots, X_r\}\right)^\perp.$$

Exercise 4

Recall that P is a polynomial if $P(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \dots + \lambda_n x^n$ for some $n \in \mathbb{N}$. In addition, it is known that a polynomial is a continuous function on $\mathbb{R}$.

If P, P' are polynomials, i.e. $P(x) = \lambda_0 + \lambda_1 x + \dots + \lambda_n x^n$ and $P'(x) = \lambda'_0 + \lambda'_1 x + \dots + \lambda'_m x^m$ for some $n, m \in \mathbb{N}$ and all $x \in \mathbb{R}$, then

$$(P + P')(x) = (\lambda_0 + \lambda'_0) + (\lambda_1 + \lambda'_1) x + \dots + (\lambda_n + \lambda'_n) x^n + \dots + \lambda'_m x^m$$

(where we have supposed that $m > n$), and $P + P'$ is again a polynomial.

Similarly, $(\lambda P)(x) = \lambda\lambda_0 + \lambda\lambda_1 x + \dots + \lambda\lambda_n x^n$ and λP is again a polynomial. Thus, the set of polynomials is a subspace of the set of all continuous functions on $\mathbb{R}$.

A generating family is provided by $\{x^n\}_{n=0}^{\infty}$. This family contains an <u>infinite</u> number of elements!

Exercise 5

Let W be a subspace of V, and let $X, Y, Z \in W$ and $\lambda, \mu \in F$ (the field). One has to check 9 conditions in order to show that W itself is a vector space.

0) $X + Y \in W$ because W is a subspace.
1) $(X+Y)+Z = X+(Y+Z)$ since this condition holds for V.
2) $X + Y = Y + X$ since this condition holds for V.
3) Since W is a subspace, $\lambda X \in W$ for any $\lambda \in F$. In particular $0X = O \in W$ and $X + O = O + X = X$, as in V.
4) Since F is a field, there exits $1 \in F$ and $-1 \in F$ such that $1 + (-1) = 0 \in F$. Then $-1X = -X \in W$ and $-X$ satisfies $X + (-X) = 1X + (-1)X = (1+(-1))X = 0X = O$. Thus, for any $X \in W$, there exits $-X \in W$ such that $X + (-X) = O$.
5) It has already been mentioned that $\lambda X \in W$ for any $\lambda \in F$ and $1X = X$, as in V.
6) $\lambda(X+Y) = \lambda X + \lambda Y$, as in V.
7) $(\lambda + \mu) X = \lambda X + \mu X$, as in V, and this has already been used in 4).
8) $(\lambda\mu) X = \lambda(\mu X)$, as in V.

Exercise 6

Consider a $n \times n$ matrix. By using row elementary operations, changing the first column of the extended matrix into the column $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ requires $n \times n$ multiplicative operations (m.op.). Then, changing the second column into $\begin{pmatrix} x \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ (x arbitrary) requires $(n-1) \times n$ m. op. Going on like this for the n columns requires thus $n \times n + (n-1) \times n + (n-2) \times n \cdots + 2 \times n + n$ $= \left[\frac{1}{2}(n+1)n\right] n$ m. op. Then, changing again the columns in order to obtain the diagonal matrix requires

$$1 \times n + 2 \times n + \cdots + (n-2) \times n + (n-1) \times n = \left[\frac{1}{2}(n)(n-1)\right] n .$$

Thus, adding these 2 numbers, one obtains that

$\frac{1}{2} n^3 + \frac{1}{2} n^2 + \frac{1}{2} n^3 - \frac{1}{2} n^2 = \underline{n^3 \text{ m. op.}}$ are necessary for diagonalizing a $n \times n$ matrix.

Thus, for $n = 3$, 27 m. op. are necessary.

For $n = 12$, $12^3 = 1728$ m. op. are necessary.

If a computer can manage 27 m. op. by second, it will be necessary to wait $\frac{1728}{27} = 64$ seconds for the inversion of a 12×12 matrix.