Exercise 1

a) $$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 7 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -3 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 & -2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -16/5 & 3/5 \\ 0 & 1 & 0 & 0 & 7/5 & -1/5 \\ 0 & 0 & 1 & 0 & -2/5 & 1/5 \end{array}\right),$$

Thus, the inverse is $\frac{1}{5}\begin{pmatrix} 5 & -16 & 3 \\ 0 & 7 & -1 \\ 0 & -2 & 1 \end{pmatrix}$.

b) $$\left(\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 4 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -1 & 0 & 1 & 0 \\ 0 & -1 & -3 & -2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -10 & -6 & 1 & 3 \\ 0 & 1 & 3 & 2 & 0 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1/2 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & -1 \\ 0 & 0 & 1 & 6/10 & -1/10 & -3/10 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1/2 & -1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 2/10 & 3/10 & -1/10 \\ 0 & 0 & 1 & 6/10 & -1/10 & -3/10 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -4/20 & -1/20 & 7/20 \\ 0 & 1 & 0 & 2/10 & 3/10 & -1/10 \\ 0 & 0 & 1 & 6/10 & -1/10 & -3/10 \end{array}\right).$$ Thus, the inverse is $\frac{1}{20}\begin{pmatrix} -4 & -1 & 7 \\ 4 & 6 & -2 \\ 12 & -2 & -6 \end{pmatrix}$.

c) $$\left(\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 10 & 3 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & 2 & -5 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 1/2 & 1 & -3/2 \\ 0 & 0 & 1 & -1/2 & -1 & 5/2 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & -9/4 \\ 0 & 1 & 0 & 1/4 & 1/2 & -3/4 \\ 0 & 0 & 1 & -1/2 & -1 & 5/2 \end{array}\right).$$

Thus, the inverse is $\frac{1}{4}\begin{pmatrix} 3 & 2 & -9 \\ 1 & 2 & -3 \\ -2 & -4 & 10 \end{pmatrix}$.

d) $$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 8 & -5 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right).$$ The inverse is $\begin{pmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{pmatrix}$.

e) $$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & -4 & 0 & 0 & 1 \end{array}\right) \sim \dots \sim \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 \end{array}\right)$$

Thus, the inverse is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}$.

Exercise 2

i) One easily finds that $c_1 = 0$, $c_5 = 0$, $c_2 = -c_4$ and $c_3 = -c_6$.
Thus, if one sets $c = c_2$ and $d = c_3$, one gets

$$f(x,y) = cx + dy - cx^2 - dy^2.$$

ii) The augmented matrix corresponding to these equations is

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 9 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

It follows that

$$c_1 = 0, \quad c_2 = 0, \quad c_4 = 0, \quad c_3 + c_5 + c_6 = 0.$$

Thus, by setting $c_5 = c$ and $c_6 = d$ one gets

$$f(x,y) = -(c+d)y + cxy + dy^2 = 0$$
$$= c(xy - y) + d(y^2 - y).$$

Exercise 3

Let $X \in \mathbb{R}^n$ satisfying $AX = \mathbb{O}$, with $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

One has $AX = \mathbb{O} \iff (AX)_i = 0 \quad \forall i \in \{1, \dots, m\}$

$\iff \sum_{j=1}^{n} a_{ij} x_j = 0 \quad \forall i \in \{1, \dots, m\}$ with $A = (a_{ij})$

$\iff {}^tA_i \cdot X = 0 \quad \forall i \in \{1, \dots, m\}$

scalar product between two elements of $\mathbb{R}^n$.

$\iff {}^tA_i \perp X$ for any $i \in \{1, \dots, m\}$

On the other hand one also has

$\sum_{j=1}^{n} a_{ij} x_j = 0 \quad \forall i \in \{1, \dots, m\}$

$\iff \sum_{j=1}^{n} x_j a_{ij} = 0 \quad \forall i \in \{1, \dots, m\}$

$\iff x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \iff \sum_{j=1}^{n} x_j A^j = \mathbb{O}.$

Exercise 4

One has $\begin{pmatrix} 4 & 3-k \\ 1-k & 2 \end{pmatrix} \sim \begin{pmatrix} 4 & 3-k \\ 0 & 2-\frac{1}{4}(1-k)(3-k) \end{pmatrix} \sim \begin{pmatrix} 1 & (3-k)/4 \\ 0 & (k-5)(k+1) \end{pmatrix}$.

Thus, this upper-triangular matrix is invertible if and only if $(k-5)(k+1) \neq 0$, i.e. if and only if $k \notin \{5, -1\}$.

Exercise 5

$A = (a_{ij})$

i) A diagonal $\Leftrightarrow$ $a_{ij} = 0 \ \forall i \neq j$.

ii) A upper-triangular $\Leftrightarrow$ $a_{ij} = 0 \ \forall i > j$.

iii) A nilpotent $\Leftrightarrow$ $\exists m \in \mathbb{N}$ such that $A^m = 0$.

iv) A symmetric $\Leftrightarrow$ $A = {}^tA \Leftrightarrow a_{ij} = a_{ji}$.

v) A skew-symmetric $\Leftrightarrow$ ${}^tA = -A \Leftrightarrow a_{ij} = -a_{ji}$.

vi) A orthogonal $\Leftrightarrow$ ${}^tA = A^{-1} \Leftrightarrow {}^tA A = \mathbb{1}_n$.

vii) A invertible $\Leftrightarrow$ $\exists A^{-1}$ such that $A A^{-1} = \mathbb{1}_n$.

viii) A diagonal and invertible $\Leftrightarrow$ $a_{ij} = 0 \ \forall i \neq j$ and $a_{ii} \neq 0 \ \forall i$.

ix) A upper-triangular and invertible $\Leftrightarrow$
$a_{ij} = 0 \ \forall i > j$ and $a_{ii} \neq 0 \ \forall i$.