Homework 6

Exercise 1 *1. Find some* $\mathcal{A} \in M_2(\mathbb{R})$ *such that* $\mathcal{A}^2 = -\mathbf{1}_2$.

2. *Determine all* $\mathcal{A} \in M_2(\mathbb{R})$ *such that* $\mathcal{A}^2 = \mathcal{O}$.

Exercise 2 *Let* a, b *be real numbers and let*

$$\mathcal{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathcal{B} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

What is $\mathcal{AB}$ *? Compute* $\mathcal{A}^2$ *and* $\mathcal{A}^3$. *What is* $\mathcal{A}^m$ *for an arbitrary integer* m, *and how to prove it ?*

Exercise 3 *One says that a matrix* $\mathcal{A} \in M_n(\mathbb{R})$ *is orthogonal if* ${}^t\mathcal{A} = \mathcal{A}^{-1}$, *or equivalently if* ${}^t\mathcal{A}\mathcal{A} = \mathbf{1}_n$. *Show that if* $\mathcal{A} \in M_n(\mathbb{R})$ *is an orthogonal matrix, then*

1. $\|\mathcal{A}X\| = \|X\|$ *for any* $X \in \mathbb{R}^n$,
2. $(\mathcal{A}X) \cdot (\mathcal{A}Y) = X \cdot Y$ *for any* $X, Y \in \mathbb{R}^n$.

In other words, orthogonal matrices preserve lengths and angles between vectors of $\mathbb{R}^n$.

Exercise 4 *A special type of* 2×2 *matrices represents rotations in the plane. For arbitrary* $\theta \in \mathbb{R}$, *consider the matrix*

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

1. *For* ${}^tX = (1, 2)$, *what are its coordinates after a rotation of* $\pi/4$ *?*
2. *For* ${}^tY = (-1, 3)$, *what are its coordinates after a rotation of* $\pi/2$ *?*
3. *Show that for arbitrary* θ_1, θ_2 *one has* $R(\theta_1)\,R(\theta_2) = R(\theta_2)\,R(\theta_1)$,
4. *Show that for arbitrary* θ_1, θ_2 *one has* $R(\theta_1)\,R(\theta_2) = R(\theta_1 + \theta_2)$,
5. *Show that for any* θ, *the matrix* $R(\theta)$ *has an inverse and write down this inverse.*

Exercise 5 *Let*

$$\mathcal{A} = \begin{pmatrix} 1&0&2&0 \\ 0&3&0&4 \\ 5&0&6&0 \\ 0&7&0&8 \end{pmatrix}$$

and let $\mathcal{U}$ *be one of the matrices shown below. Compute* $\mathcal{UA}$.

$$a)\ \mathcal{U} = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \qquad b)\ \mathcal{U} = \begin{pmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \qquad c)\ \mathcal{U} = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{pmatrix}$$

$$d)\ \mathcal{U} = \begin{pmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \qquad e)\ \mathcal{U} = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \end{pmatrix} \qquad f)\ \mathcal{U} = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}$$

Exercise 6 *Do the same exercise with the following matrices* $\mathcal{U}$ *and* $\mathcal{A}$ *as above:*

$$a)\ \mathcal{U} = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} \quad b)\ \mathcal{U} = \begin{pmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{pmatrix} \quad c)\ \mathcal{U} = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&5&0&1 \end{pmatrix} \quad d)\ \mathcal{U} = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&-2&1&0 \\ 0&0&0&1 \end{pmatrix}$$