Nagoya University, G30 program Autumn 2014

Linear algebra I Instructor : Serge Richard

Final exam

Exercise 1 *Find the equation of the plane in* $\mathbb{R}^3$ *passing through the three points* $P_1 = (1,0,0)$, $P_2 = (0,2,0)$ *and* $P_3 = (0,0,3)$.

Exercise 2 *By using Gauss elimination, find all solutions for the following systems:*

$$a)\ \begin{cases} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_1 + 2x_3 - x_4 &= 2 \\ x_1 + 2x_2 + 3x_4 &= 1 \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 3 \end{cases} \qquad b)\ \begin{cases} -2x + 3y + z + 4w &= 0 \\ x + y + 2z + 3w &= 0 \\ 2x + y + z - 2w &= 0 \end{cases}$$

Exercise 3 *By using elementary row operations, find the inverse for the following matrix:* $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{pmatrix}$.

Exercise 4 *Consider the vectors* $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $X_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $X_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, *and* $X_5 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

1. *Does the family* $\{X_1, X_2, X_3, X_4, X_5\}$ *generate* $\mathbb{R}^4$, *and why ?*
2. *Is the family* $\{X_1, X_2, X_3, X_4, X_5\}$ *a basis for* $\mathbb{R}^4$, *and why ?*
3. *Is the family* $\{X_1, X_2, X_3, X_4\}$ *a basis for* $\mathbb{R}^4$, *and why ?*

Exercise 5 *Consider the map* $F : \mathbb{R}^3 \to \mathbb{R}^2$ *defined by* $F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ y-z \end{pmatrix}$ *for all* $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$.

1. *Is* F *a linear map ? (Justify your answer)*
2. *Determine the range of* F.
3. *Determine the kernel of* F.

Exercise 6 *A square* $n \times n$ *matrix is called a* permutation matrix *if it contains a* 1 *exactly once in each row and in each column, with all other entries being* 0, *as for example in* $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. *Are permutation matrices invertible ? And if so, is the inverse of such a matrix a permutation matrix as well ? Explain your answers.*

Exercise 7 *Consider the set* $\mathcal{E}_n$ *of all* $n \times n$ **diagonal** *matrices which satisfy* $a_{11}a_{22}\dots a_{nn} = 1$ *(the product of the elements of the diagonal is equal to* 1*).*

1. *Is* $\mathcal{E}_n$ *a vector space ? (Justify your answer)*
2. *Exhibit* $n-1$ *linearly independent elements of* $\mathcal{E}_n$.
3. *If* $\mathcal{A}, \mathcal{B} \in \mathcal{E}_n$, *does* $\mathcal{AB}$ *belong to* $\mathcal{E}_n$ *? (Justify your answer)*

Final exam

Exercise 1

One looks for $N \in \mathbb{R}^3$ such that $\vec{ON} \perp \vec{P_1P_2}$ and $\vec{ON} \perp \vec{P_1P_3}$.

If $N = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$, then $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \perp \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \iff -n_1 + 2n_2 = 0$ and

$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \perp \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \iff -n_1 + 3n_3 = 0$. Thus one gets

$\begin{cases} n_1 = 2n_2 \\ n_1 = 3n_3 \end{cases}$. By choosing $n_1 = 6$, one obtains $N = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$

and $H_{N,P_1} = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \right\}$

$$= \underline{\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; 6x + 3y + 2z = 6 \right\}}.$$

Exercise 2

a) The augmented matrix satisfies:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & -1 & 2 \\ 1 & 2 & 0 & 3 & 1 \\ 1 & 2 & 3 & 4 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 3 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 & 2 \end{pmatrix}.$$

However, one gets from the third row, that $0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$, which means that this system has no solution.

b) The augmented matrix satisfies

$$\begin{pmatrix} -2 & 3 & 1 & 4 & 0 \\ 1 & 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & 3 & 0 \\ 0 & 5 & 5 & 10 & 0 \\ 0 & -1 & -3 & -8 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & -2 & -6 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \end{pmatrix}.$$

The general solution is $\begin{cases} x = 2w \\ y = w \\ z = -3w \\ w \text{ arbitrary} \end{cases}$.

Exercise 3

See Homework 10, ex. 1.e).

Exercise 4

1) Yes, $\{X_1, X_2, X_3, X_4, X_5\}$ generate $\mathbb{R}^4$ because

$E_1 = X_1$, $E_2 = X_2 - X_1$, $E_3 = X_3 - X_2 - X_1$ and $E_4 = X_4 - X_3 - X_2 - X_1$

and $\{E_1, \dots, E_4\}$ is the standard basis for $\mathbb{R}^4$.

2) No, it is not a basis because 5 vectors in $\mathbb{R}^4$ can not be linearly independent. And indeed, $X_5 = X_3 - X_1$.

3) Yes, $\{X_1, \dots, X_4\}$ is a basis of $\mathbb{R}^4$ since they generate $E_1, \dots, E_4$ by linear combinations, and these vectors generate the standard basis of $\mathbb{R}^4$.

Exercise 5

1) Yes, F is linear. Indeed, if we consider $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ belong to $\mathbb{R}^3$, then

$$F\left(\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\right) = F\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix} = \begin{pmatrix} 2(x_1+x_2) \\ (y_1+y_2)-(z_1+z_2) \end{pmatrix} = \begin{pmatrix} 2x_1 \\ y_1 - z_1 \end{pmatrix} + \begin{pmatrix} 2x_2 \\ y_2 - z_2 \end{pmatrix}$$

$$= F\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + F\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}.$$

Similarly, for $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$, $F\left(\lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = F\begin{pmatrix} \lambda x \\ \lambda y \\ \lambda z \end{pmatrix} = \begin{pmatrix} 2\lambda x \\ \lambda y - \lambda z \end{pmatrix}$

$$= \lambda \begin{pmatrix} 2x \\ y - z \end{pmatrix} = \lambda F\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

2) $\mathrm{Ran}(F) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2x \\ y-z \end{pmatrix} \text{ for any } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^2 \right\}$

$$= \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; u = x,\ v = y \text{ for any } x, y \in \mathbb{R} \right\} = \mathbb{R}^2.$$

3) $\mathrm{Ker}(F) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} 2x \\ y-z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 \\ y \\ -y \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; y \in \mathbb{R} \right\}.$

Exercise 6 :

Yes, permutation matrices are invertible because they are row equivalent to $\mathbb{1}_n$ by simple permutation of the rows. The inverse of a permutation matrix is also a permutation matrix because it is obtained from $\mathbb{1}_n$ by simple permutations of rows. In fact, for a permutation matrix A one has $A^{-1} = {}^tA$.

Exercise 7

1) No, it is not a vector space since if $A = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$ belong to $\mathcal{E}_n$, then $\lambda A = \begin{pmatrix} \lambda a_{11} & & \\ & \ddots & \\ & & \lambda a_{nn} \end{pmatrix}$ does not belong to $\mathcal{E}_n$ for any $\lambda \neq 1$. Indeed, $(\lambda a_{11})(\lambda a_{22}) \cdots (\lambda a_{nn})$ $= \lambda^n a_{11} \cdots a_{nn} = \lambda^n$ if $a_{11} \cdots a_{nn} = 1$.

2) The following matrices belong to $\mathcal{E}_n$ and are linearly independent :

$$\begin{pmatrix} 2 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & 1 & \\ & & & 1/2 \end{pmatrix}, \begin{pmatrix} 1 & & & 0 \\ & 2 & & \\ & & 1 \ddots & \\ 0 & & 1 & 1/2 \end{pmatrix} \cdots \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & 1 \; 2 & 1/2 \end{pmatrix}.$$

3) If A, B belong to $\mathcal{E}_n$, with $A = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & & 0 \\ & \ddots & \\ 0 & & b_{nn} \end{pmatrix}$, then $AB = \begin{pmatrix} a_{11}b_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn}b_{nn} \end{pmatrix}$

and $(a_{11}b_{11})(a_{22}b_{22}) \cdots (a_{nn}b_{nn}) = a_{11}a_{22} \cdots a_{nn} b_{11} \cdots b_{nn}$
$= 1 \cdot 1 = 1$.

Thus AB belongs to $\mathcal{E}_n$.