Math Tutorial I - Calculus

Final

Name: Me

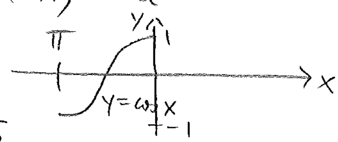
Explain your solution process clearly.
Write legible.

1. [10 points] Consider the equation

$$(*) \qquad 3x + 2\cos(x) + 5 = 0.$$

(a) Show that the equation (*) has a solution. Hint: Intermediate Value Theorem.

$f(x) = 3x + 2\cos(x) + 5$ is a continuous function for all x, since polynomials and $\cos(x)$ are continuous functions.



$f(0) = 7 > 0$, $f(-\pi) = -3\pi + 2\cos(-\pi) + 5 = -3\pi - 2 + 5$

$\Rightarrow f(-\pi) = -3\pi + 3 = 3(1-\pi) < 0$ since $\pi \approx 3.14\ldots$

Since 0 is in $(f(-\pi), f(0))$, the intermediate value theorem gives a number x_0 in $(-\pi, 0)$ s.t. $f(x_0) = 0$.

(b) Show that the equation (*) has only one solution.

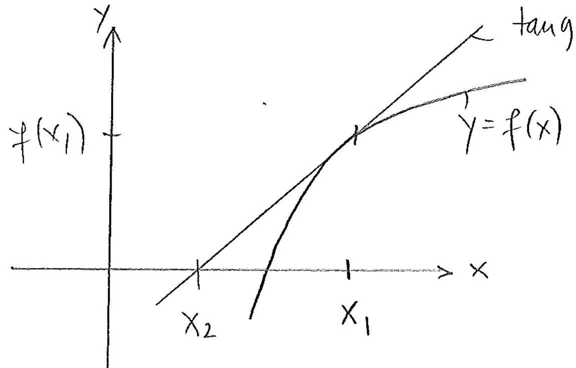
$f'(x) = 3 - 2\sin(x)$, $\sin(x) \leq 1 \Rightarrow -2\sin x \geq -2$

$\Rightarrow f'(x) \geq 1 > 0 \Rightarrow f$ is strictly increasing $\Rightarrow f$ is 1-1, hence there is no other number than x_0 which solves (*), or equivalently where f is zero.

(c) Use Newton's method with intial approximation $x_1 = -\pi$ to find the second approximation x_2 of the solution of (*).

Newton's method: $X_2 = X_1 - \dfrac{f(x_1)}{f'(x_1)} = -\pi - \dfrac{3(1-\pi)}{3} = \underline{\underline{-1}}$

(d) Explain geometrically how the second approximation x_2 in Newton's method is obtained (You may explain this in general or in regards to (*) above.)



tangent line to the graph of $f(x)$ at $(x_1, f(x_1))$

X_2 is the zero of the line tangent to the graph of $f(x)$ at $(x_1, f(x_1))$

This f is not the same as in (a) & (b)

2. [10 points] Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why and, if possible, give an example that disproves the statement.

(a) If $f'(c) = 0$, then f has a local maximum or minimum at c.

FALSE. Example: $f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f'(0) = 0$

but $f(x) > 0$ for all $x > 0$ and $f(x) < 0$ for all $x < 0$

$\Rightarrow$ f has neither a loc. max or min at $c = 0$.

(b) There exists a function f such that $f(1) = -2$, $f(3) = 0$, and $f'(x) > 1$ for all x.

FALSE: If there was, then by the Mean value theorem for derivatives there exists a c in $(1,3)$ such that

$f'(c) = \frac{f(3) - f(1)}{3 - 1} = 1$. This contradicts $f'(x) > 1$ for all x.

(c) Let f be a continuous, positive function on $[0,1]$. Then $\int_0^1 \sqrt{f(x)}\, dx = \sqrt{\int_0^1 f(x)\, dx}$.

FALSE: $f(x) = x^2 \Rightarrow \int_0^1 \sqrt{x^2}\, dx = \int_0^1 x\, dx = \frac{1}{2}$

$\sqrt{\int_0^1 x^2 dx} = \sqrt{\frac{1}{3}}$ $\neq$

(d) If f and g are continuous on $[0,1]$ then

$$\int_0^1 [f(x) \cdot g(x)]\, dx = \int_0^1 f(x)\, dx \cdot \int_0^1 g(x)\, dx.$$

FALSE: $f(x) = x = g(x)$

$\Rightarrow \int_0^1 f(x) \cdot g(x)\, dx = \frac{1}{3}$ but $\left(\int_0^1 f(x) dx\right)\left(\int_0^1 g(x)\, dx\right) = \left(\int_0^1 x\, dx\right)^2 = \frac{1}{4}$

$\frac{1}{3} \neq \frac{1}{4}$.

(e) $\int \tan(x)\, dx = \ln(\cos(x)) + C$.

FALSE: $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\int u^{-1} du = -\ln|u| + C$

$= -\ln|\cos(x)| + C$