Nagoya University, G30 program — Autumn 2013
Linear algebra I — Instructor : Serge Richard
Midterm exam

Exercise 1. Let $A = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ be points in $\mathbb{R}^3$. Compute the following expressions : (i) $A + B$, (ii) $B - A$, (iii) $B - 2A$, (iv) $C - A$.
Determine also : (v) $\|\overrightarrow{AB}\|$, (vi) $\overrightarrow{AB} \cdot \overrightarrow{AC}$, (vii) the cosine between the located vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

i) $A + B = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ ii) $B - A = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ iii) $B - 2A = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ iv) $C - A = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

v) $\|\overrightarrow{AB}\| = \|\overrightarrow{O(B-A)}\| = \sqrt{1^2 + 1^2} = \sqrt{2}$

vi) $\overrightarrow{AB} \cdot \overrightarrow{AC} = \overrightarrow{O(B-A)} \cdot \overrightarrow{O(C-A)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 2$

vii) $\cos\theta = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\|\,\|\overrightarrow{AC}\|} = \dfrac{2}{\sqrt{2}\,\sqrt{3}} = \dfrac{2}{\sqrt{6}}$.

Exercise 2. Let A, B, C, D be points in $\mathbb{R}^n$, and consider the located vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Recall the definitions of "$\overrightarrow{AB}$ is equivalent to $\overrightarrow{CD}$" and of "$\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CD}$".

"$\overrightarrow{AB}$ is equivalent to $\overrightarrow{CB}$" $\iff$ $B - A = D - C$.

"$\overrightarrow{AB}$ is perpendicular to $\overrightarrow{CB}$" $\iff$ "$\overrightarrow{O(B-A)}$ is perpendicular to $\overrightarrow{O(D-C)}$"

$\iff (B - A) \cdot (D - C) = 0$.

Exercise 3. Let $P = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \in \mathbb{R}^4$. Determine the hyperplane passing through P and which is normal to the vector $\overrightarrow{0P}$. Does the point ${}^T(0,0,0,0)$ belong to that plane? Same question with the points ${}^T(1,2,3,-2)$ and ${}^T(1,-2,2,3)$? What is the distance between the hyperplane and the point $Q = {}^T(2,2,2,2)$?

The normal to the hyperplane is $N = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Thus

$H_{P,N} = \{ X \in \mathbb{R}^4 \mid X \cdot N = P \cdot N \} = \{ {}^T(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 4 \}$.

Thus, ${}^T(0,0,0,0) \notin H_{P,N}$, but ${}^T(1,2,3,-2) \in H_{P,N}$ and ${}^T(1,-2,2,3) \in H_{P,N}$.

Then, the distance between Q and $H_{P,N}$ is given by

$$d(Q, H_{P,N}) = \frac{|(X - P) \cdot N|}{\|N\|} = \frac{|(1,1,1,1) \cdot (1,1,1,1)|}{\sqrt{4}} = \frac{4}{\sqrt{4}} = \sqrt{4}.$$

Exercise 4. Consider the matrices $\mathcal{A}, \mathcal{B} \in M_3(\mathbb{R})$ given by

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathcal{B} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

1. Compute $\mathcal{A}^2$, $\mathcal{B}^2$, $\mathcal{AB}$ and $\mathcal{BA}$.
2. Can you express $(\mathcal{A}+\mathcal{B})^2$ in terms of $\mathcal{A}^2$, $\mathcal{B}^2$, $\mathcal{AB}$ and $\mathcal{BA}$?
3. Compute $(\mathcal{A}+\mathcal{B})^2$ (you can use the previous question).

1. $\mathcal{A}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathcal{B}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\mathcal{AB} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathcal{BA} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

2. $(\mathcal{A}+\mathcal{B})^2 = (\mathcal{A}+\mathcal{B})(\mathcal{A}+\mathcal{B}) = \mathcal{A}^2 + \mathcal{AB} + \mathcal{BA} + \mathcal{B}^2$.

3. $(\mathcal{A}+\mathcal{B})^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$.

Exercise 5. Consider the matrices $\mathcal{U} \in M_4(\mathbb{R})$ given by

$$\mathcal{U} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

1. Compute ${}^T\mathcal{U}$,
2. Determine $\mathcal{U}^{-1}$,
3. Show that $\mathcal{U}$ is an orthogonal matrix,
4. Let $X \in \mathbb{R}^4$ with ${}^TX = (x_1, x_2, x_3, x_4)$. Show that $\|\mathcal{U}X\| = \|X\|$.

1. By definition ${}^T\mathcal{U} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ 2. $\mathcal{U}^{-1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ because

$$\mathcal{U}\mathcal{U}^{-1} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. Since ${}^T\mathcal{U} = \mathcal{U}^{-1}$, $\mathcal{U}$ is orthogonal. 4. cf Homework 5, ex 3, or do the computation.

Exercise 6. Consider the system of equations :

$$\begin{aligned} 2x + y + 4z + w &= 0 \\ -3x + 2y - 3z + w &= 0 \\ x + y + z &= 0 \end{aligned}$$

i) Write the corresponding augmented matrix, ii) find a row equivalent matrix in the standard form, iii) determine all solutions of that system.

i) and ii) $\begin{pmatrix} 2 & 1 & 4 & 1 & 0 \\ -3 & 2 & -3 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 5 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 10 & 6 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 3/5 & 0 \end{pmatrix}$

$\sim \begin{pmatrix} 1 & 0 & 0 & -4/5 & 0 \\ 0 & 1 & 0 & 1/5 & 0 \\ 0 & 0 & 1 & 3/5 & \end{pmatrix}$. iii) The general solution is $\begin{cases} x = 4/5\, w \\ y = -1/5\, w \\ z = -3/5\, w \\ w \text{ arbitrary} \end{cases}$