Nagoya University, G30 program — Autumn 2013

Mathematics Tutorial I (linear algebra part) — Instructor : Serge Richard

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Homework n° 8

Exercise 1. By using Gauss elimination, find all solutions for the following systems :

$$
\begin{aligned}
a) \quad x_4 + 2x_5 - x_6 &= 2\\
x_1 + 2x_2 + x_5 - x_6 &= 0\\
x_1 + 2x_2 + 2x_3 - x_5 + x_6 &= 2
\end{aligned}
$$

$$
\begin{aligned}
b) \quad x_1 + 2x_3 + 4x_4 &= -8\\
x_2 - 3x_3 - x_4 &= 6\\
3x_1 + 4x_2 - 6x_3 + 8x_4 &= 0\\
-x_2 + 3x_3 + 4x_4 &= -12
\end{aligned}
$$

Exercise 2. Find a polynomial of degree 3 whose graph goes through the points $(0,-1)$, $(1,-1)$, $(-1,-5)$ and $(2,1)$.

Exercise 3. For $r \in \{1,\dots,m\}$ and $s \in \{1,\dots,m\}$, let $I_{rs} \in M_m(\mathbb{R})$ be the matrix whose rs-component is 1 and all the other ones are equal to 0. First show that if $r,s,r',s' \in \{1,\dots,m\}$ then

$$
I_{rs}\, I_{r's'} = \begin{cases} I_{rs'} & \text{if } s = r' \\ 0 & \text{if } s \neq r' \end{cases}
$$

Then, for $c \neq 0$, consider the following 3 types of matrices :

1. $\mathbb{I}_m - I_{rr} + cI_{rr}$, the matrix obtained from the unit matrix by multiplying the r-th diagonal component by c,
2. For $r \neq s$, $(\mathbb{I}_m + I_{rs} + I_{sr} - I_{rr} - I_{ss})$, the matrix obtained from the unite matrix by interchanging the r-th row with the s-th row,
3. For $r \neq s$, $(\mathbb{I}_m + cI_{rs})$, the matrix having the rs-th component equal to c, all other components 0 except the diagonal components which are equal to 1.

Show that these matrices are invertible and exhibit their inverse. If $\mathcal{A} \in M_{mn}(\mathbb{R})$, show that multiplying the matrix $\mathcal{A}$ on the left by one of these matrices corresponds to one of the elementary row operations. For that reason, these matrices are called **elementary matrices**.

Exercise 4. Let $\mathcal{A}, \mathcal{A}' \in M_n(\mathbb{R})$ be row equivalent. With the help of the previous exercise, prove the following statements : $\mathcal{A}$ is invertible if and only if $\mathcal{A}'$ is invertible.