University of Nagoya, G30 program | Autumn 2013

Mathematics Tutorial I (linear algebra part) | Instructor : Serge Richard

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Homework n° 4

Exercise 1. Let us consider

$$\mathcal{A} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad \mathcal{B} = \begin{pmatrix} -1 & 5 & -2 \\ 1 & 1 & -1 \end{pmatrix}.$$

Compute $\mathcal{A}+\mathcal{B}$, $\mathcal{A}-2\mathcal{B}$, and ${}^T\!\mathcal{A}$.

Exercise 2. Let $\mathcal{A} \in M_n(\mathbb{R})$. Show that $\mathbb{I}_n\, \mathcal{A} = \mathcal{A} = \mathcal{A}\, \mathbb{I}_n$.

Exercise 3. Let $\mathcal{A} \in M_{mn}(\mathbb{R})$, $\mathcal{B}, \mathcal{C} \in M_{np}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Show that

1. $\mathcal{A}(\mathcal{B}+\mathcal{C}) = \mathcal{A}\mathcal{B} + \mathcal{A}\mathcal{C}$,
2. $(\lambda\mathcal{A})\mathcal{B} = \lambda(\mathcal{A}\mathcal{B}) = \mathcal{A}(\lambda\mathcal{B})$.

If $\mathcal{A} \in M_{mn}(\mathbb{R})$, $\mathcal{B} \in M_{np}(\mathbb{R})$ and $\mathcal{C} \in M_{pq}(\mathbb{R})$, show that

$$(\mathcal{A}\mathcal{B})\mathcal{C} = \mathcal{A}(\mathcal{B}\mathcal{C}).$$

If $\mathcal{A} \in M_{mn}(\mathbb{R})$ and $\mathcal{B} \in M_{np}(\mathbb{R})$, show that

$${}^T(\mathcal{A}\mathcal{B}) = {}^T\mathcal{B}\; {}^T\!\mathcal{A}.$$

Exercise 4. One says that a matrix $\mathcal{A} \in M_n(\mathbb{R})$ is *symmetric* if ${}^T\!\mathcal{A} = \mathcal{A}$ and is *skew-symmetric* if ${}^T\!\mathcal{A} = -\mathcal{A}$. Show that for an arbitrary matrix $\mathcal{A} \in M_n(\mathbb{R})$, the matrix $\mathcal{A} + {}^T\!\mathcal{A}$ is symmetric while the matrix $A - {}^T\!\mathcal{A}$ is skew-symmetric.

Exercise 5. Let $\mathcal{A} \in M_n(\mathbb{R})$.

1. If $\mathcal{A}^2 = 0$, show that $\mathbb{I}_n - \mathcal{A}$ is invertible.
2. More generally, if $\mathcal{A}$ is nilpotent, show that $\mathbb{I}_n - \mathcal{A}$ is invertible.
3. Suppose that $\mathcal{A}^2 + 2\mathcal{A} + \mathbb{I}_n = 0$. Show that $\mathcal{A}$ is invertible.

Exercise 6. If $\mathcal{A}, \mathcal{B} \in M_n(\mathbb{R})$ are two upper triangular matrices, show that the product $\mathcal{A}\mathcal{B}$ is also an upper triangular matrix.

Exercise 7.

1. Find $\mathcal{A} \in M_2(\mathbb{R})$ such that $\mathcal{A}^2 = -\mathbb{I}_2$.
2. Determine all $\mathcal{A} \in M_2(\mathbb{R})$ such that $\mathcal{A}^2 = 0$.