Nagoya University, G30 program Autumn 2013

Linear algebra I Instructor : Serge Richard

Final exam

Exercise 1. Find the equation of the plane in $\mathbb{R}^3$ passing through the three points $P_1 = (1,0,0)$, $P_2 = (0,2,0)$ and $P_3 = (0,0,3)$.

cf Homework 3, ex. 3

One looks for $N \in \mathbb{R}^3$ such that $\overrightarrow{ON} \perp \overrightarrow{P_1P_2}$ and $\overrightarrow{ON} \perp \overrightarrow{P_1P_3}$. If $N = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ this reads $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \perp \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \perp \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$

$\Leftrightarrow \begin{cases} -n_1 + 2n_2 = 0 \\ -n_1 + 3n_3 = 0 \end{cases}$. Choosing $n_1 = 6$, one gets $N = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$.

Then $\overrightarrow{ON} \cdot \overrightarrow{OP_1} = 6$, and therefore

$$H_{N,P_1} = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y,z)\cdot N = P_1 \cdot N\}$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid 6x + 3y + 2z = 6\}.$$

Exercise 2. By using Gauss elimination, find all solutions for the following systems :

$$a) \begin{cases} x_1 + 2x_3 + 4x_4 &= -8 \\ x_2 - 3x_3 - x_4 &= 6 \\ 3x_1 + 4x_2 - 6x_3 + 8x_4 &= 0 \\ -x_2 + 3x_3 + 4x_4 &= -12 \end{cases} \qquad b) \begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 &= 5 \\ x_1 + 3x_3 &= 2 \\ 2x_2 + 4x_4 &= 1 \\ x_1 + x_2 + x_3 + x_4 &= 0 \end{cases}$$

a) cf Homework 8 ex 1.(b)

b) The augmented matrix satisfies

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 3 & 0 & 2 \\ 0 & 2 & 0 & 4 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 0 & 4 & 1 \\ 0 & -2 & 0 & -4 & -3 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & -2 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

but the third row means that this system has no solution since $0 \neq -2$.

Exercise 3. By using elementary row operations, find the inverse for the following matrix : $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{pmatrix}$.

Cf Homework 10 ex. 1.(a)

Exercise 4. Consider the vectors $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $X_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $X_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, and $X_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

1. Does the family $\{X_1, X_2, X_3, X_4, X_5\}$ generate $\mathbb{R}^4$?
2. Is the family $\{X_1, X_2, X_3, X_4, X_5\}$ a basis for $\mathbb{R}^4$?
3. Is the family $\{X_2, X_3, X_4, X_5\}$ a basis for $\mathbb{R}^4$?

1) Yes, $\{X_1, X_2, X_3, X_4, X_5\}$ generates $\mathbb{R}^4$ since already $\{X_1, X_2, X_3, X_4\}$ does (it corresponds to the usual basis in $\mathbb{R}^4$).

2) No, because these vectors are not linearly independent.

3) Yes, these vectors are linearly independent and generate $\mathbb{R}^4$. The family $\{X_2, X_3, X_4, X_5\}$ is a basis for $\mathbb{R}^4$.

Exercise 5. Consider the set $\mathcal{E}_n$ of all $n \times n$ diagonal matrices, *i.e.* $\mathcal{A} = (a_{jk}) \in \mathcal{E}_n$ if and only if $\mathcal{A} \in M_n(\mathbb{R})$ and $a_{jk} = 0$ for all $j \neq k$.

1. Is $\mathcal{E}_n$ a vector space? (Justify your answer)
2. Exhibit a basis for $\mathcal{E}_n$
3. Is $\mathcal{E}_n$ a convex set? (Justify your answer)
4. How many elements of $\mathcal{E}_n$ are skew-symmetric?

1) Yes, if $\mathcal{A}, \mathcal{B} \in \mathcal{E}_n$, then $a_{jk} = 0$, $b_{jk} = 0$ $\forall j \neq k$. Then $a_{jk} + b_{jk} = 0$ and $\lambda a_{jk} = 0$ $\forall j \neq k$. Thus $\mathcal{A} + \mathcal{B} \in \mathcal{E}_n$ and $\lambda \mathcal{A} \in \mathcal{E}_n$, $\forall \lambda \in \mathbb{R}$.

2) $\begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \end{pmatrix}, \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix}, \dots \begin{pmatrix} 0 & & 0 \\ & 0 & \\ 0 & & 1 \end{pmatrix}$ are elements of $\mathcal{E}_n$ and are linearly independent. They generate a basis of $\mathcal{E}_n$.

3) Yes, similarly to 1), if $\mathcal{A}, \mathcal{B} \in \mathcal{E}_n$, then $(1-t)a_{jk} + t b_{jk} = 0$ $\forall t \in [0,1]$ and $j \neq k$. Thus $(1-t)\mathcal{A} + t\mathcal{B} \in \mathcal{E}_n$.

4) Only the element $\begin{pmatrix} 0 & & 0 \\ & 0 & \\ 0 & & 0 \end{pmatrix} = 0$ is skew-symmetric and diagonal. The answer is 1.

Exercise 6. Consider the map $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x, y, z) = (2x, y - z)$ for all $(x, y, z) \in \mathbb{R}^3$.

1. Is F a linear map? (Justify your answer)
2. Determine the range of F.
3. Determine the kernel of F.

1) Yes. Indeed, if $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$ one has

$F(x_1+x_2, y_1+y_2, z_1+z_2) = (2(x_1+x_2), (y_1+y_2)-(z_1+z_2)) =$

$= (2x_1, y_1-z_1) + (2x_2, y_2-z_2) = F(x_1, y_1, z_1) + F(x_2, y_2, z_2)$,

and $F(\lambda x_1, \lambda y_1, \lambda z_1) = (2\lambda x_1, \lambda y_1 - \lambda z_1) = \lambda(2x_1, y_1 - z_1) = \lambda F(x_1, y_1, z_1)$.

2) $\operatorname{Ran} F = \mathbb{R}^2$ since for any $(u, v) \in \mathbb{R}^2$, one has $(\frac{1}{2}u, v, 0) \in \mathbb{R}^3$ and $F(\frac{1}{2}u, v, 0) = (u, v)$.

3) $\ker F = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = (0, 0)\}$

$= \{(x, y, z) \in \mathbb{R}^3 \mid (2x, y - z) = (0, 0)\}$

$= \underline{\{(0, y, y) \in \mathbb{R}^3 \mid y \in \mathbb{R}\}}$.

Exercise 7. Let $\mathcal{A}$ be the matrix given by $\mathcal{A} = \begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{pmatrix}$ and consider the linear map $L_\mathcal{A} : \mathbb{R}^5 \to \mathbb{R}^3$ defined by $L_\mathcal{A} X = \mathcal{A}X$ for all $X \in \mathbb{R}^5$.

1. Determine the rank of $\mathcal{A}$ and the dimension of the range of $L_\mathcal{A}$.
2. Deduce the dimension of the kernel of $L_\mathcal{A}$.
3. Check that the vectors $X_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $X_2 = \begin{pmatrix} -8 \\ 0 \\ 0 \\ 1 \\ 6 \end{pmatrix}$ satisfy $\mathcal{A}X_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\mathcal{A}X_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.
4. Check that $X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ satisfies the equation $\mathcal{A}X_3 = \begin{pmatrix} 1 \\ 6 \\ 10 \end{pmatrix}$, and deduce the set of all solutions of the equation $\mathcal{A}X = \begin{pmatrix} 1 \\ 6 \\ 10 \end{pmatrix}$.

1) Rank $\mathcal{A} = 3$ because the 3 rows are linearly independent (or alternatively the columns 1, 3 and 5 are linearly independent). Since rank $\mathcal{A}$ = dim ran $L_\mathcal{A}$, it follows that dim ran $L_\mathcal{A} = 3$.

2) Since $5 = \dim \ker L_\mathcal{A} + \dim \operatorname{ran} L_\mathcal{A}$, one has $\dim \ker L_\mathcal{A} = 2$

3) Easy check.

4) X_3 is a solution of $\mathcal{A}X_3 = B$ with $B = \begin{pmatrix} 1 \\ 6 \\ 10 \end{pmatrix}$, and $\mathcal{A}X_1 = 0$ and $\mathcal{A}X_2 = 0$. Thus $X_3 + \lambda X_1 + \mu X_2$ satisfies

$\mathcal{A}(X_3 + \lambda X_1 + \mu X_2) = \mathcal{A}X_3 + \lambda \mathcal{A}X_1 + \mu \mathcal{A}X_2 = B + 0 + 0 = B, \ \forall \lambda, \mu \in \mathbb{R}$.

The solutions are $\underline{\{X_3 + \lambda X_1 + \mu X_2 \mid \lambda, \mu \in \mathbb{R}\}}$.