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Inner and outer continuity of the spectra for families of magnetic operators on $\mathbb{Z}^d$

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Abstract

In this note we consider magnetic self-adjoint operators on $\mathbb{Z}^d$ whose symbols and magnetic fields depend on a parameter ϵ. Sufficient conditions are imposed such that the spectrum of these operators varies continuously with respect to ϵ. The emphasize is put on a construction which is independent of any particular choice of the magnetic potentials.

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1 Introduction

This paper is an extended version of a presentation made at the conference *Spectral and Scattering Theory and Related Topics* at Rims in Kyoto in January 2016. The presentation was based on the paper [11] to which we refer for more details and for the proofs.

In the Hilbert space $\mathcal{H} := l^2(\mathbb{Z}^d)$ and for some fixed parameter ϵ let us consider operators of the form

$$[H^\epsilon u](x) := \sum_{y\in\mathbb{Z}^d} h^\epsilon(x; y-x)\, \mathrm{e}^{i\phi^\epsilon(x,y)}\, u(y) \tag{1.1}$$

with $u \in \mathcal{H}$ of finite support, $x \in \mathbb{Z}^d$ and where $h^\epsilon : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{C}$ and $\phi^\epsilon : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ satisfy

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(i) $\sum_{x\in\mathbb{Z}^d} \sup_{q\in\mathbb{Z}^d} |h^\epsilon(q;x)| < \infty$,

(ii) $\overline{h^\epsilon(q+x;-x)} = h^\epsilon(q;x)$ for any $q, x \in \mathbb{Z}^d$,

(iii) $\phi^\epsilon(x,y) = -\phi^\epsilon(y,x)$ for all $x, y \in \mathbb{Z}^d$.

Such operators are usually called *discrete magnetic Schrödinger operators.* Note that condition (i) ensures that H^ϵ extends continuously to a bounded operator in $\mathcal{H}$, and can be simply rewritten as $h^\epsilon \in l^1\big(\mathbb{Z}^d; l^\infty(\mathbb{Z}^d)\big)$. Conditions (ii) and (iii) imply that the operator H^ϵ is self-adjoint. Note also that a map $\phi : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ satisfying $\phi(x,y) = -\phi(y,x)$ for any $x, y \in \mathbb{Z}^d$ will be called *a magnetic potential.*

Our aim is exhibit some continuity properties of the spectrum of these operators under suitable and natural assumptions. Natural conditions on the family of symbols h^ϵ are imposed below. However, it is well-known (at least in the continuous setting) that continuity conditions should not be imposed on the magnetic potentials but rather on the magnetic fields. This requirement comes from the non-unicity for the choice of a magnetic potential. Indeed, if $\phi' : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ is defined for any $x, y \in \mathbb{Z}^d$ by

$$\phi'(x,y) \equiv [\phi + \nabla\varphi](x,y) := \phi(x,y) + \varphi(y) - \varphi(x) \tag{1.2}$$

for some $\varphi : \mathbb{Z}^d \to \mathbb{R}$, then the magnetic operators constructed with ϕ and ϕ' are known to be unitarily equivalent. This property is called *the gauge invariance* of the magnetic operators and imposes a slightly more elaborated notion of continuity for the magnetic contribution, as emphasized below.

Before explaining more in details the necessary construction, let us state a simplified version of our main theorem in which the ϵ-dependence on ϕ^ϵ is very simple. A more general setting will be introduced in the subsequent sections. The continuity we shall consider for the spectrum corresponds to the stability of the spectral gaps as well as the stability of the spectral compounds. In a more precise terminology we shall prove inner and outer continuity for the family of spectra. The following definition is borrowed from [1] but originally inspired by [3].

Definition 1.1. *Let Ω be a compact Hausdorff space, and let $\{\sigma_\epsilon\}_{\epsilon\in\Omega}$ be a family of closed subsets of $\mathbb{R}$.*

1. *The family $\{\sigma_\epsilon\}_{\epsilon\in\Omega}$ is* outer continuous *at $\epsilon_0 \in \Omega$ if for any compact subset $\mathcal{K}$ of $\mathbb{R}$ such that $\mathcal{K} \cap \sigma_{\epsilon_0} = \emptyset$ there exists a neighbourhood $\mathcal{N} = \mathcal{N}(\mathcal{K}, \epsilon_0)$ of ϵ_0 in Ω such that $\mathcal{K} \cap \sigma_\epsilon = \emptyset$ for any $\epsilon \in \mathcal{N}$,*

2. *The family $\{\sigma_\epsilon\}_{\epsilon\in\Omega}$ is* inner continuous *at $\epsilon_0 \in \Omega$ if for any open subset $\mathcal{O}$ of $\mathbb{R}$ such that $\mathcal{O} \cap \sigma_{\epsilon_0} \neq \emptyset$ there exists a neighbourhood $\mathcal{N} = \mathcal{N}(\mathcal{O}, \epsilon_0)$ of ϵ_0 in Ω such that $\mathcal{O} \cap \sigma_\epsilon \neq \emptyset$ for any $\epsilon \in \mathcal{N}$.*

With this definition at hand we can now choose $\Omega := [0,1]$ and state a special instance of our main result which is presented in Theorem 3.1. Note that the following statement is inspired from [8] and a comparison with the existing literature will be established just afterwards.

Theorem 1.2. *For each $\epsilon \in \Omega := [0,1]$ let $h^\epsilon : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{C}$ satisfy the above conditions (i) and (ii), and*

1) $\sum_{x\in\mathbb{Z}^d} \sup_{q\in\mathbb{Z}^d} \sup_{\epsilon\in[0,1]} |h^\epsilon(q;x)| < \infty$,

2) For any fixed $x \in \mathbb{Z}^d$,

$$\lim_{\epsilon'\to\epsilon} \sup_{q\in\mathbb{Z}^d} |h^{\epsilon'}(q;x) - h^\epsilon(q;x)| = 0.$$

Let also ϕ be a magnetic potential which satisfies

$$\big|\phi(x,y) + \phi(y,z) + \phi(z,x)\big| \leq \text{ area } \triangle(x,y,z),$$

where $\triangle(x,y,z)$ means the triangle in $\mathbb{R}^d$ determined by the three points $x,y,z \in \mathbb{Z}^d$. Then for H^ϵ defined on $u \in \mathcal{H}$ by

$$[H^\epsilon u](x) := \sum_{y\in\mathbb{Z}^d} h^\epsilon(x; y-x)\, \mathrm{e}^{i\epsilon\phi(x,y)}\, u(y)$$

the family of spectra $\sigma(H^\epsilon)$ forms an outer and an inner continuous family at every points $\epsilon \in \Omega$.

It is certainly impossible to mention all papers dealing with continuity properties of families of such operators, but let us cite a few of them which are relevant for our investigations. First of all, let us mention the seminal paper [3] in which the author proves the Lipschitz continuity of gap boundaries with respect to the variation of a constant magnetic field for a family of pseudodifferential operators acting on $\mathbb{Z}^2$. In [6] and based on the framework introduced in [13], similar Lipschitz continuity is proved for self-adjoint operators acting on a crystal lattice, a natural generalization of $\mathbb{Z}^d$. Papers [8] and [5] also deal with families of magnetic pseudodifferential operators on $\mathbb{Z}^2$, and the results contained in [8] partially motivated our work. Let us still mention two additional papers which are at the root of our work: [7] in which a general framework for magnetic systems, involving twisted crossed product C^*-algebras, is introduced, and the reference [1] which contains results similar to ours but in a continuous setting.

Let us now emphasize that the framework presented in Section 3 does not allow us to get any quantitative estimate, as emphasized in the recent paper [2]. Indeed, the very weak continuity requirement we impose on the ϵ-dependence on our objets can not lead to any Lipschitz or Hölder continuity. More stringent assumptions are necessary for that purpose, and such estimates certainly deserve further investigations.

Our approach relies on the concepts of twisted crossed product C^*-algebras and on a field of such algebras, mainly borrowed from [12, 14]. In the discrete setting, such algebras have already been used, for example in [3, 6, 13]. However, instead of considering a 2-cocycle with scalar values, which is sufficient for the case of a constant magnetic field, our 2-cocycles take values in the group of unitary elements of $l^\infty(\mathbb{Z}^d)$.

This allows us to consider arbitrary magnetic potential on $\mathbb{Z}^d$ and to encompass all the corresponding operators in a single algebra.

Let us finally describe the content of this paper. In Section 2 we introduce the framework for a single magnetic system, *i.e.* for a fixed ϵ. For that reason, no ϵ-dependence is indicated in this section. In Section 3 the ϵ-dependence is introduced and the continuous dependence on this parameter is studied. Our main result is presented in Theorem 3.1.

2 Discrete magnetic systems

In this section we introduce a quantity which does not depend on the choice of a particular magnetic potential. In that respect, this function depends only on the magnetic field, as emphasized below. Note that no ϵ-dependence is written in this section.

We start by recalling that a magnetic potential consists in a map $\phi : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ satisfying for any $x, y \in \mathbb{Z}^d$ the relation

$$\phi(x,y) = -\phi(y,x).$$

Then, given such a magnetic potential ϕ let us introduce and study a new map

$$\omega : \mathbb{Z}^d \times \mathbb{Z}^d \to l^\infty(\mathbb{Z}^d)$$

defined for $q, x, y \in \mathbb{Z}^d$ by

$$\big[\omega(x,y)\big](q) \equiv \omega(q;x,y) := \exp\big\{i\big[\phi(q,q+x)+\phi(q+x,q+x+y)+\phi(q+x+y,q)\big]\big\}. \quad (2.1)$$

Note that $\omega(x,y)$ is unitary-valued since $|\omega(q,;x,y)| = 1$ for any $q \in \mathbb{Z}^d$. In fact, $\omega(x,y)$ takes values in the unitary group of the algebra $l^\infty(\mathbb{Z}^d)$, which shall simply denote by $\mathscr{U}(\mathbb{Z}^d)$, *i.e.* $\mathscr{U}(\mathbb{Z}^d) = \{f : \mathbb{Z}^d \to \mathbb{T}\}$.

Let us also introduce the action θ of $\mathbb{Z}^d$ by translations, namely on any $f \in l^\infty(\mathbb{Z}^d)$ one sets

$$\theta_x f(y) = f(x+y).$$

In particular, since $\omega(x,y) \in l^\infty(\mathbb{Z}^d)$ we have

$$\big[\theta_z\omega(x,y)\big](q) := [\omega(x,y)](q+z) = \omega(q+z;x,y).$$

Based on these definitions, the following properties for ω can be obtained by straightforward computations. Recall that the notation $\phi + \nabla\varphi$ has been introduced in (1.2).

Lemma 2.1. *Let ϕ be a magnetic potential and let ω be defined by* (2.1). *Then for any $x, y, z \in \mathbb{Z}^d$ the following properties hold:*

(i) $\omega(x+y,z)\,\omega(x,y) = \theta_x\omega(y,z)\,\omega(x,y+z)$ *(2-cocycle property)*

(ii) $\omega(x,0) = \omega(0,x) = 1$ *(normalization)*

(iii) $\omega(x, -x) = 1$ *(additional property)*

(iv) For any $\varphi : \mathbb{Z}^d \to \mathbb{R}$, *the magnetic potentials* ϕ *and* $\phi + \nabla\varphi$ *define the same function* ω *(independence property).*

With the above lemma, one directly infers that ω is a normalized 2-cocycle on $\mathbb{Z}^d$ with values in $\mathscr{U}(\mathbb{Z}^d)$ and which satisfies the additional property (iii). In addition, this map depends only on equivalent classes of magnetic potentials, as emphasized in (iv). One could argue that the 2-cocycle ω depends only on the *magnetic field* as introduced in [4], and not on the choice of a magnetic potential. However, this would lead us too far from our purpose since we would have to consider $\mathbb{Z}^d$ as a graph endowed with edges between every pair of vertices.

Let us now adopt a very pragmatic point of view and recall only the strictly necessary information on twisted crossed product C^*-algebras. More can be found in the fundamental papers [9, 10] or in the review paper [7]. Since the group we are dealing with is simply $\mathbb{Z}^d$, most of the necessary information can also be found in [14].

Consider $l^1(\mathbb{Z}^d; l^\infty(\mathbb{Z}^d))$, with the norm

$$\|f\|_{1,\infty} := \sum_{x\in\mathbb{Z}^d} \sup_{q\in\mathbb{Z}^d} |f(q;x)| \qquad \forall f \in l^1(\mathbb{Z}^d; l^\infty(\mathbb{Z}^d)),$$

and with the twisted product and the involution defined for any $f, g \in l^1(\mathbb{Z}^d; l^\infty(\mathbb{Z}^d))$ by

$$[f \diamond g](q;x) := \sum_{y\in\mathbb{Z}^d} f(q;y)\; g(q+y; x-y)\; \omega(q; y, x-y)$$

and

$$f^\diamond(q;x) = \overline{f(q+x;-x)}.$$

Endowed with this multiplication and with this involution, $l^1(\mathbb{Z}^d; l^\infty(\mathbb{Z}^d))$ is a Banach $*$-algebra. The corresponding enveloping C^*-algebra will be denoted by $\mathfrak{C}(\omega)$. Recall that this algebra corresponds to the completion of $l^1(\mathbb{Z}^d; l^\infty(\mathbb{Z}^d))$ with respect to the C^*-norm defined as the supremum over all the faithful representations of $l^1(\mathbb{Z}^d; l^\infty(\mathbb{Z}^d))$. As a consequence, $l^1(\mathbb{Z}^d; l^\infty(\mathbb{Z}^d))$ is dense in $\mathfrak{C}(\omega)$ and the new C^*-norm $\|\cdot\|$ satisfies $\|f\| \leq \|f\|_{1,\infty}$.

Note that the above construction holds for any normalized 2-cocycle satisfying the additional property (iii). In fact, it is proved in [11] that any such 2-cocycle can be obtained by a magnetic potential, *i.e.* there always exists a magnetic potential ϕ satisfying (2.1).

Let us now look at a faithful representation of the algebra $\mathfrak{C}(\omega)$ in the Hilbert space $\mathcal{H} = l^2(\mathbb{Z}^d)$. For that purpose, consider any magnetic potential ϕ' satisfying

$$\exp\left\{i\left[\phi'(q, q+x) + \phi'(q+x, q+x+y) + \phi'(q+x+y, q)\right]\right\} = \omega(q; x, y). \tag{2.2}$$

Then for any $h \in l^1(\mathbb{Z}^d; l^\infty(\mathbb{Z}^d))$, any $u \in \mathcal{H}$ and any $x \in \mathbb{Z}^d$ one sets

$$[\mathfrak{Rep}^{\phi'}(h)u](x) := \sum_{y\in\mathbb{Z}^d} h(x; y-x)\; e^{i\phi'(x;y)}\; u(y).$$

Clearly, this expression is similar to the one contained in (1.1) and this fact partially justifies the entire construction. The main properties of this representation are gathered in the following statement, which corresponds to [7, Prop. 2.16 & 2.17] adapted to our setting.

Proposition 2.2. *Let ϕ' be any magnetic potential satisfying* (2.2) *for a given normalized 2-cocycle ω with the additional condition (iii) of Lemma 2.1. Then,*

(i) The representation $\mathfrak{Rep}^{\phi'}$ of $\mathfrak{C}(\omega)$ is irreducible and faithful,

(ii) Any other choice for ϕ' leads to a unitarily equivalent representation,

(iii) If $h \in l^1\big(\mathbb{Z}^d; l^\infty(\mathbb{Z}^d)\big)$, then $\mathfrak{Rep}^{\phi'}(h)$ is self-adjoint if $h^\diamond = h$.

3 A continuous field of C^*-algebras

In this section we consider a family of discrete magnetic systems which are parameterized by the elements ϵ of a compact Hausdorff space Ω. The necessary continuity relations between the various objects is encoded in the structure of a field of twisted crossed product C^*-algebras, as introduced in [12] and already used in a similar context in [1]. Again, let us be very pragmatic and introduce only the strictly necessary information, and refer to [11, Sec. 3] for more details.

For fixed $x, y \in \mathbb{Z}^d$, let us consider a continuous map

$$\Omega \ni \epsilon \mapsto \omega^\epsilon(x,y) \in \mathscr{U}(\mathbb{Z}^d) \tag{3.1}$$

such that each element $\omega^\epsilon(\cdot,\cdot)$ is a 2-cocycle on $\mathbb{Z}^d$ with values $\mathscr{U}(\mathbb{Z}^d)$ and which satisfies the additional property (iii) of Lemma 2.1. Note that equivalently one can consider a normalized 2-cocycle $\boldsymbol{\omega}$ on $\mathbb{Z}^d$ and with values in the C^*-algebra $C\big(\Omega; l^\infty(\mathbb{Z}^d)\big)$. In this setting the additional condition reads $\boldsymbol{\omega}(x,-x) = 1$ for any $x \in \mathbb{Z}^d$, and the following relation clearly holds:

$$\omega^\epsilon(q;x,y) \equiv \big[\omega^\epsilon(x,y)\big](q) = \big[\boldsymbol{\omega}(x,y)\big](\epsilon,q) \equiv \boldsymbol{\omega}(\epsilon,q;x,y).$$

Let us stress that the continuity assumption mentioned in the previous paragraph is precisely the sufficient one for the continuity of the spectrum. More precisely, this continuity condition corresponds to a gauge-independent condition, and does not correspond to any direct requirement on any magnetic potential ϕ^ϵ. So, let us now state the main result of this note:

Theorem 3.1. *Let $\{\omega^\epsilon\}_{\epsilon\in\Omega}$ be a family of normalized 2-cocycle satisfying the additional property (iii) of Lemma 2.1 and such that the map defined by* (3.1) *is continuous. Consider a family $\{h^\epsilon\}_{\epsilon\in\Omega} \subset l^1\big(\mathbb{Z}^d; l^\infty(\mathbb{Z}^d)\big)$ such that the following conditions are satisfied:*

(i) $\sum_{x\in\mathbb{Z}^d} \sup_{q\in\mathbb{Z}^d} \sup_{\epsilon\in\Omega} |h^\epsilon(q;x)| < \infty$,

(ii) For any fixed $x \in \mathbb{Z}^d$,

$$\lim_{\epsilon' \to \epsilon} \sup_{q \in \mathbb{Z}^d} |h^{\epsilon'}(q;x) - h^{\epsilon}(q;x)| = 0,$$

(iii) $(h^\epsilon)^\diamond = h^\epsilon$.

Then, for any family of magnetic potential ϕ^ϵ *satisfying*

$$\big[\omega^\epsilon(x,y)\big](q) = \exp\big\{i\big[\phi^\epsilon(q,q+x) + \phi^\epsilon(q+x,q+x+y) + \phi^\epsilon(q+x+y,q)\big]\big\},$$

the family of spectra $\left\{\sigma\big(\mathfrak{Rep}^{\phi^\epsilon}(h^\epsilon)\big)\right\}_{\epsilon\in\Omega}$ *forms an outer and an inner continuous family at every point* ϵ *of* Ω.

The proof of this statement as well as the proof of Theorem 1.2 are given in [11]. Let us simply mention that it is based on the construction of the enveloping C^*-algebra $\mathfrak{C}(\boldsymbol{\omega})$ of the Banach $*$-algebra $l^1\big(\mathbb{Z}^d; C(\Omega; l^\infty(\mathbb{Z}^d))\big)$ endowed with a product and an involution defined in terms of $\boldsymbol{\omega}$. Properties of the evaluation maps $e_\epsilon : \mathfrak{C}(\boldsymbol{\omega}) \to \mathfrak{C}(\omega^\epsilon)$ at any $\epsilon \in \Omega$ lead rather straightforwardly to our main theorem, once the abstract results obtained in [12] are taken into account.

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