

Spectral and scattering theory of one-dimensional coupled photonic crystals

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We study the spectral and scattering theory of light transmission in a system consisting of two asymptotically periodic waveguides, also known as one-dimensional photonic crystals, coupled by a junction. Using analyticity techniques and commutator methods in a two-Hilbert spaces setting, we determine the nature of the spectrum and prove the existence and completeness of the wave operators of the system.

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1. Introduction and Main Results

In this paper, we study the propagation of an electromagnetic field (\vec{E}, \vec{H}) in an infinite one-dimensional waveguide. We assume that (i) the waveguide is parallel to the x -axis of the Cartesian coordinate system; (ii) the electric field varies along the y -axis and is constant on the planes perpendicular to the x -axis, i.e. $\vec{E}(x, y, z, t) = \varphi_E(x, t)\hat{y}$; (iii) the magnetic field varies along the z -axis and is constant on the planes perpendicular to the x -axis, i.e. $\vec{H}(x, y, z, t) = \varphi_H(x, t)\hat{z}$; (iv) the waveguide

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is made of isotropic medium.^a Under these assumptions, one has $\nabla \times \vec{E} = (\partial_x \varphi_E) \hat{z}$ and $\nabla \times \vec{H} = (-\partial_x \varphi_H) \hat{y}$ and the dynamical sourceless Maxwell equations [14] read as

$$\begin{cases} \varepsilon \partial_t \varphi_E = -\partial_x \varphi_H, \\ \mu \partial_t \varphi_H = -\partial_x \varphi_E. \end{cases} \quad (1.1)$$

The scalar quantities ε and μ in (1.1) are the electric permittivity and magnetic permeability, respectively. They are strictly positive functions on \mathbb{R} describing the interaction of the waveguide with the electromagnetic field. One can also include in the model effects associated to bi-anisotropic media [21]. In our case, this is achieved by modifying the system (1.1) as follows:

$$\begin{cases} \varepsilon \partial_t \varphi_E + \chi \partial_t \varphi_H = -\partial_x \varphi_H, \\ \mu \partial_t \varphi_H + \chi^* \partial_t \varphi_E = -\partial_x \varphi_E. \end{cases} \quad (1.2)$$

The (possibly complex-valued) function χ is called bi-anisotropic coupling term.^b In the sequel, we will refer to the triple (ε, μ, χ) as the *constitutive* functions of the waveguide.

Let us first discuss the case of periodic waveguides, also known as one-dimensional photonic crystals, consisting in one-dimensional media with dielectric properties which vary periodically in space [15, 25, 44]. Mathematically, this translates into the fact that the functions ε, μ and χ in (1.2) are periodic, all with the same period. This makes (1.2) into a coupled system of differential equations with periodic coefficients, and standard techniques like Bloch–Floquet theory (see, e.g., [19]) can be used to study the propagation of solutions (or modes). One of the fundamental properties of periodic waveguides is the presence of a frequency spectrum made of bands and gaps. This implies that not all the modes can propagate along the medium, since the propagation of modes associated to frequencies inside a gap is forbidden by the “geometry” of the system. This phenomenon is similar to the one appearing in the theory of periodic Schrödinger operators, where one has electronic energy bands instead of frequency bands [28, Sec. XIII.16].

The study of the propagation of light in a periodic waveguide can be performed using Bloch–Floquet theory. The situation becomes more complicated when one wants to study the propagation of light through two periodic waveguides of different periods that are connected by a junction. Such a system is schematically represented in Fig. 1. The asymptotic behavior of the system on the left is characterized by the periodic constitutive functions $(\varepsilon_\ell, \mu_\ell, \chi_\ell)$, whereas the asymptotic behavior of the

^aThe interaction between the electromagnetic field and the dielectric medium is characterized by the electric permittivity tensor ε and the magnetic permeability tensor μ . In an isotropic medium these tensors are multiple of the identity, and thus determined by two scalars.

^bIn the general theory of bi-anisotropic media, χ is a tensor rather than a scalar. The system of equations (1.2) corresponds to a particular choice of the form of this tensor. For more details on the theory of bi-anisotropic media, we refer the interested reader to the monograph [21].

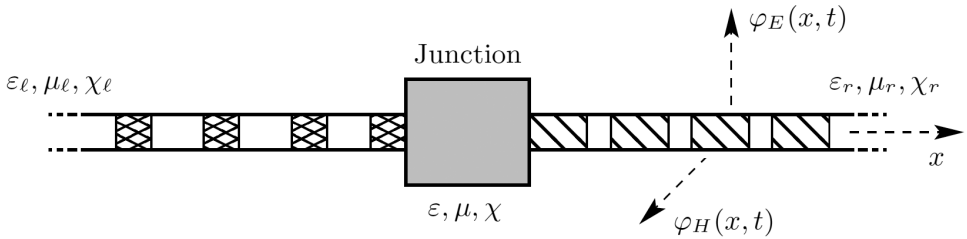


Fig. 1. Two periodic waveguides (one-dimensional photonic crystals) connected by a junction.

system on the right is characterized by periodic constitutive functions $(\varepsilon_r, \mu_r, \chi_r)$. Namely, $\varepsilon \rightarrow \varepsilon_\ell$ when $x \rightarrow -\infty$ and $\varepsilon \rightarrow \varepsilon_r$ when $x \rightarrow +\infty$, and similarly for the other two functions μ and χ (see Assumption 2.2 for a precise statement). The full system represented in Fig. 1 can therefore be interpreted as a perturbation of a “free” system obtained by glueing together two purely periodic systems, one with periodicity of type ℓ on the left and the other with periodicity of type r on the right. Accordingly, the analysis of the dynamics of the full system can be performed with the tools of spectral and scattering theories, leading us exactly to the main goal of this work: *the spectral and scattering analysis of one-dimensional coupled photonic crystals*.

Since quantum mechanics provides a rich toolbox for the study of problems associated to Schrödinger equations, we recast our equations of motion in a Schrödinger form to take advantage of these tools, in particular commutator methods which will be used extensively in this paper. Namely, with the notation $w := \begin{pmatrix} \varepsilon & \chi \\ \chi^* & \mu \end{pmatrix}^{-1}$ for the positive-definite matrix of weights associated to the constitutive functions ε , μ , χ (Maxwell weight for short), we rewrite the system of equations (1.2) in the matrix form

$$i\partial_t \begin{pmatrix} \varphi_E \\ \varphi_H \end{pmatrix} = w \begin{pmatrix} 0 & -i\partial_x \\ -i\partial_x & 0 \end{pmatrix} \begin{pmatrix} \varphi_E \\ \varphi_H \end{pmatrix}, \quad (1.3)$$

so that it can be considered as a Schrödinger equation for the state $(\varphi_E, \varphi_H)^T$ in the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^2)$. This observation is by no means new. Since the dawn of quantum mechanics, the founding fathers were well aware that the Maxwell equations in vacuum are relativistically covariant equations for a massless spin-1 particle [39, pp. 151, 198]. Moreover, similar Schrödinger formulations have already been employed in the literature to study the quantum scattering theory of electromagnetic waves and other classical waves in homogeneous media [5, 16, 27, 37, 41], and to study the propagation of light in periodic media [8, 9, 11, 20], among other things. However, to the best of our knowledge, the specific problem we want to tackle in this work has never been considered in the literature.

The papers [27, 37, 41] deal with the scattering theory of three-dimensional electromagnetic waves in a homogeneous medium. In that setup, the constitutive tensors ε , μ , χ are asymptotically constant. In contrast, in our one-dimensional setup

the constitutive functions ε , μ , χ are only assumed to be asymptotically periodic. This introduces a significant complication and novelty to the model, even though it has lower dimension than the three-dimensional models. Also, several works dealing with the scattering theory of electromagnetic waves are conducted under the simplifying assumption that $\chi = 0$ (absence of bi-anisotropic effects), an assumption that we do not make in this work. The papers [4, 40] deal with the transmission of the electric field and voltage along lines, also called one-dimensional Ohmic conductors. Mathematically, this problem is described by a system of differential equations similar to (1.2) or (1.3). However, in these papers, the constitutive quantities, namely the self-inductance and capacitance, are once again assumed to be asymptotically constant, in contrast with our less restrictive assumption of asymptotic periodicity. Finally, in the paper [43], almost no restrictions are imposed on the asymptotic behavior of the constitutive functions, but a stronger condition (invertibility) is imposed on the operator modeling the junction. Here, we do not assume that this operator is invertible or isometric, since we want to describe the scattering effects produced by the introduction of the junction itself, without imposing unnecessary conditions on the relation between the free dynamics without interface and the full dynamics in presence of the interface (see Remark 2.5 for more details). Also, even though the results of that paper hold in any space dimension, our results for one-dimensional photonic crystals are more detailed.

To conclude our overview of the literature, we point out that the dynamical equations describing our model are common to other physical systems. This is for instance the case of the equations describing the propagation of an Alfvén wave in a periodically stratified stationary plasma [2], the propagation of linearized water waves in a periodic bottom topography [7], or the propagation of harmonic acoustic waves in periodic waveguides [3]. In consequence, the results of our analysis here can be applied to all these models by reinterpreting in an appropriate way the necessary quantities.

Here is a description of our results. In Sec. 2.1, we introduce our assumption on the Maxwell weight w (Assumption 2.2) and we define the full Hamiltonian M in the Hilbert space \mathcal{H}_w describing the one-dimensional coupled photonic crystal. In Sec. 2.2, we define the free Hamiltonian M_0 in the Hilbert space \mathcal{H}_0 associated to M , and we define the operator $J : \mathcal{H}_0 \rightarrow \mathcal{H}_w$ modeling the junction depicted in Fig. 1 (Definition 2.4). The operator M_0 is the direct sum of an Hamiltonian M_ℓ describing the periodic waveguide asymptotically on the left and an Hamiltonian M_r describing the periodic waveguide asymptotically on the right. In Sec. 2.3, we use Bloch–Floquet theory to show that the asymptotic Hamiltonians M_ℓ and M_r fiber analytically in the sense of Gérard and Nier [12] (Proposition 2.6). As a by-product, we prove that M_ℓ and M_r do not possess flat bands, and thus have purely absolutely continuous spectra (Proposition 2.8). The analytic fibration of M_ℓ and M_r provides also a natural definition for the set \mathcal{T}_M of thresholds in the spectrum of M (Eqs. (2.5) and (2.6)). In Sec. 3.1, we recall from [1, 33] the necessary abstract results on commutator methods for self-adjoint operators. In Sec. 3.2, we

construct for each compact interval $I \subset \mathbb{R} \setminus \mathcal{T}_M$ a conjugate operator $A_{0,I}$ for the free Hamiltonian M_0 and use it to prove a limiting absorption principle for M_0 in I (Theorem 3.3 and the discussion that follows). In Sec. 3.3, we use the fact that $A_{0,I}$ is a conjugate operator for M_0 and abstract results on the Mourre theory in a two-Hilbert spaces setting [32] to show that the operator $A_I := JA_{0,I}J^*$ is a conjugate operator for M (Theorem 3.9). In Sec. 3.4, we use the operator A_I to prove a limiting absorption principle for M in I , which implies in particular that in any compact interval $I \subset \mathbb{R} \setminus \mathcal{T}_M$ the Hamiltonian M has at most finitely many eigenvalues, each one of finite multiplicity, and no singular continuous spectrum (Theorem 3.15). Using Zhislin sequences (a particular type of Weyl sequences), we also show in Proposition 3.10 that M and M_0 have the same essential spectrum. In Sec. 4.1, we recall abstract criteria for the existence and the completeness of wave operators in a two-Hilbert spaces setting. Finally, in Sec. 4.2, we use these abstract results in conjunction with the results of the previous sections to prove the existence and the completeness of wave operators for the pair (M_0, M) (Theorem 4.6). We also give an explicit description of the initial sets of the wave operators in terms of the asymptotic velocity operators for the Hamiltonians M_ℓ and M_r (Proposition 4.8 and Theorem 4.10).

2. Model

2.1. Full Hamiltonian

In this section, we introduce the full Hamiltonian M that we will study. It is a one-dimensional Maxwell-like operator describing perturbations of an anisotropic periodic one-dimensional photonic crystal.

Throughout the paper, for any Hilbert space \mathcal{H} , we write $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ for the scalar product on \mathcal{H} , $\| \cdot \|_{\mathcal{H}}$ for the norm on \mathcal{H} , $\mathcal{B}(\mathcal{H})$ for the set of bounded operators on \mathcal{H} and $\mathcal{K}(\mathcal{H})$ for the set of compact operators on \mathcal{H} . We also use the notation $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ (respectively, $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$) for the set of bounded (respectively, compact) operators from a Hilbert space \mathcal{H}_1 to a Hilbert space \mathcal{H}_2 .

Definition 2.1 (One-Dimensional Maxwell-Like Operator). Let $0 < c_0 < c_1 < \infty$ and take a Hermitian matrix-valued function $w \in L^\infty(\mathbb{R}, \mathcal{B}(\mathbb{C}^2))$ such that $c_0 \leq w(x) \leq c_1$ for a.e. $x \in \mathbb{R}$. Let P be the momentum operator in $L^2(\mathbb{R})$, that is, $Pf := -if'$ for each $f \in \mathcal{H}^1(\mathbb{R})$, with $\mathcal{H}^1(\mathbb{R})$ the first Sobolev space on \mathbb{R} . Let

$$D\varphi := \begin{pmatrix} 0 & P \\ P & 0 \end{pmatrix} \varphi, \quad \varphi \in \mathcal{D}(D) := \mathcal{H}^1(\mathbb{R}, \mathbb{C}^2).$$

Then the Maxwell-like operator M in $L^2(\mathbb{R}, \mathbb{C}^2)$ is defined as

$$M\varphi := wD\varphi, \quad \varphi \in \mathcal{D}(M) := \mathcal{D}(D).$$

The Maxwell weight w that we consider converges at $\pm\infty$ to periodic functions in the following sense.

Assumption 2.2 (Maxwell Weight). There exist $\varepsilon > 0$ and Hermitian matrix-valued functions $w_\ell, w_r \in L^\infty(\mathbb{R}, \mathcal{B}(\mathbb{C}^2))$ periodic of periods $p_\ell, p_r > 0$ such that

$$\begin{aligned} \|w(x) - w_\ell(x)\|_{\mathcal{B}(\mathbb{C}^2)} &\leq \text{Const.} \langle x \rangle^{-1-\varepsilon}, \quad \text{a.e. } x < 0, \\ \|w(x) - w_r(x)\|_{\mathcal{B}(\mathbb{C}^2)} &\leq \text{Const.} \langle x \rangle^{-1-\varepsilon}, \quad \text{a.e. } x > 0, \end{aligned} \tag{2.1}$$

where the indexes ℓ and r stand for “left” and “right”, and $\langle x \rangle := (1 + |x|^2)^{1/2}$.

Lemma 2.3. *Let Assumption 2.2 be satisfied.*

(a) *One has for a.e. $x \in \mathbb{R}$ the inequalities*

$$c_0 \leq w_\ell(x) \leq c_1 \quad \text{and} \quad c_0 \leq w_r(x) \leq c_1,$$

with c_0, c_1 introduced in Definition 2.1.

(b) *The sesquilinear form*

$$\langle \cdot, \cdot \rangle_{\mathcal{H}_w} : L^2(\mathbb{R}, \mathbb{C}^2) \times L^2(\mathbb{R}, \mathbb{C}^2) \rightarrow \mathbb{C}, \quad (\psi, \varphi) \mapsto \langle \psi, w^{-1} \varphi \rangle_{L^2(\mathbb{R}, \mathbb{C}^2)},$$

defines a new scalar product on $L^2(\mathbb{R}, \mathbb{C}^2)$, and we denote by \mathcal{H}_w the space $L^2(\mathbb{R}, \mathbb{C}^2)$ equipped with $\langle \cdot, \cdot \rangle_{\mathcal{H}_w}$. Moreover, the norm of $L^2(\mathbb{R}, \mathbb{C}^2)$ and \mathcal{H}_w are equivalent, and the claim remains true if one replaces w with w_ℓ or w_r .

(c) *The operator M with domain $\mathcal{D}(M) := \mathcal{H}^1(\mathbb{R}, \mathbb{C}^2)$ is self-adjoint in \mathcal{H}_w .*

Proof. Point (a) is a direct consequence of the assumptions on w, w_ℓ, w_r . Point (b) follows from the bounds $c_0 \leq w(x), w_\ell(x), w_r(x) \leq c_1$ valid for a.e. $x \in \mathbb{R}$. Point (c) can be proved as in [10, Proposition 6.2]. \square

2.2. Free Hamiltonian

We now define the free Hamiltonian associated to the operator M . Due to the anisotropy of the Maxwell weight w at $\pm\infty$, it is convenient to define left and right asymptotic operators

$$M_\ell := w_\ell D \quad \text{and} \quad M_r := w_r D,$$

with w_ℓ and w_r as in Assumption 2.2. Lemma 2.3(c) implies that the operators M_ℓ and M_r are self-adjoint in the Hilbert spaces \mathcal{H}_{w_ℓ} and \mathcal{H}_{w_r} , with the same domain $\mathcal{D}(M_\ell) = \mathcal{D}(M_r) = \mathcal{D}(M)$. Then we define the free Hamiltonian as the direct sum operator

$$M_0 := M_\ell \oplus M_r$$

in the Hilbert space $\mathcal{H}_0 := \mathcal{H}_{w_\ell} \oplus \mathcal{H}_{w_r}$. Since the free Hamiltonian acts in the Hilbert space \mathcal{H}_0 and the full Hamiltonian acts in the Hilbert space \mathcal{H}_w , we need to introduce an identification operator between the spaces \mathcal{H}_0 and \mathcal{H}_w :

Definition 2.4 (Junction Operator). Let $j_\ell, j_r \in C^\infty(\mathbb{R}, [0, 1])$ be such that

$$j_\ell(x) := \begin{cases} 1 & \text{if } x \leq -1, \\ 0 & \text{if } x \geq -1/2 \end{cases} \quad \text{and} \quad j_r(x) := \begin{cases} 0 & \text{if } x \leq 1/2, \\ 1 & \text{if } x \geq 1. \end{cases}$$

Then $J : \mathcal{H}_0 \rightarrow \mathcal{H}_w$ is the bounded operator defined by

$$J(\varphi_\ell, \varphi_r) := j_\ell \varphi_\ell + j_r \varphi_r,$$

with adjoint $J^* : \mathcal{H}_w \rightarrow \mathcal{H}_0$ given by $J^* \varphi = (w_\ell w^{-1} j_\ell \varphi, w_r w^{-1} j_r \varphi)$.

Remark 2.5. We call J the junction operator because it models mathematically the junction depicted in Fig. 1. Indeed, the Hamiltonian M_0 only describes the free dynamics of the system in the bulk asymptotically on the left and in the bulk asymptotically on the right. Since M_0 is the direct sum of the operators M_ℓ and M_r , the interface effects between the left and the right parts of the system are not described by M_0 in any way. The role of the operator J is thus to map the free bulk states of the system belonging to the direct sum Hilbert space \mathcal{H}_0 onto a *joined* state belonging to the physical Hilbert space \mathcal{H}_w , where acts the full Hamiltonian M describing the interface effects.

Given a state $\varphi \in \mathcal{H}_w$, the square norm $E(\varphi) := \|\varphi\|_{\mathcal{H}_w}^2$ can be interpreted as the total energy of the electromagnetic field $\varphi \equiv (\varphi_E, \varphi_H)^T$. A direct computation shows that the total energy of a state $J(\varphi_\ell, \varphi_r)$ obtained by joining bulk states $\varphi_\ell \in \mathcal{H}_\ell$ and $\varphi_r \in \mathcal{H}_r$ satisfies

$$E(J(\varphi_\ell, \varphi_r)) = E_\ell(\varphi_\ell) + E_r(\varphi_r) + E_{\text{interface}}(\varphi_\ell, \varphi_r)$$

with $E_\ell(\varphi_\ell) := \|j_\ell \varphi_\ell\|_{\mathcal{H}_{w_\ell}}^2$ and $E_r(\varphi_r) := \|j_r \varphi_r\|_{\mathcal{H}_{w_r}}^2$ the total energies of the field $j_\ell \varphi_\ell$ on the left and the field $j_r \varphi_r$ on the right, and with

$$\begin{aligned} E_{\text{interface}}(\varphi_\ell, \varphi_r) &:= \langle \varphi_\ell, j_\ell(w^{-1} - w_\ell^{-1})j_\ell \varphi_\ell \rangle_{L^2(\mathbb{R}, \mathbb{C}^2)} \\ &\quad + \langle \varphi_r, j_r(w^{-1} - w_r^{-1})j_r \varphi_r \rangle_{L^2(\mathbb{R}, \mathbb{C}^2)} \end{aligned}$$

the energy associated with the left and right external interfaces of the junction. In particular, one notices that there is no contribution to the energy associated to the central region $(-1/2, 1/2)$ of the junction. This physical observation shows as a by-product that the operator J is neither invertible, nor isometric.

2.3. Fiberings of the free Hamiltonian

In this section, we introduce a Bloch–Floquet (or Bloch–Floquet–Zak or Floquet–Gelfand) transform to take advantage of the periodicity of the operators M_ℓ and M_r . For brevity, we use the symbol \star to denote either the index “ ℓ ” or the index “ r ”.

Let

$$\Gamma_\star := \{np_\star \mid n \in \mathbb{Z}\} \subset \mathbb{R}$$

be the one-dimensional lattice of period p_\star with fundamental cell $Y_\star := [-p_\star/2, p_\star/2]$, and let

$$\Gamma_\star^* := \{2\pi n/p_\star \mid n \in \mathbb{Z}\} \subset \mathbb{R}$$

be the reciprocal lattice of Γ_\star with fundamental cell $Y_\star^* := [-\pi/p_\star, \pi/p_\star]$. For each $t \in \mathbb{R}$, we define the translation operator

$$T_t : L_{\text{loc}}^2(\mathbb{R}, \mathbb{C}^2) \rightarrow L_{\text{loc}}^2(\mathbb{R}, \mathbb{C}^2), \quad \varphi \mapsto \varphi(\cdot - t).$$

Using this operator, we can define the Bloch–Floquet transform of a \mathbb{C}^2 -valued Schwartz function $\varphi \in \mathcal{S}(\mathbb{R}, \mathbb{C}^2)$ as

$$(\mathcal{U}_\star \varphi)(k, \theta) := \sum_{n \in \mathbb{Z}} e^{-ik(\theta - np_\star)} (T_{np_\star} \varphi)(\theta), \quad k, \theta \in \mathbb{R}.$$

One can verify that $\mathcal{U}_\star \varphi$ is p_\star -periodic in the variable θ ,

$$(\mathcal{U}_\star \varphi)(k, \theta + \gamma) = (\mathcal{U}_\star \varphi)(k, \theta), \quad \gamma \in \Gamma_\star,$$

and $2\pi/p_\star$ -pseudo-periodic in the variable k ,

$$(\mathcal{U}_\star \varphi)(k + \gamma^*, \theta) = e^{-i\theta\gamma^*} (\mathcal{U}_\star \varphi)(k, \theta), \quad \gamma^* \in \Gamma_\star^*.$$

Now, let \mathfrak{h}_\star be the Hilbert space obtained by equipping the set

$$\{\varphi \in L_{\text{loc}}^2(\mathbb{R}, \mathbb{C}^2) \mid T_\gamma \varphi = \varphi \text{ for all } \gamma \in \Gamma_\star\}$$

with the scalar product

$$\langle \varphi, \psi \rangle_{\mathfrak{h}_\star} := \int_{Y_\star} d\theta \langle \varphi(\theta), w_\star(\theta)^{-1} \psi(\theta) \rangle_{\mathbb{C}^2}.$$

Since \mathfrak{h}_\star and $L^2(Y_\star, \mathbb{C}^2)$ are isomorphic, we shall use both representations. Next, let $\tau : \Gamma_\star^* \rightarrow \mathcal{B}(\mathfrak{h}_\star)$ be the unitary representation of the dual lattice Γ_\star^* on \mathfrak{h}_\star given by

$$(\tau(\gamma^*)\varphi)(\theta) := e^{i\theta\gamma^*} \varphi(\theta), \quad \gamma^* \in \Gamma_\star^*, \varphi \in \mathfrak{h}_\star, \text{ a.e. } \theta \in \mathbb{R},$$

and let $\mathcal{H}_{\tau, \star}$ be the Hilbert space obtained by equipping the set

$$\{u \in L_{\text{loc}}^2(\mathbb{R}, \mathfrak{h}_\star) \mid u(\cdot - \gamma^*) = \tau(\gamma^*)u \text{ for all } \gamma^* \in \Gamma_\star^*\}$$

with the scalar product

$$\langle u, v \rangle_{\mathcal{H}_{\tau, \star}} := \frac{1}{|Y_\star^*|} \int_{Y_\star^*} dk \langle u(k), v(k) \rangle_{\mathfrak{h}_\star}.$$

There is a natural isomorphism from $\mathcal{H}_{\tau, \star}$ to $L^2(Y_\star^*, \mathfrak{h}_\star)$ given by the restriction from \mathbb{R} to Y_\star^* , and with inverse given by τ -equivariant continuation. However, using $\mathcal{H}_{\tau, \star}$ has various advantages and we shall stick to it in the sequel. Direct calculations show that the Bloch–Floquet transform extends to a unitary operator $\mathcal{U}_\star : \mathcal{H}_{w_\star} \rightarrow \mathcal{H}_{\tau, \star}$ with inverse

$$(\mathcal{U}_\star^{-1}u)(x) = \frac{1}{|Y_\star^*|} \int_{Y_\star^*} dk e^{ikx} (u(k))(x), \quad u \in \mathcal{H}_{\tau, \star}, \text{ a.e. } x \in \mathbb{R}.$$

Furthermore, since M_\star commutes with the translation operators T_γ ($\gamma \in \Gamma_\star$), the operator M_\star is decomposable in the Bloch–Floquet representation. Namely, we have

$$\widehat{M}_\star := \mathcal{U}_\star M_\star \mathcal{U}_\star^{-1} = \{\widehat{M}_\star(k)\}_{k \in \mathbb{R}}$$

with

$$\widehat{M}_*(k - \gamma^*) = \tau(\gamma^*)\widehat{M}_*(k)\tau(\gamma^*)^*, \quad k \in \mathbb{R}, \gamma^* \in \Gamma_*^*, \quad (2.2)$$

and

$$\widehat{M}_*(k) = w_*\widehat{D}(k) \quad \text{and} \quad \widehat{D}(k)u(k) = \begin{pmatrix} 0 & -i\partial_\theta + k \\ -i\partial_\theta + k & 0 \end{pmatrix} u(k),$$

$$k \in Y_*^*, u \in \mathcal{U}_*\mathcal{D}(M_*).$$

Here, the domain $\mathcal{U}_*\mathcal{D}(M_*)$ of $\mathcal{U}_*M_*\mathcal{U}_*^{-1}$ satisfies

$$\mathcal{U}_*\mathcal{D}(M_*) = \mathcal{U}_*\mathcal{H}^1(\mathbb{R}, \mathbb{C}^2) = \{u \in L_{\text{loc}}^2(\mathbb{R}, \mathfrak{h}_*^1) \mid u(\cdot - \gamma^*) = \tau(\gamma^*)u \text{ for all } \gamma^* \in \Gamma_*^*\}$$

with

$$\mathfrak{h}_*^1 := \{\varphi \in \mathcal{H}_{\text{loc}}^1(\mathbb{R}, \mathbb{C}^2) \mid T_\gamma\varphi = \varphi \text{ for all } \gamma \in \Gamma_*^*\}.$$

In the following proposition, we prove that the operator \widehat{M}_* is analytically fibered in the sense of [12, Definition 2.2]. For this, we need to introduce the Bloch variety

$$\Sigma_* := \{(k, \lambda) \in Y_*^* \times \mathbb{R} \mid \lambda \in \sigma(\widehat{M}_*(k))\}. \quad (2.3)$$

Proposition 2.6 (Fibering of the Asymptotic Hamiltonians). *Let*

$$\widehat{M}_*(\omega)\varphi := \left(w_*\widehat{D}(0) + w_* \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix} \right) \varphi, \quad \omega \in \mathbb{C}, \varphi \in \mathfrak{h}_*^1.$$

(a) *The set*

$$\mathcal{O}_* := \{(\omega, z) \in \mathbb{C} \times \mathbb{C} \mid z \in \rho(\widehat{M}_*(\omega))\}$$

is open in $\mathbb{C} \times \mathbb{C}$ and the map $\mathcal{O}_ \ni (\omega, z) \mapsto (\widehat{M}_*(\omega) - z)^{-1} \in \mathcal{B}(\mathfrak{h}_*)$ is analytic in the variables ω and z .*

(b) *For each $\omega \in \mathbb{C}$, the operator $\widehat{M}_*(\omega)$ has purely discrete spectrum.*

(c) *If Σ_* is equipped with the topology induced by $Y_*^* \times \mathbb{C}$, then the projection $\pi_{\mathbb{R}} : \Sigma_* \rightarrow \mathbb{R}$ given by $\pi_{\mathbb{R}}(k, \lambda) := \lambda$ is proper.*

In particular, the operator \widehat{M}_ is analytically fibered in the sense of [12, Definition 2.2].*

Proof. (a) The operator $w_*\widehat{D}(0)$ is self-adjoint on $\mathfrak{h}_*^1 \subset \mathfrak{h}_*$, and for each $\omega \in \mathbb{C}$ we have that $w_* \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix} \in \mathcal{B}(\mathfrak{h}_*)$. Hence, for each $\omega \in \mathbb{C}$ the operator $\widehat{M}_*(\omega)$ is closed in \mathfrak{h}_* and has domain \mathfrak{h}_*^1 , and for each $x \in \mathbb{R}$ the operator $\widehat{M}_*(x)$ is self-adjoint on \mathfrak{h}_*^1 . In particular, we infer by functional calculus that

$$\lim_{|t| \rightarrow \infty} \|(\widehat{M}_*(x) - it)^{-1}\|_{\mathcal{B}(\mathfrak{h}_*)} \leq \lim_{|t| \rightarrow \infty} \frac{1}{|t|} = 0 \quad (t \in \mathbb{R}).$$

Therefore, for each $y \in \mathbb{R}$ the set

$$\Omega_y := \{it \in i\mathbb{R} \mid (\|\widehat{M}_*(x) - it\|_{\mathfrak{B}(\mathfrak{h}_*)})^{-1} > |y| \|w_*\|_{\mathfrak{B}(\mathfrak{h}_*)}\}$$

is non-empty, and then the argument in [17, Remark IV.3.2] guarantees that Ω_y is contained in the resolvent set of $\widehat{M}_*(x + iy)$. Thus, for each $\omega \equiv x + iy \in \mathbb{C}$ the operator $\widehat{M}_*(\omega)$ is closed in \mathfrak{h}_* , has domain \mathfrak{h}_*^1 , and non-empty resolvent set, and for each $\varphi \in \mathfrak{h}_*^1$ the map $\mathbb{C} \ni \omega \mapsto \widehat{M}_*(\omega)\varphi \in \mathfrak{h}_*$ is linear and therefore analytic. So, the collection $\{\widehat{M}_*(\omega)\}_{\omega \in \mathbb{C}}$ is an analytic family of type (A) [28, p. 16], and thus also an analytic family in the sense of Kato [28, p. 14]. The claim is then a consequence of [28, Theorem XII.7].

(b) Since $\{\widehat{M}_*(\omega)\}_{\omega \in \mathbb{C}}$ is an analytic family of type (A), the operators $\widehat{M}_*(\omega)$ have compact resolvent (and thus purely discrete spectrum) either for all $\omega \in \mathbb{C}$ or for no $\omega \in \mathbb{C}$ [17, Theorems III.6.26 and VII.2.4]. Therefore, to prove the claim, it is sufficient to show that $\widehat{M}_*(0)$ has compact resolvent. Now, we have

$$\widehat{M}_*(0) = w_* \widehat{D}(0) = w_* \begin{pmatrix} 0 & -i\partial_\theta \\ -i\partial_\theta & 0 \end{pmatrix},$$

where $-i\partial_\theta$ is a first-order differential operator in $L^2(Y_*)$ with periodic boundary conditions, and thus with purely discrete spectrum that accumulates at $\pm\infty$. In consequence, each entry of the matrix operator

$$(\widehat{D}(0) + i)^{-1} = \begin{pmatrix} -i(1 + (i\partial_\theta)^2)^{-1} & -i\partial_\theta(1 + (i\partial_\theta)^2)^{-1} \\ -i\partial_\theta(1 + (i\partial_\theta)^2)^{-1} & -i(1 + (i\partial_\theta)^2)^{-1} \end{pmatrix}$$

is compact in $L^2(Y_*)$, so that $(\widehat{D}(0) + i)^{-1}$ is compact in $L^2(Y_*, \mathbb{C}^2)$. Since Lemma 2.3(a) implies that the norms on $L^2(Y_*, \mathbb{C}^2)$ and \mathfrak{h}_* are equivalent, we infer that $(\widehat{D}(0) + i)^{-1}$ is also compact in \mathfrak{h}_* . Finally, since

$$\begin{aligned} (\widehat{M}_*(0) + i)^{-1} &= (\widehat{M}_*(0) + iw_*)^{-1} + (\widehat{M}_*(0) + i)^{-1} - (\widehat{M}_*(0) + iw_*)^{-1} \\ &= (\widehat{D}(0) + i)^{-1} w_*^{-1} - i(\widehat{M}_*(0) + i)^{-1} (1 - w_*) (\widehat{D}(0) + i)^{-1} w_*^{-1}, \end{aligned} \tag{2.4}$$

with w_*^{-1} and $(1 - w_*)$ bounded in \mathfrak{h}_* , we obtain that $(\widehat{M}_*(0) + i)^{-1}$ is compact in \mathfrak{h}_* .

(c) Let $Y_*^* \times \mathbb{C}$ be endowed with the topology induced by $\mathbb{C} \times \mathbb{C}$. Point (a) implies that the set

$$\Sigma_*^c := \{(k, z) \in Y_*^* \times \mathbb{C} \mid z \in \rho(\widehat{M}_*(k))\}$$

is open in $Y_*^* \times \mathbb{C}$. Therefore, the set Σ_* is closed in $Y_*^* \times \mathbb{C}$ and the inclusion $\iota : \Sigma_* \rightarrow Y_*^* \times \mathbb{C}$ is a closed map. Since the projection $\pi_{\mathbb{C}} : Y_*^* \times \mathbb{C} \rightarrow \mathbb{C}$ given by $\pi_{\mathbb{C}}(k, z) := z$ is also a closed map (because Y_*^* is compact, see [24, Exercise 7, p. 171]) and $\pi_{\mathbb{R}} = \pi_{\mathbb{C}} \circ \iota$, we infer that $\pi_{\mathbb{R}}$ is a closed map. Moreover, $\pi_{\mathbb{R}}$ is continuous because it is the restriction to the subset Σ_* of the continuous projection $\pi_{\mathbb{C}} : Y_*^* \times \mathbb{C} \rightarrow \mathbb{C}$.

In consequence, in order to prove that $\pi_{\mathbb{R}}$ is proper it is sufficient to show that $\pi_{\mathbb{R}}^{-1}(\{\lambda\})$ is compact in Σ_{\star} for each $\lambda \in \pi_{\mathbb{R}}(\Sigma_{\star})$. But since

$$\pi_{\mathbb{R}}^{-1}(\{\lambda\}) = (\iota^{-1} \circ \pi_{\mathbb{C}}^{-1})(\{\lambda\}) = \iota^{-1}(Y_{\star}^* \times \{\lambda\}) = (Y_{\star}^* \times \{\lambda\}) \cap \Sigma_{\star},$$

this follows from compactness of Y_{\star}^* and the closedness of Σ_{\star} in $Y_{\star}^* \times \mathbb{C}$. □

Proposition 2.6 can be combined with the theorem of Rellich [17, Theorem VII.3.9] which, adapted to our notations, states:

Theorem 2.7 (Rellich). *Let $\Omega \subset \mathbb{C}$ be a neighborhood of an interval $I_0 \subset \mathbb{R}$ and let $\{T(\omega)\}_{\omega \in \Omega}$ be a self-adjoint analytic family of type (A), with each $T(\omega)$ having compact resolvent. Then there is a sequence of scalar-valued functions λ_n and a sequence of vector-valued functions u_n , all analytic on I_0 , such that for $\omega \in I_0$ the $\lambda_n(\omega)$ are the repeated eigenvalues of $T(\omega)$ and the $u_n(\omega)$ form a complete orthonormal family of the associated eigenvectors of $T(\omega)$.*

By applying this theorem to the family $\{\widehat{M}_{\star}(\omega)\}_{\omega \in \mathbb{C}}$, we infer the existence of analytic eigenvalue functions $\lambda_{\star,n} : Y_{\star}^* \rightarrow \mathbb{R}$ and analytic orthonormal eigenvector functions $u_{\star,n} : Y_{\star}^* \rightarrow \mathfrak{h}_{\star}$. We call band the graph $\{(k, \lambda_{\star,n}(k)) | k \in Y_{\star}^*\}$ of the eigenvalue function $\lambda_{\star,n}$, so that the Bloch variety Σ_{\star} coincides with the countable union of the bands (see (2.3)). Since the derivative $\lambda'_{\star,n}$ of $\lambda_{\star,n}$ exists and is analytic, it is natural to define the set of thresholds of the operator M_{\star} as

$$\mathcal{T}_{\star} := \bigcup_{n \in \mathbb{N}} \{\lambda \in \mathbb{R} \mid \exists k \in Y_{\star}^* \text{ such that } \lambda = \lambda_{\star,n}(k) \text{ and } \lambda'_{\star,n}(k) = 0\}, \quad (2.5)$$

and the set of thresholds of both M_{ℓ} and M_r as

$$\mathcal{T}_M := \mathcal{T}_{\ell} \cup \mathcal{T}_r. \quad (2.6)$$

Proposition 2.6(b), together with the analyticity of the functions $\lambda_{\star,n}$, implies that the set \mathcal{T}_{\star} is discrete, with only possible accumulation point at infinity. Furthermore, [28, Theorem XIII.85(e)] implies that the possible eigenvalues of M_{\star} are contained in \mathcal{T}_{\star} . However, these eigenvalues should be generated by locally (hence globally) flat bands, and one can show their absence by adapting Thomas' argument [38, Sec. II] to our setup.

Proposition 2.8 (Spectrum of the Asymptotic Hamiltonians). *The spectrum of M_{\star} is purely absolutely continuous. In particular,*

$$\sigma(M_{\star}) = \sigma_{\text{ac}}(M_{\star}) = \sigma_{\text{ess}}(M_{\star}),$$

with $\sigma_{\text{ac}}(M_{\star})$ the absolutely continuous spectrum of M_{\star} and $\sigma_{\text{ess}}(M_{\star})$ the essential spectrum of M_{\star} .

Proof. In view of [28, Theorem XIII.86], the claim follows once we prove the absence of flat bands for M_\star . For this purpose, we use the version of Thomas' argument as presented in [35, Sec. 1.3]. Accordingly, we first need to show that, for $\omega = i\rho$ with $\rho \in \mathbb{R}$ large enough, the operator $\widehat{M}_\star(i\rho)$ is invertible and satisfies

$$\lim_{|\rho| \rightarrow \infty} \|\widehat{M}_\star(i\rho)^{-1}\|_{\mathcal{B}(\mathfrak{h}_\star)} = 0. \quad (2.7)$$

Let us start with the operator

$$\widehat{D}(i\rho) = \begin{pmatrix} 0 & -i\partial_\theta + i\rho \\ -i\partial_\theta + i\rho & 0 \end{pmatrix}$$

acting on $\mathfrak{h}_\star^1 \subset \mathfrak{h}_\star$. Since the family of functions $\{e_n^\pm\}_{n \in \mathbb{Z}}$ given by

$$e_n^+(\theta) := \frac{1}{\sqrt{p_\star}} e^{2\pi i n \theta / p_\star} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_n^-(\theta) := \frac{1}{\sqrt{p_\star}} e^{2\pi i n \theta / p_\star} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \theta \in Y_\star$$

is an orthonormal basis of $L^2(Y_\star, \mathbb{C}^2)$, and since \mathfrak{h}_\star and $L^2(Y_\star, \mathbb{C}^2)$ have equivalent norms, the family $\{e_n^\pm\}_{n \in \mathbb{Z}}$, with extended variable $\theta \in \mathbb{R}$, is also a (non-orthogonal) basis for \mathfrak{h}_\star , and thus any $\varphi \in \mathfrak{h}_\star^1$ can be expanded in \mathfrak{h}_\star as

$$\varphi = \sum_{n \in \mathbb{Z}} (\widehat{\varphi}_n^+ e_n^+ + \widehat{\varphi}_n^- e_n^-) \quad \text{with } \widehat{\varphi}_n^\pm := \langle \varphi, e_n^\pm \rangle_{L^2(Y_\star, \mathbb{C}^2)}.$$

It follows that

$$\begin{aligned} \|\widehat{D}(\pm i\rho)\varphi\|_{\mathfrak{h}_\star}^2 &= \left\| \sum_{n \in \mathbb{Z}} \left(\frac{2\pi n}{p_\star} \pm i\rho \right) (\widehat{\varphi}_n^+ e_n^- + \widehat{\varphi}_n^- e_n^+) \right\|_{\mathfrak{h}_\star}^2 \\ &\geq \text{Const.} \left\| \sum_{n \in \mathbb{Z}} \left(\frac{2\pi n}{p_\star} \pm i\rho \right) (\widehat{\varphi}_n^+ e_n^- + \widehat{\varphi}_n^- e_n^+) \right\|_{L^2(Y_\star, \mathbb{C}^2)}^2 \\ &= \text{Const.} \sum_{n \in \mathbb{Z}} \left| \frac{2\pi n}{p_\star} \pm i\rho \right|^2 (|\widehat{\varphi}_n^+|^2 + |\widehat{\varphi}_n^-|^2) \\ &\geq \text{Const.} |\rho|^2 \|\varphi\|_{L^2(Y_\star, \mathbb{C}^2)}^2 \\ &\geq \text{Const.} |\rho|^2 \|\varphi\|_{\mathfrak{h}_\star}^2. \end{aligned}$$

Thus, the operators $\widehat{D}(\pm i\rho)$ are injective with closed range and satisfy in \mathfrak{h}_\star the relations

$$(\text{Ran } \widehat{D}(\pm i\rho))^\perp = \text{Ker}(\widehat{D}(\pm i\rho)^*) = \text{Ker}(w_\star \widehat{D}(\mp i\rho) w_\star^{-1}) = 0.$$

In consequence $\text{Ran } \widehat{D}(\pm i\rho) = \mathfrak{h}_\star$, and the operators $\widehat{D}(\pm i\rho)$ are invertible with

$$\|\widehat{D}(\pm i\rho)^{-1}\|_{\mathcal{B}(\mathfrak{h}_\star)} \leq \text{Const.} |\rho|^{-1}.$$

It follows that $\widehat{M}_\star(i\rho)$ is invertible too, with

$$\|\widehat{M}_\star(i\rho)^{-1}\|_{\mathcal{B}(\mathfrak{h}_\star)} = \|\widehat{D}(i\rho)^{-1}w_\star^{-1}\|_{\mathcal{B}(\mathfrak{h}_\star)} \leq \text{Const.}|\rho|^{-1},$$

which implies (2.7).

Now, let us assume by contradiction that there exists $n \in \mathbb{N}$ such that $\lambda_{\star,n}(k)$ is equal to a constant $c \in \mathbb{R}$ for all $k \in Y_\star^*$. Then using the analyticity properties of \widehat{M}_\star (Proposition 2.6) in conjunction with the analytic Fredholm alternative, one infers that c is an eigenvalue of $\widehat{M}_\star(\omega)$ for all $\omega \in \mathbb{C}$. Letting $u(\omega)$ be the corresponding eigenfunction for $\widehat{M}_\star(\omega)$, one obtains that $\widehat{M}_\star(\omega)u(\omega) = cu(\omega)$ for all $\omega \in \mathbb{C}$. Choosing $\omega = i\rho$ with $\rho \in \mathbb{R}$ and using the fact that $\widehat{M}_\star(i\rho)$ is invertible, one thus obtains that

$$u(i\rho) = c\widehat{M}_\star(i\rho)^{-1}u(i\rho) \quad \text{with } \|u(i\rho)\|_{\mathfrak{h}_\star} = 1,$$

which contradicts (2.7). □

Remark 2.9. The absence of flat bands for the three-dimensional Maxwell operator has been discussed in [35, Sec. 5]. However, the results of [35] do not cover the result of Proposition 2.8 since the weights considered in [35] are block diagonal and smooth while in Proposition 2.8 the weights are L^∞ positive-definite 2×2 matrices. Neither diagonality, nor smoothness is assumed.

3. Mourre Theory and Spectral Results

3.1. Commutators

In this section, we recall some definitions appearing in Mourre theory and provide a precise meaning to the commutators mentioned in Sec. 1. We refer to [1, 33] for more information and details.

Let A be a self-adjoint operator in a Hilbert space \mathcal{H} with domain $\mathcal{D}(A)$, and let $T \in \mathcal{B}(\mathcal{H})$. For any $k \in \mathbb{N}$, we say that T belongs to $C^k(A)$, with notation $T \in C^k(A)$, if the map

$$\mathbb{R} \ni t \mapsto e^{-itA} T e^{itA} \in \mathcal{B}(\mathcal{H}) \tag{3.1}$$

is strongly of class C^k . In the case $k = 1$, one has $T \in C^1(A)$ if and only if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle \varphi, TA\varphi \rangle_{\mathcal{H}} - \langle A\varphi, T\varphi \rangle_{\mathcal{H}} \in \mathbb{C}$$

is continuous for the norm topology induced by \mathcal{H} on $\mathcal{D}(A)$. We denote by $[T, A]$ the bounded operator associated with the continuous extension of this form, or equivalently $-i$ times the strong derivative of the function (3.1) at $t = 0$.

If H is a self-adjoint operator in \mathcal{H} with domain $\mathcal{D}(H)$ and spectrum $\sigma(H)$, we say that H is of class $C^k(A)$ if $(H - z)^{-1} \in C^k(A)$ for some $z \in \mathbb{C} \setminus \sigma(H)$. In particular, H is of class $C^1(A)$ if and only if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle \varphi, (H - z)^{-1}A\varphi \rangle_{\mathcal{H}} - \langle A\varphi, (H - z)^{-1}\varphi \rangle_{\mathcal{H}} \in \mathbb{C}$$

extends continuously to a bounded form defined by the operator $[(H - z)^{-1}, A] \in \mathcal{B}(\mathcal{H})$. In such a case, the set $\mathcal{D}(H) \cap \mathcal{D}(A)$ is a core for H and the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi, A\varphi \rangle_{\mathcal{H}} - \langle A\varphi, H\varphi \rangle_{\mathcal{H}} \in \mathbb{C}$$

is continuous in the natural topology of $\mathcal{D}(H)$ (i.e. the topology of the graph-norm) [1, Theorem 6.2.10(b)]. This form then extends uniquely to a continuous quadratic form on $\mathcal{D}(H)$ which can be identified with a continuous operator $[H, A]$ from $\mathcal{D}(H)$ to the adjoint space $\mathcal{D}(H)^*$. In addition, one has the identity

$$[(H - z)^{-1}, A] = -(H - z)^{-1}[H, A](H - z)^{-1}, \quad (3.2)$$

and the following result is verified [1, Theorem 6.2.15]: If H is of class $C^k(A)$ for some $k \in \mathbb{N}$ and $\eta \in \mathcal{S}(\mathbb{R})$ is a Schwartz function, then $\eta(H) \in C^k(A)$.

A regularity condition slightly stronger than being of class $C^1(A)$ is defined as follows: H is of class $C^{1+\varepsilon}(A)$ for some $\varepsilon \in (0, 1)$ if H is of class $C^1(A)$ and if for some $z \in \mathbb{C} \setminus \sigma(H)$

$$\|e^{-itA}[(H - z)^{-1}, A]e^{itA} - [(H - z)^{-1}, A]\|_{\mathcal{B}(\mathcal{H})} \leq \text{Const. } t^\varepsilon \quad \text{for all } t \in (0, 1).$$

The condition $C^2(A)$ is stronger than $C^{1+\varepsilon}(A)$, which in turn is stronger than $C^1(A)$.

We now recall the definition of two useful functions introduced in [1, Sec. 7.2]. For this, we need the following conventions: if $E^H(\cdot)$ denotes the spectral projection-valued measure of H , then we set $E^H(\lambda; \varepsilon) := E^H((\lambda - \varepsilon, \lambda + \varepsilon))$ for any $\lambda \in \mathbb{R}$ and $\varepsilon > 0$, and if $S, T \in \mathcal{B}(\mathcal{H})$, then we write $S \approx T$ if $S - T$ is compact, and $S \lesssim T$ if there exists a compact operator K such that $S \leq T + K$. With these conventions, we define for H of class $C^1(A)$ the function $\varrho_H^A : \mathbb{R} \rightarrow (-\infty, \infty]$ by

$$\varrho_H^A(\lambda) := \sup\{a \in \mathbb{R} \mid \exists \varepsilon > 0 \text{ such that } aE^H(\lambda; \varepsilon) \leq E^H(\lambda; \varepsilon)[iH, A]E^H(\lambda; \varepsilon)\},$$

and we define the function $\tilde{\varrho}_H^A : \mathbb{R} \rightarrow (-\infty, \infty]$ by

$$\tilde{\varrho}_H^A(\lambda) := \sup\{a \in \mathbb{R} \mid \exists \varepsilon > 0 \text{ such that } aE^H(\lambda; \varepsilon) \lesssim E^H(\lambda; \varepsilon)[iH, A]E^H(\lambda; \varepsilon)\}.$$

Note that the following equivalent definition of the function $\tilde{\varrho}_H^A$ is often useful:

$$\begin{aligned} \tilde{\varrho}_H^A(\lambda) &= \sup\{a \in \mathbb{R} \mid \exists \eta \in C_c^\infty(\mathbb{R}, \mathbb{R}) \text{ such that } \eta(\lambda) \neq 0 \text{ and} \\ & a\eta(H)^2 \lesssim \eta(H)[iH, A]\eta(H)\}. \end{aligned} \quad (3.3)$$

One says that A is conjugate to H at a point $\lambda \in \mathbb{R}$ if $\tilde{\varrho}_H^A(\lambda) > 0$, and that A is strictly conjugate to H at λ if $\varrho_H^A(\lambda) > 0$. It is shown in [1, Proposition 7.2.6] that the function $\tilde{\varrho}_H^A : \mathbb{R} \rightarrow (-\infty, \infty]$ is lower semicontinuous, that $\tilde{\varrho}_H^A \geq \varrho_H^A$, and that $\tilde{\varrho}_H^A(\lambda) < \infty$ if and only if $\lambda \in \sigma_{\text{ess}}(H)$. In particular, the set of points where A is conjugate to H ,

$$\tilde{\mu}^A(H) := \{\lambda \in \mathbb{R} \mid \tilde{\varrho}_H^A(\lambda) > 0\}$$

is open in \mathbb{R} .

The main consequences of the existence of a conjugate operator A for H are given in the following theorem, which is a particular case of [33, Theorems 0.1 and 0.2]. For its statement, we use the notation $\sigma_p(H)$ for the point spectrum of H , and we recall that if \mathcal{G} is an auxiliary Hilbert space, then an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ is locally H -smooth on an open set $I \subset \mathbb{R}$ if for each compact set $I_0 \subset I$ there exists $c_{I_0} \geq 0$ such that

$$\int_{\mathbb{R}} dt \|T e^{-itH} E^H(I_0)\varphi\|_{\mathcal{G}}^2 \leq c_{I_0} \|\varphi\|_{\mathcal{H}}^2 \quad \text{for each } \varphi \in \mathcal{H}, \quad (3.4)$$

and T is (globally) H -smooth if (3.4) is satisfied with $E^H(I_0)$ replaced by the identity 1.

Theorem 3.1 (Spectrum of H). *Let H, A be self-adjoint operators in a Hilbert space \mathcal{H} , let \mathcal{G} be an auxiliary Hilbert space, assume that H is of class $C^{1+\varepsilon}(A)$ for some $\varepsilon \in (0, 1)$, and suppose there exist an open set $I \subset \mathbb{R}$, a number $a > 0$ and an operator $K \in \mathcal{X}(\mathcal{H})$ such that*

$$E^H(I)[iH, A]E^H(I) \geq aE^H(I) + K. \quad (3.5)$$

Then

- (a) each operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ which extends continuously to an element of $\mathcal{B}(\mathcal{D}((A)^s)^*, \mathcal{G})$ for some $s > 1/2$ is locally H -smooth on $I \setminus \sigma_p(H)$;
- (b) H has at most finitely many eigenvalues in I , each one of finite multiplicity, and H has no singular continuous spectrum in I .

3.2. Conjugate operator for the free Hamiltonian

With the definitions of Sec. 2.3 at hand, we can construct a conjugate operator for the operator \widehat{M}_\star . Our construction follows from the one given in [12, Sec. 3], but it is simpler because our base manifold Y_\star^* is one-dimensional. Indeed, thanks to Theorem 2.7, it is sufficient to construct the conjugate operator band by band.

So, for each $n \in \mathbb{N}$, let $\widehat{\Pi}_{\star,n} := \{\widehat{\Pi}_{\star,n}(k)\}_{k \in \mathbb{R}}$ and $\widehat{\lambda}'_{\star,n} := \{\widehat{\lambda}'_{\star,n}(k)\}_{k \in \mathbb{R}}$ be the bounded decomposable self-adjoint operators in $\mathcal{H}_{\tau,\star}$ defined by τ -equivariant continuation as in (2.2) and by the relations

$$\widehat{\Pi}_{\star,n}(k)\varphi := \langle u_{\star,n}(k), \varphi \rangle_{\mathfrak{h}_\star} u_{\star,n}(k) \quad \text{and} \quad \widehat{\lambda}'_{\star,n}(k)\varphi := \lambda'_{\star,n}(k)\varphi, \quad k \in Y_\star^*, \varphi \in \mathfrak{h}_\star.$$

Set also $\Pi_{\star,n} := \mathcal{U}_\star^{-1} \widehat{\Pi}_{\star,n} \mathcal{U}_\star$ and $\widehat{Q}_\star := \mathcal{U}_\star Q_\star \mathcal{U}_\star^{-1}$, with Q_\star the operator of multiplication by the variable in \mathcal{H}_{w_\star}

$$(Q_\star \varphi)(x) := x\varphi(x), \quad \varphi \in \mathcal{D}(Q_\star) := \{\varphi \in \mathcal{H}_{w_\star} \mid \|Q_\star \varphi\|_{\mathcal{H}_{w_\star}} < \infty\}.$$

Remark 3.2. Since Q_\star commutes with w_\star^{-1} , the operator Q_\star is self-adjoint in \mathcal{H}_{w_\star} and essentially self-adjoint on $\mathcal{S}(\mathbb{R}, \mathbb{C}^2) \subset \mathcal{H}_{w_\star}$. The definition and the domain of Q_\star are independent of the specific weight w_\star^{-1} appearing in the scalar product of \mathcal{H}_{w_\star} . The insistence on the label $\star = \ell, r$ is only motivated by a notational need that will result helpful in the following sections.

For any compact interval $I \subset \mathbb{R} \setminus \mathcal{T}_*$, we define the finite set $\mathbb{N}(I) := \{n \in \mathbb{N} \mid \lambda_{*,n}^{-1}(I) \neq \emptyset\}$. Finally, we set

$$\mathcal{D}_* := \mathcal{U}_* \mathcal{S}(\mathbb{R}, \mathbb{C}^2) \subset \{u \in C^\infty(\mathbb{R}, \mathfrak{h}_*) \mid u(\cdot - \gamma^*) = \tau(\gamma^*)u \text{ for all } \gamma^* \in \Gamma_*^*\}. \quad (3.6)$$

Then we can define the symmetric operator $\widehat{A}_{*,I}$ in $\mathcal{H}_{\tau,*}$ by

$$\widehat{A}_{*,I}u := \frac{1}{2} \sum_{n \in \mathbb{N}(I)} \widehat{\Pi}_{*,n}(\widehat{\lambda}'_{*,n} \widehat{Q}_* + \widehat{Q}_* \widehat{\lambda}'_{*,n}) \widehat{\Pi}_{*,n}u, \quad u \in \widehat{\mathcal{D}}_*. \quad (3.7)$$

Theorem 3.3 (Mourre Estimate for \widehat{M}_*). *Let $I \subset \mathbb{R} \setminus \mathcal{T}_*$ be a compact interval. Then*

- (a) *the operator $\widehat{A}_{*,I}$ is essentially self-adjoint on \mathcal{D}_* and on any other core for \widehat{Q}_*^2 , with closure denoted by the same symbol;*
- (b) *the operator \widehat{M}_* is of class $C^2(\widehat{A}_{*,I})$;*
- (c) *there exists $c_I > 0$ such that $\varrho_{\widehat{M}_*}^{\widehat{A}_{*,I}} \geq c_I$.*

Proof. (a) The claim is a consequence of Nelson's criterion of self-adjointness [26, Theorem X.37] applied to the triple $(\widehat{A}_{*,I}, N_*, \mathcal{D}_*)$, where $N_* := \widehat{Q}_*^2 + 1$ and $\widehat{Q}_*^2 := \mathcal{U}_* Q_*^2 \mathcal{U}_*^{-1}$. Indeed, the operator N_* is essentially self-adjoint on $\mathcal{D}_* = \mathcal{U}_* \mathcal{S}(\mathbb{R}, \mathbb{C}^2)$ since Q_*^2 is essentially self-adjoint on $\mathcal{S}(\mathbb{R}, \mathbb{C}^2)$. In addition, since $\widehat{A}_{*,I}$ is composed of the bounded operators $\widehat{\Pi}_{*,n}$ and $\widehat{\lambda}'_{*,n}$ which are analytic in the variable $k \in Y_*^*$ and \widehat{Q}_* acts as $i\partial_k$ in $\mathcal{H}_{\tau,*}$, a direct computation gives

$$\|\widehat{A}_{*,I}u\|_{\mathcal{H}_{\tau,*}} \leq \text{Const.} \|N_*u\|_{\mathcal{H}_{\tau,*}}, \quad u \in \mathcal{D}_*.$$

Similarly, a direct computation using the boundedness and the analyticity of $\widehat{\Pi}_{*,n}$ and $\widehat{\lambda}'_{*,n}$ implies that

$$|\langle \widehat{A}_{*,I}u, N_*u \rangle_{\mathcal{H}_{\tau,*}} - \langle N_*u, \widehat{A}_{*,I}u \rangle_{\mathcal{H}_{\tau,*}}| \leq \text{Const.} \langle N_*u, u \rangle_{\mathcal{H}_{\tau,*}}, \quad u \in \mathcal{D}_*.$$

In both inequalities, we used the fact that $\mathcal{D}(\widehat{Q}_*^2) \subset \mathcal{D}(\widehat{Q}_*)$. As a consequence, $\widehat{A}_{*,I}$ is essentially self-adjoint on \mathcal{D}_* and on any other core for N_* .

(b) The set

$$\mathcal{E}_* := \{u \in C^\infty(\mathbb{R}, \mathfrak{h}_*) \mid u(\cdot - \gamma^*) = \tau(\gamma^*)u \text{ for all } \gamma^* \in \Gamma_*^*\} \supset \mathcal{D}_*$$

is a core for N_* . So, it follows from point (a) that $\widehat{A}_{*,I}$ is essentially self-adjoint on \mathcal{E}_* . Moreover, since $\widehat{M}_*(k)$ is analytic in $k \in \mathbb{R}$ and satisfies the covariance relation (2.2), we obtain that $(\widehat{M}_* - z)\mathcal{E}_* \subset \mathcal{E}_*$ for any $z \in \mathbb{C} \setminus \sigma(\widehat{M}_*)$. Since the same argument applies to the resolvent, we obtain that $(\widehat{M}_* - z)^{-1}\mathcal{E}_* = \mathcal{E}_*$. Therefore, we have the inclusion $(\widehat{M}_* - z)^{-1}u \in \mathcal{D}(\widehat{A}_{*,I})$ for each $u \in \mathcal{E}_*$, and a calculation

using (3.2) gives

$$\begin{aligned}
 & \langle u, [i(\widehat{M}_\star - z)^{-1}, \widehat{A}_{\star,I}]u \rangle_{\mathcal{H}_{\tau,\star}} \\
 &= \langle u, -(\widehat{M}_\star - z)^{-1} [i\widehat{M}_\star, \widehat{A}_{\star,I}](\widehat{M}_\star - z)^{-1}u \rangle_{\mathcal{H}_{\tau,\star}} \\
 &= \left\langle u, -(\widehat{M}_\star - z)^{-1} \sum_{n \in \mathbb{N}(I)} \widehat{\Pi}_{\star,n} |\widehat{\chi}'_{\star,n}|^2 \widehat{\Pi}_{\star,n} (\widehat{M}_\star - z)^{-1}u \right\rangle_{\mathcal{H}_{\tau,\star}}.
 \end{aligned}$$

Since $\sum_{n \in \mathbb{N}(I)} \widehat{\Pi}_{\star,n} |\widehat{\chi}'_{\star,n}|^2 \widehat{\Pi}_{\star,n} \in \mathcal{B}(\mathcal{H}_{\tau,\star})$, it follows that \widehat{M}_\star is of class $C^1(\widehat{A}_{\star,I})$ with

$$[i\widehat{M}_\star, \widehat{A}_{\star,I}] = \sum_{n \in \mathbb{N}(I)} \widehat{\Pi}_{\star,n} |\widehat{\chi}'_{\star,n}|^2 \widehat{\Pi}_{\star,n}. \quad (3.8)$$

Finally, since $\widehat{\Pi}_{\star,n} \in C^1(\widehat{A}_{\star,I})$ and $\widehat{\chi}'_{\star,n} \in C^1(\widehat{A}_{\star,I})$ for each $n \in \mathbb{N}$, we infer from (3.8) and [1, Proposition 5.1.5] that \widehat{M}_\star is of class $C^2(\widehat{A}_{\star,I})$.

(c) Using point (b) and the definition of the operators $\widehat{\Pi}_{\star,n}$, we obtain for all $\eta \in C_c^\infty(I, \mathbb{R})$ and $k \in Y_\star^*$ that

$$\begin{aligned}
 & \eta(\widehat{M}_\star(k)) [i\widehat{M}_\star, \widehat{A}_{\star,I}](k) \eta(\widehat{M}_\star(k)) \\
 &= \eta(\widehat{M}_\star(k)) \left(\sum_{n \in \mathbb{N}(I)} \widehat{\Pi}_{\star,n}(k) |\widehat{\chi}'_{\star,n}(k)|^2 \widehat{\Pi}_{\star,n}(k) \right) \eta(\widehat{M}_\star(k)) \\
 &\geq c_I \eta(\widehat{M}_\star(k)) \left(\sum_{n \in \mathbb{N}(I)} \widehat{\Pi}_{\star,n}(k)^2 \right) \eta(\widehat{M}_\star(k)) \\
 &= c_I \eta(\widehat{M}_\star(k))^2
 \end{aligned}$$

with $c_I := \min_{n \in \mathbb{N}(I)} \min_{\{k \in Y_\star^* \mid \lambda_{\star,n}(k) \in I\}} |\lambda'_{\star,n}(k)|^2$. Thus, by using the definition of the scalar product in $\mathcal{H}_{\tau,\star}$, we infer that

$$\eta(\widehat{M}_\star) [i\widehat{M}_\star, \widehat{A}_{\star,I}] \eta(\widehat{M}_\star) \geq c_I \eta(\widehat{M}_\star)^2,$$

which, together with the definition (3.3), implies the claim. \square

Since the operator $\widehat{A}_{\star,I}$ is essentially self-adjoint on $\mathcal{D}_\star = \mathcal{U}_\star \mathcal{S}(\mathbb{R}, \mathbb{C}^2)$ and on any other core for \widehat{Q}_\star^2 , it follows by Theorem 3.3(a) that the inverse Bloch–Floquet transform of $\widehat{A}_{\star,I}$,

$$A_{\star,I} := \mathcal{U}_\star^{-1} \widehat{A}_{\star,I} \mathcal{U}_\star$$

is essentially self-adjoint on $\mathcal{S}(\mathbb{R}, \mathbb{C}^2)$ and on any other core for Q_\star^2 . Therefore, the results (b) and (c) of Theorem 3.3 can be restated as follows: the operator M_\star is of class $C^2(A_{\star,I})$ and there exists $c_I > 0$ such that $\varrho_{M_\star}^{A_{\star,I}} \geq c_I$. Combining these results for $\star = \ell$ and $\star = r$, one obtains a conjugate operator for the free Hamiltonian

$M_0 = M_\ell \oplus M_r$ introduced in Sec. 2.2. Namely, for any compact interval $I \subset \mathbb{R} \setminus \mathcal{T}_M$, the operator

$$A_{0,I} := A_{\ell,I} \oplus A_{r,I}$$

satisfies the following properties:

- (a') the operator $A_{0,I}$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}, \mathbb{C}^2) \oplus \mathcal{S}(\mathbb{R}, \mathbb{C}^2)$ and on any set $\mathcal{E} \oplus \mathcal{E}$ with \mathcal{E} a core for Q_\star^2 , with closure denoted by the same symbol;
- (b') the operator M_0 is of class $C^2(A_{0,I})$;
- (c') there exists $c_I > 0$ such that $\varrho_{M_0}^{A_{0,I}} \geq c_I$.

Remark 3.4. What precedes implies in particular that the free Hamiltonian M_0 has purely absolutely continuous spectrum except at the points of \mathcal{T}_M , where it may have eigenvalues. However, we already know from Proposition 2.8 that this does not occur. Therefore,

$$\sigma(M_0) = \sigma_{\text{ac}}(M_0) = \sigma_{\text{ess}}(M_0) = \sigma_{\text{ess}}(M_\ell) \cup \sigma_{\text{ess}}(M_r).$$

3.3. Conjugate operator for the full Hamiltonian

In this section, we show that the operator $JA_{0,I}J^*$ is a conjugate operator for the full Hamiltonian M introduced in Sec. 2.1. We start with the proof of the essential self-adjointness of $JA_{0,I}J^*$ in \mathcal{H}_w . We use the notation Q (see Remark 3.2) for the operator of multiplication by the variable in \mathcal{H}_w ,

$$(Q\varphi)(x) := x\varphi(x), \quad \varphi \in \mathcal{D}(Q) := \{\varphi \in \mathcal{H}_w \mid \|Q\varphi\|_{\mathcal{H}_w} < \infty\}.$$

Proposition 3.5. *For each compact interval $I \subset \mathbb{R} \setminus \mathcal{T}_M$ the operator $A_I := JA_{0,I}J^*$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}, \mathbb{C}^2)$ and on any other core for Q^2 , with closure denoted by the same symbol.*

Proof. First, we observe that since $J^*\mathcal{S}(\mathbb{R}, \mathbb{C}^2) \subset \mathcal{D}(Q_0^2)$ with $Q_0 := Q_\ell \oplus Q_r$ the operator A_I is well defined and symmetric on $\mathcal{S}(\mathbb{R}, \mathbb{C}^2) \subset \mathcal{H}_w$ due to point (a'). Next, to prove the claim, we use Nelson's criterion of essential self-adjointness [26, Theorem X.37] applied to the triple $(A_I, N, \mathcal{S}(\mathbb{R}, \mathbb{C}^2))$ with $N := Q^2 + 1$.

For this, we note that $\mathcal{S}(\mathbb{R}, \mathbb{C}^2)$ is a core for N and that the operators $\frac{Q_\star}{Q^2+1}$, j_\star , $w_\star w^{-1} j_\star$, $\Pi_{\star,n}$ and $\mathcal{U}_\star^{-1} \widehat{\lambda}'_{\star,n} \mathcal{U}_\star$ are bounded in \mathcal{H}_w . Moreover, we verify with direct calculations on $\mathcal{S}(\mathbb{R}, \mathbb{C}^2)$ that the operators $\Pi_{\star,n}$ and $\mathcal{U}_\star^{-1} \widehat{\lambda}'_{\star,n} \mathcal{U}_\star$ belong to $C^1(Q_\star)$ (in \mathcal{H}_{w_\star}), and that their commutators $[\Pi_{\star,n}, Q_\star]$ and $[\mathcal{U}_\star^{-1} \widehat{\lambda}'_{\star,n} \mathcal{U}_\star, Q_\star]$ belong to $C^1(Q)$ (in \mathcal{H}_w). Then a short computation using these properties gives the bound

$$\|A_I \varphi\|_{\mathcal{H}_w} \leq \text{Const.} \|N \varphi\|_{\mathcal{H}_w}, \quad \varphi \in \mathcal{S}(\mathbb{R}, \mathbb{C}^2),$$

and a slightly longer computation using the same properties shows that

$$|\langle A_I \varphi, N \varphi \rangle_{\mathcal{H}_w} - \langle N \varphi, A_I \varphi \rangle_{\mathcal{H}_w}| \leq \text{Const.} \langle N \varphi, \varphi \rangle_{\mathcal{H}_w}, \quad \varphi \in \mathcal{S}(\mathbb{R}, \mathbb{C}^2).$$

Thus, the hypotheses of Nelson's criterion are satisfied, and the claim follows. \square

In order to prove that A_I is a conjugate operator for M , we need two preliminary lemmas. They involve the two-Hilbert spaces difference of resolvents of M_0 and M :

$$B(z) := J(M_0 - z)^{-1} - (M - z)^{-1}J, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Lemma 3.6. *For each $z \in \mathbb{C} \setminus \mathbb{R}$, one has the inclusion $B(z) \in \mathcal{K}(\mathcal{H}_0, \mathcal{H}_w)$.*

Proof. One has for $(\varphi_\ell, \varphi_r) \in \mathcal{H}_0$

$$\begin{aligned} B(z)(\varphi_\ell, \varphi_r) &= \sum_{\star \in \{\ell, r\}} (j_\star(M_\star - z)^{-1} - (M - z)^{-1}j_\star)\varphi_\star \\ &= \sum_{\star \in \{\ell, r\}} \{((M_\star - z)^{-1} - (M - z)^{-1})j_\star\varphi_\star + [j_\star, (M_\star - z)^{-1}]\varphi_\star\}. \end{aligned} \tag{3.9}$$

Thus, an application of the standard result [34, Theorem 4.1] taking into account the properties of j_\star implies that the operator $[j_\star, (M_\star - z)^{-1}]$ is compact. This proves the claim for the second term in (3.9).

For the first term in (3.9), we have the equalities

$$\begin{aligned} &((M_\star - z)^{-1} - (M - z)^{-1})j_\star \\ &= (M - z)^{-1}(M - M_\star)j_\star(M_\star - z)^{-1} \\ &\quad + (M - z)^{-1}(M - M_\star)[(M_\star - z)^{-1}, j_\star] \\ &= (M - z)^{-1}j_\star(w - w_\star)D(M_\star - z)^{-1} + (M - z)^{-1}(w - w_\star)[D, j_\star] \\ &\quad \times (M_\star - z)^{-1} + (M - z)^{-1}(M - M_\star)[(M_\star - z)^{-1}, j_\star], \end{aligned} \tag{3.10}$$

with $j_\star(w - w_\star)$ and $[D, j_\star]$ matrix-valued functions vanishing at $\pm\infty$. Thus, the operator in the first term in (3.9) is also compact, which concludes the proof. \square

Lemma 3.7. *For each $z \in \mathbb{C} \setminus \mathbb{R}$ and each compact interval $I \subset \mathbb{R} \setminus \mathcal{T}_M$, one has the inclusion*

$$\overline{B(z)A_{0,I} \upharpoonright \mathcal{D}(A_{0,I})} \in \mathcal{K}(\mathcal{H}_0, \mathcal{H}_w).$$

Proof. Since $A_{0,I}$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R}, \mathbb{C}^2) \oplus \mathcal{S}(\mathbb{R}, \mathbb{C}^2)$, it is sufficient to show that

$$\overline{B(z)A_{0,I} \upharpoonright (\mathcal{S}(\mathbb{R}, \mathbb{C}^2) \oplus \mathcal{S}(\mathbb{R}, \mathbb{C}^2))} \in \mathcal{K}(\mathcal{H}_0, \mathcal{H}_w).$$

Furthermore, we have $A_{0,I} = A_{\ell,I} \oplus A_{r,I}$, and each operator $A_{\star,I}$ acts on $\mathcal{S}(\mathbb{R}, \mathbb{C}^2)$ as a sum $Q_\star F_{\star,I} + G_{\star,I}$, with $F_{\star,I}, G_{\star,I}$ bounded operators in \mathcal{H}_w , mapping the

set $\mathcal{S}(\mathbb{R}, \mathbb{C}^2)$ into $\mathcal{D}(Q_\star)$. These facts, together with the compactness result of Lemma 3.6 and (3.9), imply that it is sufficient to show that

$$\overline{((M_\star - z)^{-1} - (M - z)^{-1})j_\star Q_\star \upharpoonright \mathcal{S}(\mathbb{R}, \mathbb{C}^2)} \in \mathcal{K}(\mathcal{H}_{w_\star}, \mathcal{H}_w)$$

and

$$\overline{[j_\star, (M_\star - z)^{-1}]Q_\star \upharpoonright \mathcal{S}(\mathbb{R}, \mathbb{C}^2)} \in \mathcal{K}(\mathcal{H}_{w_\star}, \mathcal{H}_w).$$

Now, if one takes Assumption 2.2 into account, the proof of these inclusions is similar to the proof of Lemma 3.6. We leave the details to the reader. \square

Next, we will need the following theorem which is a direct consequence of [32, Theorem 3.1 and Corollaries 3.7, 3.8].

Theorem 3.8. *Let H_0, A_0 be self-adjoint operators in a Hilbert space \mathcal{H}_0 , let H be a self-adjoint operator in a Hilbert space \mathcal{H} , let $J \in \mathcal{B}(\mathcal{H}_0, \mathcal{H})$, and let*

$$\mathcal{B}(z) := J(H_0 - z)^{-1} - (H - z)^{-1}J, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Suppose there exists a set $\mathcal{D} \subset \mathcal{D}(A_0 J^)$ such that $J A_0 J^*$ is essentially self-adjoint on \mathcal{D} , with A its self-adjoint extension. Finally, assume that*

- (i) H_0 is of class $C^1(A_0)$;
- (ii) for each $z \in \mathbb{C} \setminus \mathbb{R}$, one has $\mathcal{B}(z) \in \mathcal{K}(\mathcal{H}_0, \mathcal{H})$;
- (iii) for each $z \in \mathbb{C} \setminus \mathbb{R}$, one has $\overline{\mathcal{B}(z)A_0} \upharpoonright \mathcal{D}(A_0) \in \mathcal{K}(\mathcal{H}_0, \mathcal{H})$;
- (iv) for each $\eta \in C_c^\infty(\mathbb{R})$, one has $\eta(H)(JJ^* - 1)\eta(H) \in \mathcal{K}(\mathcal{H})$.

Then H is of class $C^1(A)$ and $\tilde{\varrho}_H^A \geq \tilde{\varrho}_{H_0}^{A_0}$. In particular, if A_0 is conjugate to H_0 at $\lambda \in \mathbb{R}$, then A is conjugate to H at λ .

We are now ready to prove a Mourre estimate for M .

Theorem 3.9 (Mourre Estimate for M). *Let $I \subset \mathbb{R} \setminus \mathcal{T}_M$ be a compact interval. Then M is of class $C^1(A_I)$, and*

$$\tilde{\varrho}_M^{A_I}(\lambda) \geq \tilde{\varrho}_{M_0}^{A_0, I}(\lambda) = \min\{\tilde{\varrho}_{M_\ell}^{A_{\ell, I}}(\lambda), \tilde{\varrho}_{M_r}^{A_{r, I}}(\lambda)\} > 0 \quad \text{for every } \lambda \in I.$$

Proof. Theorem 3.3 and its restatement at the end of Sec. 3.2 give us the estimate

$$\min\{\tilde{\varrho}_{M_\ell}^{A_{\ell, I}}(\lambda), \tilde{\varrho}_{M_r}^{A_{r, I}}(\lambda)\} > 0 \quad \text{for every } \lambda \in I.$$

In addition, the equality $\tilde{\varrho}_{M_0}^{A_0, I} = \min\{\tilde{\varrho}_{M_\ell}^{A_{\ell, I}}, \tilde{\varrho}_{M_r}^{A_{r, I}}\}$ is a consequence of the definition of $A_{0, I}$ as a direct sum of $A_{\ell, I}$ and $A_{r, I}$ (see [1, Proposition 8.3.5]).

So, it only remains to show the inequality $\tilde{\varrho}_M^{A_I} \geq \tilde{\varrho}_{M_0}^{A_0, I}$ to prove the claim. For this, we apply Theorem 3.8 with $H_0 = M_0$, $H = M$ and $A_0 = A_{0, I}$, starting with the verification of its assumptions (i)–(iv): the assumptions (i), (ii) and (iii) of Theorem 3.8 follow from point (b'), Lemma 3.6 and Lemma 3.7, respectively.

Furthermore, the assumption (iv) of Theorem 3.8 follows from the fact that for any $\eta \in C_c^\infty(\mathbb{R})$ we have the inclusion

$$\eta(M)(JJ^* - 1)\eta(M) = \eta(M)(w_\ell w^{-1}j_\ell^2 + w_r w^{-1}j_r^2 - 1)(Q)\eta(M) \in \mathcal{K}(\mathcal{H}_w),$$

since

$$w_\ell w^{-1}j_\ell^2 + w_r w^{-1}j_r^2 - 1 = (w_\ell - w)j_\ell^2 w^{-1} + (w_r - w)j_r^2 w^{-1} + (j_\ell^2 + j_r^2 - 1)$$

is a matrix-valued function vanishing at $\pm\infty$. These facts, together with Proposition 3.5 and the inclusion $\mathcal{S}(\mathbb{R}, \mathbb{C}^2) \subset \mathcal{D}(A_0 J^*)$, imply that all the assumptions of Theorem 3.8 are satisfied. We thus obtain that $\tilde{\varrho}_M^{A_I} \geq \tilde{\varrho}_{M_0}^{A_{0,I}}$, as desired. \square

3.4. Spectral properties of the full Hamiltonian

In this section, we determine the spectral properties of the full Hamiltonian M . We start by proving that M has the same essential spectrum as the free Hamiltonian M_0 .

Proposition 3.10. *One has*

$$\sigma_{\text{ess}}(M) = \sigma_{\text{ess}}(M_0) = \sigma(M_\ell) \cup \sigma(M_r).$$

To prove Proposition 3.10, we first need two preliminary lemmas. In the first lemma, we use the notation χ_Λ for the characteristic function of a Borel set $\Lambda \subset \mathbb{R}$.

Lemma 3.11. (a) *The operator M is locally compact in \mathcal{H}_w , that is, $\chi_B(Q)(M - i)^{-1} \in \mathcal{K}(\mathcal{H}_w)$ for each bounded Borel set $B \subset \mathbb{R}$.*

(b) *Let $\zeta \in C_c^\infty(\mathbb{R}, [0, \infty))$ satisfy $\zeta(x) = 1$ for $|x| \leq 1$ and $\zeta(x) = 0$ for $|x| \geq 2$, and set $\zeta_n(x) := \zeta(x/n)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N} \setminus \{0\}$. Then*

$$\lim_{n \rightarrow \infty} \|[M, \zeta_n(Q)](M - i)^{-1}\|_{\mathcal{B}(\mathcal{H}_w)} = 0.$$

Moreover, the results of (a) and (b) also hold true for the operators M_0 and Q_0 in the Hilbert space \mathcal{H}_0 .

Proof. (a) A direct computation shows that

$$\chi_B(Q)(D - i)^{-1} = \begin{pmatrix} i\chi_B(Q)(1 + P^2)^{-1} & \chi_B(Q)P(1 + P^2)^{-1} \\ \chi_B(Q)P(1 + P^2)^{-1} & i\chi_B(Q)(1 + P^2)^{-1} \end{pmatrix},$$

which implies that $\chi_B(Q)(D - i)^{-1}$ is compact in $L^2(\mathbb{R}, \mathbb{C}^2)$ since every entry of the matrix is compact in $L^2(\mathbb{R})$ (see [34, Theorem 4.1]). Given that $L^2(\mathbb{R}, \mathbb{C}^2)$ and \mathcal{H}_w have equivalent norms by Lemma 2.3(b), it follows that $\chi_B(Q)(D - i)^{-1}$ is also compact in \mathcal{H}_w . Finally, the resolvent identity (similar to (2.4))

$$(M - i)^{-1} = (D - i)^{-1}w^{-1} + i(D - i)^{-1}(w^{-1} - 1)(M - i)^{-1}$$

shows that $\chi_B(Q)(M - i)^{-1}$ is the sum of two compact operators in \mathcal{H}_w , and hence compact in \mathcal{H}_w . The same argument also shows that the operators M_\star are

locally compact in \mathcal{H}_{w_\star} , and thus that $M_0 = M_\ell \oplus M_r$ is locally compact in $\mathcal{H}_0 = \mathcal{H}_{w_\ell} \oplus \mathcal{H}_{w_r}$.

(b) Let $\varphi \in \mathcal{D}(M) = \mathcal{H}^1(\mathbb{R}, \mathbb{C}^2)$. Then a direct computation taking into account the inclusion $\zeta_n(Q)\varphi \in \mathcal{D}(M)$ gives

$$[M, \zeta_n(Q)]\varphi = w \begin{pmatrix} 0 & [P, \zeta_n(Q)] \\ [P, \zeta_n(Q)] & 0 \end{pmatrix} \varphi = -\frac{i}{n} w \zeta_n'(Q) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \varphi.$$

In consequence, we obtain that $\|[M, \zeta_n(Q)](M - i)^{-1}\|_{\mathcal{B}(\mathcal{H}_w)} \leq \text{Const. } n^{-1}$ which proves the claim. As before, the same argument also applies to the operators M_\star in \mathcal{H}_{w_\star} , and thus to the operator $M_0 = M_\ell \oplus M_r$ in $\mathcal{H}_0 = \mathcal{H}_{w_\ell} \oplus \mathcal{H}_{w_r}$. \square

Lemma 3.11 is needed to prove that the essential spectra of M and M_0 can be characterized in terms of *Zhislin sequences* (see [13, Definition 10.4]). Zhislin sequences are particular types of Weyl sequences supported at infinity as in the following lemma.

Lemma 3.12 (Zhislin Sequences). *Let $\lambda \in \mathbb{R}$. Then $\lambda \in \sigma_{\text{ess}}(M)$ if and only if there exists a sequence $\{\phi_m\}_{m \in \mathbb{N} \setminus \{0\}} \subset \mathcal{D}(M)$, called *Zhislin sequence*, such that:*

- (a) $\|\phi_m\|_{\mathcal{H}_w} = 1$ for all $m \in \mathbb{N} \setminus \{0\}$;
- (b) for each $m \in \mathbb{N} \setminus \{0\}$, one has $\phi_m(x) = 0$ if $|x| \leq m$;
- (c) $\lim_{m \rightarrow \infty} \|(M - \lambda)\phi_m\|_{\mathcal{H}_w} = 0$.

Similarly, $\lambda \in \sigma_{\text{ess}}(M_0)$ if and only if there exists a sequence $\{\phi_m^0\}_{m \in \mathbb{N} \setminus \{0\}} \subset \mathcal{D}(M_0)$ which meets the properties (a)–(c) relative to the operator M_0 .

Proof. In view of Lemma 3.11, the claim can be proved by repeating step by step the arguments in the proof of [13, Theorem 10.6]. \square

We are now ready to complete the description of the essential spectrum of M .

Proof of Proposition 3.10. Take $\lambda \in \sigma_{\text{ess}}(M)$, let $\{\phi_m\}_{m \in \mathbb{N} \setminus \{0\}} \subset \mathcal{D}(M)$ be an associated Zhislin sequence, and define for each $m \in \mathbb{N} \setminus \{0\}$

$$\phi_m^0 := c_m(j_\ell \phi_m, j_r \phi_m) \in \mathcal{D}(M_0) \quad \text{with } c_m := \|(j_\ell \phi_m, j_r \phi_m)\|_{\mathcal{H}_0}^{-1}.$$

Then one has $\|\phi_m^0\|_{\mathcal{H}_0} = 1$ for all $m \in \mathbb{N} \setminus \{0\}$ and $\phi_m^0(x) = 0$ if $|x| \leq m$. Furthermore, using successively the facts that $j_\ell j_r = 0$, that $(j_\ell + j_r)\phi_m = \phi_m$, and that $1 = \|\phi_m\|_{\mathcal{H}_w}^2 = \langle \phi_m, w^{-1} \phi_m \rangle_{L^2(\mathbb{R}, \mathbb{C}^2)}$, one obtains that

$$\begin{aligned} c_m^{-2} &= \langle \phi_m, (w_\ell^{-1} j_\ell + w_r^{-1} j_r) \phi_m \rangle_{L^2(\mathbb{R}, \mathbb{C}^2)} \\ &= 1 + \langle \phi_m, (w_\ell^{-1} j_\ell + w_r^{-1} j_r - w^{-1}) \phi_m \rangle_{L^2(\mathbb{R}, \mathbb{C}^2)}, \end{aligned}$$

which implies that $\lim_{m \rightarrow \infty} c_m = 1$ due to Assumption 2.2.

Now, consider the inequalities

$$\begin{aligned}
 \|(M_0 - \lambda)\phi_m^0\|_{\mathcal{H}_0}^2 &= c_m^2 \sum_{\star \in \{\ell, r\}} \|(M_\star - \lambda)j_\star \phi_m\|_{\mathcal{H}_{w_\star}}^2 \\
 &\leq c_m^2 \sum_{\star \in \{\ell, r\}} (\|(M - \lambda)j_\star \phi_m\|_{\mathcal{H}_{w_\star}} + \|(M_\star - M)j_\star \phi_m\|_{\mathcal{H}_{w_\star}})^2 \\
 &\leq c_m^2 \sum_{\star \in \{\ell, r\}} (\|j_\star(M - \lambda)\phi_m\|_{\mathcal{H}_{w_\star}} \\
 &\quad + \|[M, j_\star]\phi_m\|_{\mathcal{H}_{w_\star}} + \|(w_\star - w)Dj_\star \phi_m\|_{\mathcal{H}_{w_\star}})^2.
 \end{aligned}$$

From the property (c) of Zhislin sequences, the boundedness of j_\star , and the equivalence of the norms of \mathcal{H}_{w_\star} and \mathcal{H}_w , one gets that

$$\lim_{m \rightarrow \infty} \|j_\star(M - \lambda)\phi_m\|_{\mathcal{H}_{w_\star}} \leq \text{Const.} \quad \lim_{m \rightarrow \infty} \|(M - \lambda)\phi_m\|_{\mathcal{H}_w} = 0.$$

Moreover, one has $[M, j_\star]\phi_m = -iwj_\star' \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \phi_m$, with j_\star' supported in $[-1, 1]$. This implies that $[M, j_\star]\phi_m = 0$. For the same reason, one has $Dj_\star \phi_m = j_\star D\phi_m$, with the latter vector supported in $x \leq -m$ if $\star = \ell$ and in $x \geq m$ if $\star = r$. This, together with Assumption 2.2, implies that

$$\begin{aligned}
 \|(w_\star - w)Dj_\star \phi_m\|_{\mathcal{H}_{w_\star}} &\leq \text{Const.} \langle m \rangle^{-1-\varepsilon} \|D\phi_m\|_{\mathcal{H}_{w_\star}} \\
 &\leq \text{Const.} \langle m \rangle^{-1-\varepsilon} \|M\phi_m\|_{\mathcal{H}_w}.
 \end{aligned}$$

The last inequality, along with the equality $\lim_{m \rightarrow \infty} \|M\phi_m\|_{\mathcal{H}_w} = |\lambda|$ (which follows from the property (c) of Zhislin sequences), gives

$$\lim_{m \rightarrow \infty} \|(w_\star - w)Dj_\star \phi_m\|_{\mathcal{H}_{w_\star}} = 0.$$

Putting all the pieces together, we obtain that $\lim_{m \rightarrow \infty} \|(M_0 - \lambda)\phi_m^0\|_{\mathcal{H}_0} = 0$. This concludes the proof that $\{\phi_m^0\}_{m \in \mathbb{N} \setminus \{0\}}$ is a Zhislin sequence for the operator M_0 and the point $\lambda \in \sigma_{\text{ess}}(M)$, and thus that $\sigma_{\text{ess}}(M) \subset \sigma_{\text{ess}}(M_0)$.

For the opposite inclusion, take $\{\phi_m^0\}_{m \in \mathbb{N} \setminus \{0\}} \subset \mathcal{D}(M_0)$ a Zhislin sequence for the operator M_0 and the point $\lambda \in \sigma_{\text{ess}}(M_0)$, and assume that $\lambda \in \sigma_{\text{ess}}(M_r)$ (if $\lambda \notin \sigma_{\text{ess}}(M_r)$, then $\lambda \in \sigma_{\text{ess}}(M_\ell)$ and the same proof applies if one replaces “right” with “left”). By extracting the non-zero right components from ϕ_m^0 and normalizing, we can form a new Zhislin sequence $\{(0, \phi_m^r)\}_{m \in \mathbb{N} \setminus \{0\}}$ for M_0 with $\{\phi_m^r\}_{m \in \mathbb{N} \setminus \{0\}} \subset \mathcal{D}(M_r)$ a Zhislin sequence for M_r . Then we can construct as follows a new Zhislin sequence for M_r with vectors supported in $[m, \infty)$: Let $\zeta^r \in C_c^\infty(\mathbb{R}, [0, 1])$ satisfy $\zeta^r(x) = 0$ for $x \leq 1$ and $\zeta^r(x) = 1$ for $x \geq 2$, set $\zeta_m^r(x) := \zeta^r(x/m)$ for all $x \in \mathbb{R}$ and $m \in \mathbb{N} \setminus \{0\}$, and choose $n_m^r \in \mathbb{N}$ such that $\|\chi_{[-n_m^r, \infty)}(Q_r)\phi_m^r\|_{\mathcal{H}_{w_r}} \in (1 - 1/m, 1]$. Next, define for each $m \in \mathbb{N} \setminus \{0\}$

$$\tilde{\phi}_m^r := d_m \zeta_m^r(Q_r) T_{k_m^r p_r} \phi_m^r \in \mathcal{D}(M_r),$$

with $d_m := \|\zeta_m^r(Q_r)T_{k_m^r p_r} \phi_m^r\|_{\mathcal{H}_{w_r}}^{-1}$, $k_m^r \in \mathbb{N}$ such that $k_m^r p_r \geq n_m^r + 2m$, and $T_{k_m^r p_r}$ the operator of translation by $k_m^r p_r$. One verifies easily that $\lim_{m \rightarrow \infty} d_m = 1$ and that $\tilde{\phi}_m^r$ has support in $[m, \infty)$. Furthermore, since the operators M_r and $T_{k_m^r p_r}$ commute, one also has

$$\begin{aligned} \|(M_r - \lambda)\tilde{\phi}_m^r\|_{\mathcal{H}_{w_r}} &\leq d_m \|\zeta_m^r(Q_r)T_{k_m^r p_r}(M_r - \lambda)\phi_m^r\|_{\mathcal{H}_{w_r}} \\ &\quad + d_m \|[M_r, \zeta_m^r(Q_r)]T_{k_m^r p_r}\phi_m^r\|_{\mathcal{H}_{w_r}} \\ &\leq d_m \|(M_r - \lambda)\phi_m^r\|_{\mathcal{H}_{w_r}} + \text{Const. } m^{-1} \|\phi_m^r\|_{\mathcal{H}_{w_r}}, \end{aligned}$$

which implies that $\lim_{m \rightarrow \infty} \|(M_r - \lambda)\tilde{\phi}_m^r\|_{\mathcal{H}_{w_r}} = 0$. Thus, $\{\tilde{\phi}_m^r\}_{m \in \mathbb{N} \setminus \{0\}} \subset \mathcal{D}(M_r)$ is a new Zhislin sequence for M_r with $\tilde{\phi}_m^r$ supported in $[m, \infty)$.

Now, define for each $m \in \mathbb{N} \setminus \{0\}$

$$\phi_m := b_m \tilde{\phi}_m^r \in \mathcal{D}(M) \quad \text{with } b_m := \|\tilde{\phi}_m^r\|_{\mathcal{H}_w}^{-1}.$$

Then one has $\|\phi_m\|_{\mathcal{H}_w} = 1$ for all $m \in \mathbb{N} \setminus \{0\}$ and $\phi_m(x) = 0$ if $x \leq m$. Furthermore, using that

$$1 = \|\tilde{\phi}_m^r\|_{\mathcal{H}_{w_r}}^2 = \langle \tilde{\phi}_m^r, w_r^{-1} \tilde{\phi}_m^r \rangle_{L^2(\mathbb{R}, \mathbb{C}^2)},$$

one obtains that

$$b_m^{-2} := \langle \tilde{\phi}_m^r, w^{-1} \tilde{\phi}_m^r \rangle_{L^2(\mathbb{R}, \mathbb{C}^2)} = 1 + \langle \tilde{\phi}_m^r, (w^{-1} - w_r^{-1}) \tilde{\phi}_m^r \rangle_{L^2(\mathbb{R}, \mathbb{C}^2)},$$

which implies that $\lim_{m \rightarrow \infty} b_m = 1$ due to Assumption 2.2. Now, consider the inequality

$$\begin{aligned} \|(M - \lambda)\phi_m\|_{\mathcal{H}_w} &= b_m \|(M - \lambda)\tilde{\phi}_m^r\|_{\mathcal{H}_w} \\ &\leq b_m \|(M_r - \lambda)\tilde{\phi}_m^r\|_{\mathcal{H}_w} + b_m \|(w - w_r)D\tilde{\phi}_m^r\|_{\mathcal{H}_w}. \end{aligned}$$

From the property (c) of Zhislin sequences and the equivalence of the norms of \mathcal{H}_{w_r} and \mathcal{H}_w , one gets that

$$\lim_{m \rightarrow \infty} \|(M_r - \lambda)\tilde{\phi}_m^r\|_{\mathcal{H}_w} \leq \text{Const.} \lim_{m \rightarrow \infty} \|(M_r - \lambda)\tilde{\phi}_m^r\|_{\mathcal{H}_{w_r}} = 0.$$

Moreover, since $D\tilde{\phi}_m^r$ is supported in $[m, \infty)$, one infers again from Assumption 2.2 that

$$\|(w - w_r)D\tilde{\phi}_m^r\|_{\mathcal{H}_w} \leq \text{Const.} \langle m \rangle^{-1-\varepsilon} \|D\tilde{\phi}_m^r\|_{\mathcal{H}_w} \leq \text{Const.} \langle m \rangle^{-1-\varepsilon} \|M_r \tilde{\phi}_m^r\|_{\mathcal{H}_{w_r}}.$$

The last inequality, along with the equality $\lim_{m \rightarrow \infty} \|M_r \tilde{\phi}_m^r\|_{\mathcal{H}_{w_r}} = |\lambda|$, gives

$$\lim_{m \rightarrow \infty} \|(w - w_r)D\tilde{\phi}_m^r\|_{\mathcal{H}_w} = 0.$$

Putting all the pieces together, we obtain that $\lim_{m \rightarrow \infty} \|(M - \lambda)\phi_m\|_{\mathcal{H}_w} = 0$. This concludes the proof that $\{\phi_m\}_{m \in \mathbb{N} \setminus \{0\}}$ is a Zhislin sequence for the operator M and the point $\lambda \in \sigma_{\text{ess}}(M_0)$, and thus that $\sigma_{\text{ess}}(M_0) \subset \sigma_{\text{ess}}(M)$. In consequence, we obtained that $\sigma_{\text{ess}}(M) = \sigma_{\text{ess}}(M_0)$, which completes the proof in view of Remark 3.4. \square

In order to determine more precise spectral properties of M , we now prove that for each compact interval $I \subset \mathbb{R} \setminus \mathcal{T}_M$ the Hamiltonian M is of class $C^{1+\varepsilon}(A_I)$ for some $\varepsilon \in (0, 1)$, which is a regularity condition slightly stronger than the condition M of class $C^1(A_I)$ already established in Theorem 3.9. We start by giving a convenient formula for the commutator $[(M - z)^{-1}, A_I]$, $z \in \mathbb{C} \setminus \mathbb{R}$, in the form sense on $\mathcal{S}(\mathbb{R}, \mathbb{C}^2)$:

$$\begin{aligned}
 & [(M - z)^{-1}, A_I] \\
 &= ((M - z)^{-1}J - J(M_0 - z)^{-1})A_{0,I}J^* - JA_{0,I}(J^*(M - z)^{-1} \\
 &\quad - (M_0 - z)^{-1}J^*) + J[(M_0 - z)^{-1}, A_{0,I}]J^* \\
 &= \sum_{\star \in \{\ell, r\}} \{((M - z)^{-1} - (M_\star - z)^{-1})j_\star A_{\star, I}Z^\star + [(M_\star - z)^{-1}, j_\star]A_{\star, I}Z^\star \\
 &\quad - j_\star A_{\star, I}Z^\star((M - z)^{-1} - (M_\star - z)^{-1}) + j_\star A_{\star, I}[(M_\star - z)^{-1}, Z^\star] \\
 &\quad + j_\star[(M_\star - z)^{-1}, A_{\star, I}]Z^\star\} \\
 &= \sum_{\star \in \{\ell, r\}} (C_\star + j_\star[(M_\star - z)^{-1}, A_{\star, I}]Z^\star)
 \end{aligned}$$

with

$$Z^\star := w_\star w^{-1} j_\star = j_\star - (w - w_\star) j_\star w^{-1},$$

and

$$\begin{aligned}
 C_\star &:= ((M - z)^{-1} - (M_\star - z)^{-1})j_\star A_{\star, I}Z^\star + [(M_\star - z)^{-1}, j_\star]A_{\star, I}Z^\star \\
 &\quad - j_\star A_{\star, I}Z^\star((M - z)^{-1} - (M_\star - z)^{-1}) + j_\star A_{\star, I}[(M_\star - z)^{-1}, Z^\star].
 \end{aligned}$$

As already shown in the previous section, all the terms in C_\star extend to bounded operators, and we keep the same notation for these extensions.

In order to show that $(M - z)^{-1} \in C^{1+\varepsilon}(A_I)$, it is enough to prove that $j_\star[(M_\star - z)^{-1}, A_{\star, I}]Z^\star \in C^1(A_I)$ and to check that

$$\|e^{-itA_I} C_\star e^{itA_I} - C_\star\|_{\mathcal{B}(\mathcal{H}_w)} \leq \text{Const. } t^\varepsilon \quad \text{for all } t \in (0, 1). \quad (3.11)$$

Since the first proof reduces to computations similar to the ones presented in the previous section, we shall concentrate on the proof of (3.11). First of all, algebraic manipulations as presented in [1, pp. 325–326] or [30, Sec. 4.3] show that for all $t \in (0, 1)$

$$\begin{aligned}
 \|e^{-itA_I} C_\star e^{itA_I} - C_\star\|_{\mathcal{B}(\mathcal{H}_w)} &\leq \text{Const.} (\|\sin(tA_I)C_\star\|_{\mathcal{B}(\mathcal{H}_w)} \\
 &\quad + \|\sin(tA_I)(C_\star)^*\|_{\mathcal{B}(\mathcal{H}_w)}) \\
 &\leq \text{Const.} (\|tA_I(tA_I + i)^{-1}C_\star\|_{\mathcal{B}(\mathcal{H}_w)} \\
 &\quad + \|tA_I(tA_I + i)^{-1}(C_\star)^*\|_{\mathcal{B}(\mathcal{H}_w)}).
 \end{aligned}$$

Furthermore, if we set $A_t := tA_I(tA_I + i)^{-1}$ and $\Lambda_t := t\langle Q \rangle(t\langle Q \rangle + i)^{-1}$, we obtain that

$$A_t = (A_t + i(tA_I + i)^{-1}A_I\langle Q \rangle^{-1})\Lambda_t$$

with $A_I\langle Q \rangle^{-1} \in \mathcal{B}(\mathcal{H}_w)$ due to (3.7). Thus, since $\|A_t + i(tA_I + i)^{-1}A_I\langle Q \rangle^{-1}\|_{\mathcal{B}(\mathcal{H}_w)}$ is bounded by a constant independent of $t \in (0, 1)$, it is sufficient to prove that

$$\|\Lambda_t C_\star\|_{\mathcal{B}(\mathcal{H}_w)} + \|\Lambda_t (C_\star)^*\|_{\mathcal{B}(\mathcal{H}_w)} \leq \text{Const. } t^\varepsilon \quad \text{for all } t \in (0, 1),$$

and to prove this estimate it is sufficient to show that the operators $\langle Q \rangle^\varepsilon C_\star$ and $\langle Q \rangle^\varepsilon (C_\star)^*$ defined in the form sense on $\mathcal{S}(\mathbb{R}, \mathbb{C}^2)$ extend continuously to elements of $\mathcal{B}(\mathcal{H}_w)$. Finally, some lengthy but straightforward computations show that these two last conditions are implied by the following two lemmas.

Lemma 3.13. *M_\star is of class $C^1(\langle Q \rangle^\alpha)$ for each $\star \in \{\ell, r\}$ and $\alpha \in [0, 1]$.*

Proof. One can verify directly that the unitary group generated by the operator $\langle Q \rangle^\alpha$ leaves the domain $\mathcal{D}(M_\star) = \mathcal{H}^1(\mathbb{R}, \mathbb{C}^2)$ invariant and that the commutator $[M_\star, \langle Q \rangle^\alpha]$ defined in the form sense on $\mathcal{S}(\mathbb{R}, \mathbb{C}^2)$ extends continuously to a bounded operator. Since the set $\mathcal{S}(\mathbb{R}, \mathbb{C}^2)$ is a core for M_\star , these properties together with [1, Theorem 6.3.4(a)] imply the claim. \square

Lemma 3.14. *One has $j_\star \in C^1(A_{\star, I})$ for each $\star \in \{\ell, r\}$ and each compact interval $I \subset \mathbb{R} \setminus \mathcal{T}_M$.*

Proof. By using the commutator expansions [1, Theorem 5.5.3] and (3.7), one gets the following equalities in form sense on $\mathcal{S}(\mathbb{R}, \mathbb{C}^2)$:

$$\begin{aligned} [j_\star, A_{\star, I}] &= \frac{1}{\sqrt{2\pi}} \int_0^1 d\tau \int_{\mathbb{R}} dx e^{i\tau x Q_\star} [Q_\star, A_{\star, I}] e^{i(1-\tau)x Q_\star} \widehat{j}'_\star(x) \\ &= \frac{1}{2\sqrt{2\pi}} \sum_{n \in \mathbb{N}(I)} \int_0^1 d\tau \int_{\mathbb{R}} dx e^{i\tau x Q_\star} [Q_\star, \Pi_{\star, n}(\check{\lambda}'_{\star, n} Q_\star + Q_\star \check{\lambda}'_{\star, n}) \Pi_{\star, n}] \\ &\quad \times e^{i(1-\tau)x Q_\star} \widehat{j}'_\star(x) \\ &= \frac{1}{2\sqrt{2\pi}} \sum_{n \in \mathbb{N}(I)} \int_0^1 d\tau \int_{\mathbb{R}} dx e^{i\tau x Q_\star} \{ [Q_\star, \Pi_{\star, n} \check{\lambda}'_{\star, n} \Pi_{\star, n}] Q_\star \\ &\quad + Q_\star [Q_\star, \Pi_{\star, n} \check{\lambda}'_{\star, n} \Pi_{\star, n}] + [Q_\star, \Pi_{\star, n} \check{\lambda}'_{\star, n} [Q_\star, \Pi_{\star, n}]] \\ &\quad - [Q_\star, [Q_\star, \Pi_{\star, n} \check{\lambda}'_{\star, n} \Pi_{\star, n}]] \} e^{i(1-\tau)x Q_\star} \widehat{j}'_\star(x) \end{aligned}$$

with $\check{\lambda}'_{\star, n} := \mathcal{U}_\star^{-1} \widehat{\lambda}'_{\star, n} \mathcal{U}_\star$ and with each commutator in the last equality extending continuously to a bounded operator. Since \widehat{j}'_\star is integrable, the last two terms give bounded contributions. Furthermore, the first two terms can be rewritten as

$$f([Q_\star, \Pi_{\star, n} \check{\lambda}'_{\star, n} \Pi_{\star, n}]) Q_\star + Q_\star f([Q_\star, \Pi_{\star, n} \check{\lambda}'_{\star, n} \Pi_{\star, n}])$$

with

$$f : \mathcal{B}(\mathcal{H}_{w_*}) \rightarrow \mathcal{B}(\mathcal{H}_{w_*}),$$

$$B \mapsto f(B) := \frac{1}{2\sqrt{2\pi}} \sum_{n \in \mathbb{N}(I)} \int_0^1 d\tau \int_{\mathbb{R}} dx e^{i\tau x Q_*} B e^{i(1-\tau)x Q_*} \widehat{j}'_*(x).$$

But, since $[Q_*, \Pi_{*,n} \check{\lambda}'_{*,n} \Pi_{*,n}] \in C^k(Q_*)$ for each $k \in \mathbb{N}$, and since \widehat{j}'_* is a Schwartz function, one infers from [1, Theorem 5.5.3] that the operator $f([Q_*, \Pi_{*,n} \check{\lambda}'_{*,n} \Pi_{*,n}])$ is regularizing in the Besov scale associated to the operator Q_* . This implies in particular that the operators $f([Q_*, \Pi_{*,n} \check{\lambda}'_{*,n} \Pi_{*,n}])Q_*$ and $Q_*f([Q_*, \Pi_{*,n} \check{\lambda}'_{*,n} \Pi_{*,n}])$ extend continuously to bounded operators, as desired. \square

We can now give in the following theorem a description of the structure of the spectrum of the full Hamiltonian M . The next theorem also shows that the set \mathcal{T}_M can be interpreted as the set of thresholds in the spectrum of M .

Theorem 3.15. *In any compact interval $I \subset \mathbb{R} \setminus \mathcal{T}_M$, the operator M has at most finitely many eigenvalues, each one of finite multiplicity, and no singular continuous spectrum.*

Proof. The computations at the beginning of this section together with Lemmas 3.13 and 3.14 imply that M is of class $C^{1+\varepsilon}(A_I)$ for some $\varepsilon \in (0, 1)$, and Theorem 3.9 implies that the condition (3.5) of Theorem 3.1 is satisfied on I . So, one can apply Theorem 3.1(b) to conclude. \square

Remark 3.16. As a final remark about the spectral properties of the operator M , let us mention that the techniques used in this work do not provide any further information about the existence of eigenvalues at thresholds or embedded in the continuous spectrum. For additive perturbations, powerful techniques have been developed over the last decades, and these methods can be applied to several Schrödinger-type operators. On the other hand, for multiplicative perturbations, abstract methods have not been developed so far. To the best of our knowledge, there exist only a few results about embedded eigenvalues in the case of quantum walks with multiplicative perturbations as for example in [18, 22, 23], or more abstractly in [6]. For photonic crystals, especially in the presence of bi-anisotropic media, eigenvalues at thresholds or embedded in the continuous spectrum certainly deserve an independent study and this could be the subject of future investigations.

4. Scattering Theory

4.1. Scattering theory in a two-Hilbert spaces setting

We discuss in this section the existence and the completeness, under smooth perturbations, of the local wave operators for self-adjoint operators in a two-Hilbert spaces setting. Namely, given two self-adjoint operators H_0, H in Hilbert spaces

$\mathcal{H}_0, \mathcal{H}$ with spectral measures E^{H_0}, E^H , an identification operator $J \in \mathcal{B}(\mathcal{H}_0, \mathcal{H})$, and an open set $I \subset \mathbb{R}$, we recall criteria for the existence and the completeness of the strong limits

$$W_{\pm}(H, H_0, J, I) := \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} J e^{-itH_0} E^{H_0}(I)$$

under the assumption that the two-Hilbert spaces difference of resolvents

$$J(H_0 - z)^{-1} - (H - z)^{-1}J, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

factorizes as a product of a locally H -smooth operator on I and a locally H_0 -smooth operator on I .

We start by recalling some facts related to the notion of J -completeness. Let $\mathfrak{N}_{\pm}(H, J, I)$ be the subsets of \mathcal{H} defined by

$$\mathfrak{N}_{\pm}(H, J, I) := \{\varphi \in \mathcal{H} \mid \lim_{t \rightarrow \pm\infty} \|J^* e^{-itH} E^H(I)\varphi\|_{\mathcal{H}_0} = 0\}.$$

Then it is clear that $\mathfrak{N}_{\pm}(H, J, I)$ are closed subspaces of \mathcal{H} and that $E^H(\mathbb{R} \setminus I)\mathcal{H} \subset \mathfrak{N}_{\pm}(H, J, I)$, and it is shown in [42, Sec. 3.2] that H is reduced by $\mathfrak{N}_{\pm}(H, J, I)$ and that

$$\overline{\text{Ran}(W_{\pm}(H, H_0, J, I))} \perp \mathfrak{N}_{\pm}(H, J, I).$$

In particular, one has the inclusion

$$\overline{\text{Ran}(W_{\pm}(H, H_0, J, I))} \subset E^H(I)\mathcal{H} \ominus \mathfrak{N}_{\pm}(H, J, I),$$

which motivates the following definition.

Definition 4.1 (J -Completeness). Assume that the local wave operators $W_{\pm}(H, H_0, J, I)$ exist. Then $W_{\pm}(H, H_0, J, I)$ are J -complete on I if

$$\overline{\text{Ran}(W_{\pm}(H, H_0, J, I))} = E^H(I)\mathcal{H} \ominus \mathfrak{N}_{\pm}(H, J, I).$$

Remark 4.2. In the particular case $\mathcal{H}_0 = \mathcal{H}$ and $J = 1_{\mathcal{H}}$, the J -completeness on I reduces to the completeness of the local wave operators $W_{\pm}(H, H_0, J, I)$ on I in the usual sense. Namely, $\overline{\text{Ran}(W_{\pm}(H, H_0, 1_{\mathcal{H}}, I))} = E^H(I)\mathcal{H}$, and the operators $W_{\pm}(H, H_0, 1_{\mathcal{H}}, I)$ are unitary from $E^{H_0}(I)\mathcal{H}$ to $E^H(I)\mathcal{H}$.

The following criterion for J -completeness has been established in [42, Theorem 3.2.4].

Lemma 4.3. *If the local wave operators $W_{\pm}(H, H_0, J, I)$ and $W_{\pm}(H_0, H, J^*, I)$ exist, then $W_{\pm}(H, H_0, J, I)$ are J -complete on I .*

For the following theorem, we recall that the spectral support $\text{supp}_H(\varphi)$ of a vector $\varphi \in \mathcal{H}$ with respect to H is the smallest closed set $I \subset \mathbb{R}$ such that $E^H(I)\varphi = \varphi$.

Theorem 4.4. *Let H_0, H be self-adjoint operators in Hilbert spaces $\mathcal{H}_0, \mathcal{H}$ with spectral measures E^{H_0}, E^H , $J \in \mathcal{B}(\mathcal{H}_0, \mathcal{H})$, $I \subset \mathbb{R}$ an open set, and \mathcal{G} an auxiliary*

Hilbert space. For each $z \in \mathbb{C} \setminus \mathbb{R}$, assume there exist $T_0(z) \in \mathcal{B}(\mathcal{H}_0, \mathcal{G})$ locally H_0 -smooth on I and $T(z) \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ locally H -smooth on I such that

$$J(H_0 - z)^{-1} - (H - z)^{-1}J = T(z)^*T_0(z).$$

Then the local wave operators

$$W_{\pm}(H, H_0, J, I) = \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} J e^{-itH_0} E^{H_0}(I) \quad (4.1)$$

exist, are J -complete on I , and satisfy the relations

$$\begin{aligned} W_{\pm}(H, H_0, J, I)^* &= W_{\pm}(H_0, H, J^*, I) \quad \text{and} \\ W_{\pm}(H, H_0, J, I)\eta(H_0) &= \eta(H)W_{\pm}(H, H_0, J, I) \end{aligned}$$

for each bounded Borel function $\eta : \mathbb{R} \rightarrow \mathbb{C}$.

Proof. We adapt the proof of [1, Theorem 7.1.4] to the two-Hilbert spaces setting. The existence of the limits (4.1) is a direct consequence of the following claims: for each $\varphi_0 \in \mathcal{H}_0$ with $I_0 := \text{supp}_{H_0}(\varphi_0) \subset I$ compact, and for each $\eta \in C_c^\infty(I)$ such that $\eta \equiv 1$ on a neighborhood of I_0 , we have that

$$\begin{aligned} \text{s-lim}_{t \rightarrow \pm\infty} \eta(H) e^{itH} J e^{-itH_0} \varphi_0 &\text{ exist and} \\ \lim_{t \rightarrow \pm\infty} \|(1 - \eta(H)) e^{itH} J e^{-itH_0} \varphi_0\|_{\mathcal{H}} &= 0. \end{aligned} \quad (4.2)$$

To prove the first claim in (4.2), take $\varphi \in \mathcal{H}$ and $t \in \mathbb{R}$, and observe that the operators $W(t) := \eta(H) e^{itH} J e^{-itH_0}$ satisfy for $z \in \mathbb{C} \setminus \mathbb{R}$ and $s \leq t$

$$\begin{aligned} &| \langle (H - \bar{z})^{-1} \varphi, (W(t) - W(s))(H_0 - z)^{-1} \varphi_0 \rangle_{\mathcal{H}} | \\ &= \left| \int_s^t du \frac{d}{du} \langle e^{-iuH} \bar{\eta}(H) \varphi, (H - z)^{-1} J (H_0 - z)^{-1} e^{-iuH_0} \varphi_0 \rangle_{\mathcal{H}} \right| \\ &= \left| \int_s^t du \langle e^{-iuH} \bar{\eta}(H) \varphi, (H - z)^{-1} (HJ - JH_0) (H_0 - z)^{-1} e^{-iuH_0} \varphi_0 \rangle_{\mathcal{H}} \right| \\ &= \left| \int_s^t du \langle e^{-iuH} \bar{\eta}(H) \varphi, (J(H_0 - z)^{-1} - (H - z)^{-1} J) e^{-iuH_0} \varphi_0 \rangle_{\mathcal{H}} \right| \\ &= \left| \int_s^t du \langle T(z) e^{-iuH} \bar{\eta}(H) \varphi, T_0(z) e^{-iuH_0} \varphi_0 \rangle_{\mathcal{G}} \right| \\ &\leq \left(\int_s^t du \|T(z) e^{-iuH} \bar{\eta}(H) \varphi\|_{\mathcal{G}}^2 \right)^{1/2} \left(\int_s^t du \|T_0(z) e^{-iuH_0} \varphi_0\|_{\mathcal{G}}^2 \right)^{1/2} \\ &\leq c_{I_1}^{1/2} \|\varphi\|_{\mathcal{H}} \left(\int_s^t du \|T_0(z) e^{-iuH_0} \varphi_0\|_{\mathcal{G}}^2 \right)^{1/2}, \end{aligned}$$

with $I_1 := \text{supp}(\eta)$ and c_{I_1} the constant appearing in the definition (3.4) of a locally H -smooth operator. Since the set $(H - \bar{z})^{-1}\mathcal{H}$ is dense in \mathcal{H} and T_0 is locally H_0 -smooth on I , it follows that $\|(W(t) - W(s))(H_0 - z)^{-1}\varphi_0\|_{\mathcal{H}} \rightarrow 0$ as $s \rightarrow \infty$ or $t \rightarrow -\infty$. Applying this result with φ_0 replaced by $(H_0 - z)\varphi_0$ we infer that

$$\|(W(t) - W(s))\varphi_0\|_{\mathcal{H}} = \|(W(t) - W(s))(H_0 - z)^{-1}(H_0 - z)\varphi_0\|_{\mathcal{H}} \rightarrow 0$$

as $s \rightarrow \infty$ or $t \rightarrow -\infty$, which proves the first claim in (4.2).

To prove the second claim in (4.2), we take $\eta_0 \in C_c^\infty(I)$ such that $\eta_0 \equiv 1$ on I_0 and $\eta\eta_0 = \eta_0$. Then we have $\varphi_0 = \eta_0(H_0)\varphi_0$ and

$$(1 - \eta(H))J\eta_0(H_0) = (1 - \eta(H))(J\eta_0(H_0) - \eta_0(H)J),$$

and thus the second claim in (4.2) follows from

$$\lim_{t \rightarrow \pm\infty} \|(J\eta_0(H_0) - \eta_0(H)J)e^{-itH_0}\varphi_0\|_{\mathcal{H}} = 0.$$

Since the vector space generated by the functions $\mathbb{R} \ni x \mapsto (x - z)^{-1} \in \mathbb{C}$, $z \in \mathbb{C} \setminus \mathbb{R}$, is dense in $C_0(\mathbb{R})$, it is sufficient to show that

$$\lim_{t \rightarrow \pm\infty} \|(J(H_0 - z)^{-1} - (H - z)^{-1}J)e^{-itH_0}\varphi_0\|_{\mathcal{H}} = 0, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Now, we have for every $\varphi \in \mathcal{H}$

$$\begin{aligned} |\langle \varphi, (J(H_0 - z)^{-1} - (H - z)^{-1}J)e^{-itH_0}\varphi_0 \rangle_{\mathcal{H}}| &= |\langle T(z)\varphi, T_0(z)e^{-itH_0}\varphi_0 \rangle_{\mathcal{G}}| \\ &\leq \|T(z)\varphi\|_{\mathcal{G}}\|T_0(z)e^{-itH_0}\varphi_0\|_{\mathcal{G}}. \end{aligned}$$

Therefore, it is enough to prove that $\|T_0(z)e^{-itH_0}\varphi_0\|_{\mathcal{G}} \rightarrow 0$ as $|t| \rightarrow \infty$. But since $T_0(z)e^{-itH_0}\varphi_0$ and its derivative are square integrable in t , this follows from a standard Sobolev embedding argument. So, the existence of the limits (4.1) has been established. Similar arguments, using the relation

$$(H_0 - \bar{z})^{-1}J^* - J^*(H - \bar{z})^{-1} = T_0(z)^*T(z)$$

instead of

$$J(H_0 - z)^{-1} - (H - z)^{-1}J = T(z)^*T_0(z),$$

show that $W_{\pm}(H_0, H, J^*, I)$ exists too. This, together with standard arguments in scattering theory, implies the claims that follow (4.1). \square

As a consequence of Theorems 3.1(a) and 4.4, we obtain the following criterion for the existence and completeness of the local wave operators.

Corollary 4.5. *Let H_0, H be self-adjoint operators in Hilbert spaces $\mathcal{H}_0, \mathcal{H}$ with spectral measures E^{H_0}, E^H and A_0, A self-adjoint operators in $\mathcal{H}_0, \mathcal{H}$. Assume that H_0, H are of class $C^{1+\varepsilon}(A_0), C^{1+\varepsilon}(A)$ for some $\varepsilon \in (0, 1)$. Let*

$$I := \{\tilde{\mu}^{A_0}(H_0) \setminus \sigma_p(H_0)\} \cap \{\tilde{\mu}^A(H) \setminus \sigma_p(H)\},$$

$J \in \mathcal{B}(\mathcal{H}_0, \mathcal{H})$, \mathcal{G} an auxiliary Hilbert space, and for each $z \in \mathbb{C} \setminus \mathbb{R}$ suppose there exist $T_0(z) \in \mathcal{B}(\mathcal{H}_0, \mathcal{G})$ and $T(z) \in \mathcal{B}(\mathcal{H}, \mathcal{G})$ with

$$J(H_0 - z)^{-1} - (H - z)^{-1}J = T(z)^*T_0(z) \tag{4.3}$$

and such that $T_0(z)$ extends continuously to an element of $\mathcal{B}(\mathcal{D}(\langle A_0 \rangle^s)^*, \mathcal{G})$ and $T(z)$ extends continuously to an element of $\mathcal{B}(\mathcal{D}(\langle A \rangle^s)^*, \mathcal{G})$ for some $s > 1/2$. Then the local wave operators

$$W_{\pm}(H, H_0, J, I) = \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} J e^{-itH_0} E^{H_0}(I)$$

exist, are J -complete on I , and satisfy the relations

$$W_{\pm}(H, H_0, J, I)^* = W_{\pm}(H_0, H, J^*, I) \quad \text{and}$$

$$W_{\pm}(H, H_0, J, I)\eta(H_0) = \eta(H)W_{\pm}(H, H_0, J, I)$$

for each bounded Borel function $\eta : \mathbb{R} \rightarrow \mathbb{C}$.

4.2. Scattering theory for one-dimensional coupled photonic crystals

In the case of the pair (M_0, M) , we obtain the following result on the existence and completeness of the wave operators; we use the notation E_{ac}^M for the orthogonal projection on the absolutely continuous subspace of M .

Theorem 4.6. *Let $I_{\text{max}} := \sigma(M_0) \setminus \{\mathcal{T}_M \cup \sigma_p(M)\}$. Then the local wave operators*

$$W_{\pm}(M, M_0, J, I_{\text{max}}) := \text{s-lim}_{t \rightarrow \pm\infty} e^{itM} J e^{-itM_0} E^{M_0}(I_{\text{max}})$$

exist and satisfy $\overline{\text{Ran}(W_{\pm}(M, M_0, J, I_{\text{max}}))} = E_{\text{ac}}^M \mathcal{H}_w$. In addition, the relations

$$W_{\pm}(M, M_0, J, I_{\text{max}})^* = W_{\pm}(M_0, M, J^*, I_{\text{max}})$$

and

$$W_{\pm}(M, M_0, J, I_{\text{max}})\eta(M_0) = \eta(M)W_{\pm}(M, M_0, J, I_{\text{max}})$$

hold for each bounded Borel function $\eta : \mathbb{R} \rightarrow \mathbb{C}$.

Proof. All the claims except the equality $\overline{\text{Ran}(W_{\pm}(M, M_0, J, I_{\text{max}}))} = E_{\text{ac}}^M \mathcal{H}_w$ follow from Corollary 4.5 whose assumptions are verified below.

Let $I \subset \sigma(M_0) \setminus \{\mathcal{T}_M \cup \sigma_p(M)\}$ be a compact interval. Then we know from Sec. 3.2 that M_0 is of class $C^2(A_{0,I})$ and from Sec. 3.4 that M is of class $C^{1+\varepsilon}(A_I)$ for some $\varepsilon \in (0, 1)$. Moreover, Theorems 3.3 and 3.9 imply that

$$I \subset \tilde{\mu}^{A_{0,I}}(M_0) \cap \{\tilde{\mu}^{A_I}(M) \setminus \sigma_p(M)\}.$$

Therefore, in order to apply Corollary 4.5, it is sufficient to prove that for any $z \in \mathbb{C} \setminus \mathbb{R}$ the operator

$$B(z) = J(M_0 - z)^{-1} - (M - z)^{-1}J$$

factorizes as a product of two locally smooth operators as in (4.3). For that purpose, we set $s := \frac{1+\tilde{\varepsilon}}{2}$ with $\tilde{\varepsilon} \in (0, \varepsilon)$, we define

$$\mathcal{D} := \{\mathcal{S}(\mathbb{R}, \mathbb{C}^2) \oplus \mathcal{S}(\mathbb{R}, \mathbb{C}^2)\} \times \mathcal{S}(\mathbb{R}, \mathbb{C}^2) \subset \mathcal{H}_0 \times \mathcal{H}_w,$$

and we consider the sesquilinear form

$$\mathcal{D} \ni ((\varphi_\ell, \varphi_r), \varphi) \mapsto \langle \langle Q \rangle^s \varphi, B(z)(\langle Q_\ell \rangle^s \varphi_\ell, \langle Q_r \rangle^s \varphi_r) \rangle_{\mathcal{H}_w} \in \mathbb{C}. \quad (4.4)$$

Our first goal is to show that this sesquilinear form is continuous for the topology of $\mathcal{H}_0 \times \mathcal{H}_w$. However, since the necessary computations are similar to the ones presented in Secs. 3.3 and 3.4, we only sketch them. We know from (3.9) that

$$\begin{aligned} B(z)(\langle Q_\ell \rangle^s \varphi_\ell, \langle Q_r \rangle^s \varphi_r) &= \sum_{\star \in \{\ell, r\}} \{((M_\star - z)^{-1} - (M - z)^{-1})j_\star \langle Q_\star \rangle^s \varphi_\star \\ &\quad + [j_\star, (M_\star - z)^{-1}] \langle Q_\star \rangle^s \varphi_\star\}. \end{aligned}$$

So, we have to establish the continuity of the sesquilinear forms

$$\begin{aligned} \mathcal{S}(\mathbb{R}, \mathbb{C}^2) \times \mathcal{S}(\mathbb{R}, \mathbb{C}^2) \ni (\varphi_\star, \varphi) \mapsto \langle \langle Q \rangle^s \varphi, ((M_\star - z)^{-1} - (M - z)^{-1})j_\star \\ \times \langle Q_\star \rangle^s \varphi_\star \rangle_{\mathcal{H}_w} \in \mathbb{C} \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} \mathcal{S}(\mathbb{R}, \mathbb{C}^2) \times \mathcal{S}(\mathbb{R}, \mathbb{C}^2) \ni (\varphi_\star, \varphi) \mapsto \langle \langle Q \rangle^s \varphi, [j_\star, (M_\star - z)^{-1}] \\ \times \langle Q_\star \rangle^s \varphi_\star \rangle_{\mathcal{H}_w} \in \mathbb{C}. \end{aligned} \quad (4.6)$$

For the first one, we know from (3.10) that

$$\begin{aligned} &((M_\star - z)^{-1} - (M - z)^{-1})j_\star \langle Q_\star \rangle^s \\ &= (M - z)^{-1}j_\star(w - w_\star)D(M_\star - z)^{-1} \langle Q_\star \rangle^s + (M - z)^{-1}(w - w_\star) \\ &\quad \times [D, j_\star](M_\star - z)^{-1} \langle Q_\star \rangle^s + (M - z)^{-1}(M - M_\star)[(M_\star - z)^{-1}, j_\star] \langle Q_\star \rangle^s. \end{aligned} \quad (4.7)$$

By inserting this expression into (4.5), by taking the $C^1(\langle Q \rangle^\alpha)$ -property of M and M_\star into account, and by observing that the operators $[D, \langle Q_\star \rangle^s]$, $\langle Q \rangle^s j_\star (w - w_\star) \langle Q_\star \rangle^s$ and $\langle Q \rangle^s [D, j_\star] \langle Q_\star \rangle^s$ defined on $\mathcal{S}(\mathbb{R}, \mathbb{C}^2)$ extend continuously to elements of $\mathcal{B}(\mathcal{H}_w)$, one obtains that the sesquilinear forms defined by the first two terms in (4.7) are continuous for the topology of $\mathcal{H}_{w_\star} \times \mathcal{H}_w$. The sesquilinear form defined by the third term in (4.7) and the sesquilinear form (4.6) can be treated simultaneously. Indeed, the factor $[j_\star, (M_\star - z)^{-1}]$ can be computed explicitly and contains a factor j'_\star which has compact support. Therefore, since $\langle Q \rangle^s j'_\star \langle Q_\star \rangle^s \in \mathcal{B}(\mathcal{H}_w)$, a few more commutator computations show that the two remaining sesquilinear forms are continuous for the topology of $\mathcal{H}_{w_\star} \times \mathcal{H}_w$.

In consequence, the sesquilinear form (4.4) is continuous for the topology of $\mathcal{H}_0 \times \mathcal{H}_w$, and thus corresponds to a bounded operator $F_z \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_w)$. Therefore, if we set

$$T_0(z) := \langle Q_\ell \rangle^{-s} \oplus \langle Q_r \rangle^{-s} \in \mathcal{B}(\mathcal{H}_0) \quad \text{and} \quad T(z) := F_z^* \langle Q \rangle^{-s} \in \mathcal{B}(\mathcal{H}_w, \mathcal{H}_0),$$

we obtain that $B(z) = T(z)^* T_0(z)$. On another hand, we know from computations presented in Sec. 3.4 that

$$\langle Q \rangle^{-s} \in \mathcal{B}(\mathcal{D}(\langle Q \rangle^s)^*, \mathcal{H}_w) \subset \mathcal{B}(\mathcal{D}(\langle A_I \rangle^s)^*, \mathcal{H}_w),$$

and

$$\langle Q_\ell \rangle^{-s} \oplus \langle Q_r \rangle^{-s} \in \mathcal{B}(\mathcal{D}(\langle Q_\ell \rangle^s \oplus \langle Q_r \rangle^s)^*, \mathcal{H}_0) \subset \mathcal{B}(\mathcal{D}(\langle A_{0,I} \rangle^s)^*, \mathcal{H}_0).$$

So, we have the inclusions

$$T(z) \in \mathcal{B}(\mathcal{D}(\langle A_I \rangle^s)^*, \mathcal{H}_0) \quad \text{and} \quad T_0(z) \in \mathcal{B}(\mathcal{D}(\langle A_{0,I} \rangle^s)^*, \mathcal{H}_0),$$

and thus all the assumptions of Corollary 4.5 are verified.

Hence, it only remains to show that $\overline{\text{Ran}(W_\pm(M, M_0, J, I_{\max}))} = E_{\text{ac}}^M \mathcal{H}_w$. For that purpose, we first recall from the proof of Theorem 3.9 that $E^M(I)(JJ^* - 1)E^M(I) \in \mathcal{K}(\mathcal{H}_w)$. Then since M has purely absolutely continuous spectrum in I one infers from the RAGE theorem that

$$\text{s-lim}_{t \rightarrow \pm\infty} E^M(I) e^{itM} (JJ^* - 1) e^{-itM} E^M(I) = 0,$$

and consequently that $\mathfrak{N}_\pm(M, J, I) = E^M(\mathbb{R} \setminus I) \mathcal{H}_w$. By using the J -completeness on I of the local wave operators and that M has purely absolutely continuous spectrum in I , we thus obtain

$$\begin{aligned} \overline{\text{Ran}(W_\pm(M, M_0, J, I))} &= E^M(I) \mathcal{H} \ominus \mathfrak{N}_\pm(M, J, I) = E^M(I) \mathcal{H}_w \\ &= E^M(I) \mathcal{H}_w \cap E_{\text{ac}}^M \mathcal{H}_w. \end{aligned}$$

By putting together these results for different intervals I and by using Proposition 3.10, we thus get that

$$\begin{aligned} \overline{\text{Ran}(W_\pm(M, M_0, J, I_{\max}))} &= E^M(I_{\max}) \mathcal{H}_w \cap E_{\text{ac}}^M \mathcal{H}_w \\ &= E^M(\sigma_{\text{ess}}(M)) \mathcal{H}_w \cap E_{\text{ac}}^M \mathcal{H}_w \\ &= E_{\text{ac}}^M \mathcal{H}_w, \end{aligned}$$

which concludes the proof. \square

Remark 4.7. Let $I \subset \sigma(M_0) \setminus \{\mathcal{T}_M \cup \sigma_p(M)\}$ be a compact interval and let $(\varphi_\ell, \varphi_r) \in \mathcal{H}_0$. Then we have

$$\begin{aligned} W_\pm(M, M_0, J, I)(\varphi_\ell, \varphi_r) &= \text{s-lim}_{t \rightarrow \pm\infty} e^{itM} J e^{-itM_0} E^{M_0}(I)(\varphi_\ell, \varphi_r) \\ &= \text{s-lim}_{t \rightarrow \pm\infty} e^{itM} (J_\ell e^{-itM_\ell} E^{M_\ell}(I) \varphi_\ell + J_r e^{-itM_r} E^{M_r}(I) \varphi_r) \\ &= W_\pm(M, M_\ell, J_\ell, I) \varphi_\ell + W_\pm(M, M_r, J_r, I) \varphi_r \end{aligned}$$

with

$$W_{\pm}(M, M_{\star}, J_{\star}, I) := \text{s-lim}_{t \rightarrow \pm\infty} e^{itM} J_{\star} e^{-itM_{\star}} E^{M_{\star}}(I) \quad (4.8)$$

and $J_{\star} \in \mathcal{B}(\mathcal{H}_{w_{\star}}, \mathcal{H}_w)$ given by

$$J_{\star} \varphi_{\star} := j_{\star} \varphi_{\star}, \quad \varphi_{\star} \in \mathcal{H}_{w_{\star}}. \quad (4.9)$$

That is, the operators $W_{\pm}(M, M_0, J, I)$ act as the sum of the local wave operators $W_{\pm}(M, M_{\star}, J_{\star}, I)$:

$$W_{\pm}(M, M_0, J, I)(\varphi_{\ell}, \varphi_r) = W_{\pm}(M, M_{\ell}, J_{\ell}, I)\varphi_{\ell} + W_{\pm}(M, M_r, J_r, I)\varphi_r.$$

In order to get a better understanding of the initial sets of the partial isometries $W_{\pm}(M, M_0, J, I_{\max})$ some preliminary considerations on the asymptotic velocity operators for M_{ℓ} and M_r are necessary. First, we define for each $\star \in \{\ell, r\}$ and $n \in \mathbb{N}$ the spaces

$$\mathcal{H}_{\star, n} := \widehat{\Pi}_{\star, n} \mathcal{H}_{\tau, \star} \quad \text{and} \quad \mathcal{H}_{\star, n}^{\infty} := \widehat{\Pi}_{\star, n} \{ \mathcal{H}_{\tau, \star} \cap C^{\infty}(\mathbb{R}, \mathfrak{h}_{\star}) \},$$

and note that $\mathcal{H}_{\tau, \star}$ decomposes into the internal direct sum $\mathcal{H}_{\tau, \star} = \bigoplus_{n \in \mathbb{N}} \widehat{\mathcal{H}}_{\star, n}$ and that the operator \widehat{M}_{\star} is reduced by this decomposition, namely, $\widehat{M}_{\star} = \sum_{n \in \mathbb{N}} \widehat{\lambda}_{\star, n} \widehat{\Pi}_{\star, n}$. Next, we introduce the self-adjoint operator \widehat{V}_{\star} in $\mathcal{H}_{\tau, \star}$

$$\widehat{V}_{\star} := \sum_{n \in \mathbb{N}} \widehat{\lambda}'_{\star, n} \widehat{\Pi}_{\star, n}, \quad \mathcal{D}(\widehat{V}_{\star}) := \left\{ u \in \mathcal{H}_{\tau, \star} \mid \|\widehat{V}_{\star} u\|_{\mathcal{H}_{\tau, \star}}^2 = \sum_{n \in \mathbb{N}} \|\widehat{\lambda}'_{\star, n} \widehat{\Pi}_{\star, n} u\|_{\mathcal{H}_{\tau, \star}}^2 < \infty \right\}.$$

Then it is natural to define the asymptotic velocity operator V_{\star} for M_{\star} in $\mathcal{H}_{w_{\star}}$ as

$$V_{\star} := \mathcal{U}_{\star}^{-1} \widehat{V}_{\star} \mathcal{U}_{\star}, \quad \mathcal{D}(V_{\star}) := \mathcal{U}_{\star}^{-1} \mathcal{D}(\widehat{V}_{\star}),$$

and the asymptotic velocity operator V_0 for M_0 in \mathcal{H}_0 as the direct sum

$$V_0 := V_{\ell} \oplus V_r.$$

Additionally, we define the family of self-adjoint operators in $\mathcal{H}_{w_{\star}}$

$$Q_{\star}(t) := e^{itM_{\star}} Q_{\star} e^{-itM_{\star}}, \quad t \in \mathbb{R}, \mathcal{D}(Q_{\star}(t)) := e^{itM_{\star}} \mathcal{D}(Q_{\star}),$$

and the corresponding family of self-adjoint operators in \mathcal{H}_0

$$Q_0(t) := Q_{\ell}(t) \oplus Q_r(t), \quad t \in \mathbb{R}.$$

Our next result is inspired by the result of [36, Theorem 4.1] in the setup of quantum walks. In the proof, we use the linear span $\mathcal{H}_{\star, \tau}^{\text{fin}}$ of elements of $\mathcal{H}_{\star, n}^{\infty}$:

$$\mathcal{H}_{\star, \tau}^{\text{fin}} := \left\{ \sum_{n \in \mathbb{N}} u_n \in \bigoplus_{n \in \mathbb{N}} \mathcal{H}_{\star, n}^{\infty} \mid u_n \neq 0 \text{ for only a finite number of } n \right\}.$$

Proposition 4.8. *For each $\star \in \{\ell, r\}$ and $z \in \mathbb{C} \setminus \mathbb{R}$, we have*

$$\text{s-lim}_{t \rightarrow \pm\infty} \left(\frac{Q_{\star}(t)}{t} - z \right)^{-1} = (V_{\star} - z)^{-1}.$$

Proof. For each $t \in \mathbb{R}$, we have the inclusion $\mathcal{U}_\star^{-1}\mathcal{H}_{\star,\tau}^{\text{fin}} \subset \{\mathcal{D}(V_\star) \cap \mathcal{D}(Q_\star(t))\}$. Furthermore, if $u \in \mathcal{H}_{\star,\tau}^{\text{fin}}$, then we have

$$(V_\star - z)^{-1}\mathcal{U}_\star^{-1}u = \mathcal{U}_\star^{-1}(\widehat{V}_\star - z)^{-1}u \in \mathcal{U}_\star^{-1}\mathcal{H}_{\star,\tau}^{\text{fin}}.$$

As a consequence, the following equality holds for all $t \in \mathbb{R} \setminus \{0\}$ and $u \in \mathcal{H}_{\star,\tau}^{\text{fin}}$:

$$\begin{aligned} \left(\left(\frac{Q_\star(t)}{t} - z \right)^{-1} - (V_\star - z)^{-1} \right) \mathcal{U}_\star^{-1}u &= \left(\frac{Q_\star(t)}{t} - z \right)^{-1} \left(V_\star - \frac{Q_\star(t)}{t} \right) \\ &\quad \times (V_\star - z)^{-1}\mathcal{U}_\star^{-1}u. \end{aligned}$$

Since $\|(\frac{Q_\star(t)}{t} - z)^{-1}\|_{\mathcal{B}(\mathcal{H}_{w_\star})} \leq |\text{Im}(z)|^{-1}$ and $\|(V_\star - z)^{-1}\|_{\mathcal{B}(\mathcal{H}_{w_\star})} \leq |\text{Im}(z)|^{-1}$, and since $\mathcal{U}_\star^{-1}\mathcal{H}_{\star,\tau}^{\text{fin}}$ is dense in \mathcal{H}_{w_\star} , it follows that it is sufficient to prove that

$$\lim_{t \rightarrow \pm\infty} \left\| \left(V_\star - \frac{Q_\star(t)}{t} \right) \varphi_\star \right\|_{\mathcal{H}_{w_\star}} = 0 \quad \text{for all } \varphi_\star \in \mathcal{U}_\star^{-1}\mathcal{H}_{\star,\tau}^{\text{fin}}.$$

Now, a direct calculation using the Bloch–Floquet transform gives for $\varphi_\star = \mathcal{U}_\star^{-1}u$ with $u \in \mathcal{H}_{\star,\tau}^{\text{fin}}$

$$\begin{aligned} &\left\| \left(V_\star - \frac{Q_\star(t)}{t} \right) \varphi_\star \right\|_{\mathcal{H}_{w_\star}}^2 \\ &= \int_{Y_\star^*} dk \left\| \sum_{n \in \mathbb{N}} (\widehat{\lambda}'_{\star,n} \widehat{\Pi}_{\star,n} u)(k) - \left(e^{it\widehat{M}_\star} \frac{\widehat{Q}_\star}{t} \sum_{n \in \mathbb{N}} e^{-it\widehat{\lambda}_{\star,n}} \widehat{\Pi}_{\star,n} u \right) (k) \right\|_{\mathfrak{h}_\star}^2 \\ &= \frac{1}{t^2} \int_{Y_\star^*} dk \left\| \sum_{n \in \mathbb{N}} (\widehat{Q}_\star \widehat{\Pi}_{\star,n} u)(k) \right\|_{\mathfrak{h}_\star}^2, \end{aligned}$$

where in the last equation we have used that \widehat{Q}_\star acts as $i\partial_k$ in $\mathcal{H}_{\tau,\star}$. Since $u \in \mathcal{H}_{\star,\tau}^{\text{fin}}$, the summation over $n \in \mathbb{N}$ is finite, and since the map $Y_\star^* \ni k \mapsto (\widehat{Q}_\star \widehat{\Pi}_{\star,n} u)(k) \in \mathfrak{h}_\star$ is bounded, one deduces that

$$\left\| \left(V_\star - \frac{Q_\star(t)}{t} \right) \varphi_\star \right\|_{\mathcal{H}_{w_\star}} = \mathcal{O}(t^{-1}),$$

which implies the claim. \square

In the following proposition, we determine the initial sets of the isometries $W_\pm(M, M_\star, J_\star, I) : \mathcal{H}_{w_\star} \rightarrow \mathcal{H}_w$ defined in (4.8). In the statement, we use the fact that the operators M_\star and V_\star strongly commute. We also use the notations χ_+ and χ_- for the characteristic functions of the intervals $(0, \infty)$ and $(-\infty, 0)$, respectively.

Proposition 4.9. (a) *Let $I \subset \sigma(M_\ell) \setminus \mathcal{T}_\ell$ be a compact interval, then the operators $W_\pm(M, M_\ell, J_\ell, I) : \mathcal{H}_{w_\ell} \rightarrow \mathcal{H}_w$ are partial isometries with initial sets $\chi_\mp(V_\ell)E^{M_\ell}(I)\mathcal{H}_{w_\ell}$.*

- (b) Let $I \subset \sigma(M_r) \setminus \mathcal{T}_r$ be a compact interval, then the operators $W_{\pm}(M, M_r, J_r, I) : \mathcal{H}_{w_r} \rightarrow \mathcal{H}_w$ are partial isometries with initial sets $\chi_{\pm}(V_r)E^{M_r}(I)\mathcal{H}_{w_r}$.

Before the proof, let us observe that if $I \subset \sigma(M_{\star}) \setminus \mathcal{T}_{\star}$ is a compact interval, then we have the equalities

$$\begin{aligned} \chi_{-}(V_{\star})E^{M_{\star}}(I) &= \chi_{(-\infty, 0]}(V_{\star})E^{M_{\star}}(I) \quad \text{and} \\ \chi_{+}(V_{\star})E^{M_{\star}}(I) &= \chi_{[0, +\infty)}(V_{\star})E^{M_{\star}}(I) \end{aligned} \tag{4.10}$$

due to the definition of the set \mathcal{T}_{\star} .

Proof. Our proof is inspired by the proof of [31, Proposition 3.4]. We first show the claim for $W_{+}(M, M_{\ell}, J_{\ell}, I)$. So, let $\varphi_{\ell} \in \mathcal{H}_{w_{\ell}}$. If $\varphi_{\ell} \perp E^{M_{\ell}}(I)\mathcal{H}_{w_{\ell}}$, then $\varphi_{\ell} \in \ker(W_{+}(M, M_{\ell}, J_{\ell}, I))$. Thus, we can assume that $\varphi_{\ell} \in E^{M_{\ell}}(I)\mathcal{H}_{w_{\ell}}$. Next, let us show that if $\varphi_{\ell} \in \chi_{+}(V_{\ell})\mathcal{H}_{w_{\ell}}$ then again $\varphi_{\ell} \in \ker(W_{+}(M, M_{\ell}, J_{\ell}, I))$. For this, assume that $\chi_{[\varepsilon, \infty)}(V_{\ell})\varphi_{\ell} = \varphi_{\ell}$ for some $\varepsilon > 0$. Then it follows from (4.8) and (4.9) that

$$\begin{aligned} \|W_{+}(M, M_{\ell}, J_{\ell}, I)\varphi_{\ell}\|_{\mathcal{H}_w} &= \|\text{s-lim}_{t \rightarrow +\infty} e^{itM} J_{\ell} e^{-itM_{\ell}} \varphi_{\ell}\|_{\mathcal{H}_w} \\ &= \lim_{t \rightarrow +\infty} \|e^{itM} J_{\ell} e^{-itM_{\ell}} \varphi_{\ell}\|_{\mathcal{H}_w} \\ &\leq \text{Const.} \lim_{t \rightarrow +\infty} \|e^{itM_{\ell}} j_{\ell} e^{-itM_{\ell}} \varphi_{\ell}\|_{\mathcal{H}_{w_{\ell}}}. \end{aligned}$$

Now, let $\eta_{\ell} \in C(\mathbb{R}, [0, 1])$ satisfy $\eta_{\ell}(x) = 1$ if $x < 0$ and $\eta_{\ell}(x) = 0$ if $x \geq \varepsilon$. Then one has for each $t > 0$ the inequality

$$\|e^{itM_{\ell}} j_{\ell} e^{-itM_{\ell}} \varphi_{\ell}\|_{\mathcal{H}_{w_{\ell}}} \leq \|e^{itM_{\ell}} \eta_{\ell}(Q_{\ell}/t) e^{-itM_{\ell}} \varphi_{\ell}\|_{\mathcal{H}_{w_{\ell}}}.$$

Furthermore, since $\eta_{\ell}(V_{\ell})\varphi_{\ell} = \eta_{\ell}(V_{\ell})\chi_{[\varepsilon, \infty)}(V_{\ell})\varphi_{\ell} = 0$, one infers from Proposition 4.8 and from a standard result on strong resolvent convergence [29, Theorem VIII.20(b)] that

$$\lim_{t \rightarrow +\infty} \|e^{itM_{\ell}} \eta_{\ell}(Q_{\ell}/t) e^{-itM_{\ell}} \varphi_{\ell}\|_{\mathcal{H}_{w_{\ell}}} = \|\eta_{\ell}(V_{\ell})\varphi_{\ell}\|_{\mathcal{H}_{w_{\ell}}} = 0.$$

Putting together what precedes, one obtains that $\varphi_{\ell} = \chi_{[\varepsilon, \infty)}(V_{\ell})\varphi_{\ell} \in \ker(W_{+}(M, M_{\ell}, J_{\ell}, I))$, and then a density argument taking into account the second equation in (4.10) implies that

$$\chi_{+}(V_{\ell})\mathcal{H}_{w_{\ell}} \subset \ker(W_{+}(M, M_{\ell}, J_{\ell}, I)).$$

To show that $W_{+}(M, M_{\ell}, J_{\ell}, I)$ is an isometry on $\chi_{-}(V_{\ell})E^{M_{\ell}}(I)\mathcal{H}_{w_{\ell}}$, take $\varphi_{\ell} \in \chi_{-}(V_{\ell})E^{M_{\ell}}(I)\mathcal{H}_{w_{\ell}}$ with $\chi_{(-\infty, -\varepsilon]}(V_{\ell})\varphi_{\ell} = \varphi_{\ell}$ for some $\varepsilon > 0$, and let $\zeta_{\ell} \in C(\mathbb{R}, [0, 1])$ satisfy $\zeta_{\ell}(x) = 0$ if $x \leq -\varepsilon$ and $\zeta_{\ell}(x) = 1$ if $x > -\varepsilon/2$. Then using successively the identity $E^{M_{\ell}}(I)\varphi_{\ell} = \varphi_{\ell}$, the unitarity of e^{itM} in \mathcal{H}_w and

of e^{-itM_ℓ} in \mathcal{H}_{w_ℓ} , the definition (4.9) of J_ℓ , the definition of V_ℓ , and the fact that $\chi_{(-\infty, -\varepsilon]}(V_\ell)\varphi_\ell = \varphi_\ell$, one gets

$$\begin{aligned}
 & \left| \|W_+(M, M_\ell, J_\ell, I)\varphi_\ell\|_{\mathcal{H}_w}^2 - \|\varphi_\ell\|_{\mathcal{H}_{w_\ell}}^2 \right| \\
 &= \lim_{t \rightarrow +\infty} \left| \|e^{itM} J_\ell e^{-itM_\ell} \varphi_\ell\|_{\mathcal{H}_w}^2 - \|\varphi_\ell\|_{\mathcal{H}_{w_\ell}}^2 \right| \\
 &= \lim_{t \rightarrow +\infty} \left| \|J_\ell e^{-itM_\ell} \varphi_\ell\|_{\mathcal{H}_w}^2 - \|e^{-itM_\ell} \varphi_\ell\|_{\mathcal{H}_{w_\ell}}^2 \right| \\
 &= \lim_{t \rightarrow +\infty} \left| \langle e^{-itM_\ell} \varphi_\ell, (1 - w_\ell w^{-1} j_\ell^2) e^{-itM_\ell} \varphi_\ell \rangle_{\mathcal{H}_{w_\ell}} \right| \\
 &\leq \lim_{t \rightarrow +\infty} \left| \langle \varphi_\ell, e^{itM_\ell} (1 - j_\ell^2) e^{-itM_\ell} \varphi_\ell \rangle_{\mathcal{H}_{w_\ell}} \right| \\
 &\quad + \lim_{t \rightarrow +\infty} \left| \langle \varphi_\ell, e^{itM_\ell} (w - w_\ell) j_\ell^2 w^{-1} e^{-itM_\ell} \varphi_\ell \rangle_{\mathcal{H}_{w_\ell}} \right|.
 \end{aligned}$$

For the first term one has

$$\begin{aligned}
 \lim_{t \rightarrow +\infty} \langle \varphi_\ell, e^{itM_\ell} (1 - j_\ell^2) e^{-itM_\ell} \varphi_\ell \rangle_{\mathcal{H}_{w_\ell}} &\leq \lim_{t \rightarrow +\infty} \langle \varphi_\ell, e^{itM_\ell} \zeta_\ell(Q_\ell/t) e^{-itM_\ell} \varphi_\ell \rangle_{\mathcal{H}_{w_\ell}} \\
 &= \langle \varphi_\ell, \zeta_\ell(V_\ell)\varphi_\ell \rangle_{\mathcal{H}_{w_\ell}} \\
 &= 0,
 \end{aligned}$$

while the second term also vanishes by an application of the RAGE theorem. It follows that $W_+(M, M_\ell, J_\ell, I)$ is isometric on $\varphi_\ell = \chi_{(-\infty, -\varepsilon]}(V_\ell)\varphi_\ell$, and then a density argument taking into account the first equation in (4.10) implies that $W_+(M, M_\ell, J_\ell, I)$ is isometric on $\chi_-(V_\ell)E^{M_\ell}(I)\mathcal{H}_{w_\ell}$.

A similar proof works for the claims about $W_-(M, M_\ell, J_\ell, I)$ and $W_\pm(M, M_r, J_r, I)$. The functions η_ℓ and ζ_ℓ have to be adapted and the possible negative sign of the variable t has to be taken into account, otherwise the argument can be copied *mutatis mutandis*. \square

By collecting the results of Theorem 4.6, Remark 4.7, Proposition 4.9, and by using the fact that M_ℓ and M_r have purely absolutely continuous spectrum, one finally obtains a description of the initial sets of the local wave operators $W_\pm(M, M_0, J, I_{\max})$.

Theorem 4.10. *Let $I_{\max} := \sigma(M_0) \setminus \{\mathcal{T}_M \cup \sigma_p(M)\}$ and $I_\star := \sigma(M_\star) \setminus \mathcal{T}_\star$ ($\star = \ell, r$). Then the local wave operators $W_\pm(M, M_0, J, I_{\max}) : \mathcal{H}_0 \rightarrow \mathcal{H}_w$ are partial isometries with initial sets*

$$\mathcal{H}_0^\pm := \chi_{\mp}(V_\ell)E^{M_\ell}(I_\ell)\mathcal{H}_{w_\ell} \oplus \chi_\pm(V_r)E^{M_r}(I_r)\mathcal{H}_{w_r}.$$

Remark 4.11. One has $E^{M_\star}(I_\star) = 1$ because \mathcal{T}_\star is discrete and $\sigma_p(M_\star) = \emptyset$ (see Proposition 2.8 and the paragraph that precedes). Therefore, the spectral projections $E^{M_\star}(I_\star)$ in the statement of Theorem 4.10 can be removed if desired.

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