JOURNAL OF MATHEMATICAL PHYSICS **58**, 102101 (2017)

Does Levinson's theorem count complex eigenvalues?

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(Received 17 January 2017; accepted 13 September 2017; published online 2 October 2017)

By considering a quantum-mechanical system with complex eigenvalues, we show that indeed Levinson's theorem extends to the non self-adjoint setting. The perturbed system corresponds to a realization of the Schrödinger operator with inverse square potential on the half-line, while the Dirichlet Laplacian on the half-line is chosen for the reference system. The resulting relation is an equality between the number of eigenvalues of the perturbed system and the winding number of the scattering system together with additional operators living at 0-energy and at infinite energy. *Published by AIP Publishing.* https://doi.org/10.1063/1.5004574

I. INTRODUCTION

Since its discovery by Levinson in 1949, the so-called *Levinson's theorem* has attracted a lot of interest, and many researchers have generalized this relation between spectral and scattering theories. In the simplest situations, it corresponds to an equality between the number of bound states of a quantum mechanical system and an expression based on the scattering part of the system. Very often, this latter expression involves a regularization procedure, and in many cases, some corrections must also be taken into account. Since the literature on the subject is very vast, we simply refer to the review papers[9,10] and to the references mentioned therein.

Up to our knowledge, all these investigations have taken place in the context of self-adjoint operators in a Hilbert space since this framework is the natural one for quantum mechanics. On the other hand, non-self-adjoint operators have recently been deeply studied, and in this context, an extension of Levinson's theorem to non-real eigenvalues seems a natural question. The purpose of this note is precisely to exhibit such a relation for a system involving non-self-adjoint operators with a finite family of complex eigenvalues.

The system we consider consists in the operator

$$H = -\partial_x^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{x^2}$$

in $L^2(\mathbb{R}_+)$ with a complex parameter m and with a possibly complex boundary condition at $x = 0$. A large family of such operators has been recently analyzed in Ref. 3, where their spectral and scattering theories have been constructed. For the reference system H_0, we simply consider the Dirichlet Laplacian on $\mathbb{R}_+$, but other choices are possible by using the chain rule. By particularizing some expressions obtained in Ref. 3, explicit formulas for the wave operators and for the scattering operator for the pair (H, H_0) are directly available. Note that if H is not self-adjoint, the wave operators are not partial isometries and the scattering operator is not a unitary operator. Nevertheless, these operators can be defined and their properties are studied.

Our approach for proving a Levinson's type theorem is based on the topological approach first introduced[7,8] and extensively reviewed in Ref. 10. In this framework, Levinson's theorem corresponds

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to an index theorem in scattering theory, and the corrections come from the contribution of newly introduced operators living at threshold energies. For the model presented here, these corrections appear both at 0-energy and at energy corresponding to $+\infty$. Accordingly, the number of eigenvalues of H will be equal to the winding number of the scattering operator together with the contributions of an operator living at 0-energy and of an operator living at ∞-energy. This relation and its explanation correspond to our main result.

This note is organized as follows: In Sec. II, we introduce the model and recall some relations obtained in Ref. 3. The main result is presented in Sec. III which also contains some additional information on the wave operators. Finally, the algebraic framework and the few analytical proofs are provided in Sec. IV.

II. THE MODEL

The material of this section is borrowed from Ref. 3. For any $m \in \mathbb{C}$, we consider the differential expression

$$L_{m^2} := -\partial_x^2 + (m^2 - \frac{1}{4})\frac{1}{x^2}\ .$$

The maximal and minimal operators associated with it in $L^2(\mathbb{R}_+)$ are given by $\mathcal{D}(L_{m^2}^{\max}) := \{f \in L^2(\mathbb{R}_+) \mid L_{m^2}f \in L^2(\mathbb{R}_+)\}$ and by $\mathcal{D}(L_{m^2}^{\min})$ which is the closure of the restriction of L_{m^2} to $C_c^\infty(\mathbb{R}_+)$. These realizations satisfy the relation $(L_{m^2}^{\min})^* = L_{\overline{m}^2}^{\max}$. In addition, if $|\mathrm{Re}(m)| < 1$, then $L_{m^2}^{\min} \subsetneq L_{m^2}^{\max}$, and $\mathcal{D}(L_{m^2}^{\min})$ is a closed subspace of codimension 2 of $\mathcal{D}(L_{m^2}^{\max})$. More precisely, if $|\mathrm{Re}(m)| \in (0,1)$ and if $f \in \mathcal{D}(L_{m^2}^{\max})$, then there exists $a, b \in \mathbb{C}$ such that

$$f(x) - a x^{1/2-m} - b x^{1/2+m} \in \mathcal{D}(L_{m^2}^{\min}) \text{ around } 0.$$

Here the expression $g(x) \in \mathcal{D}(L_{m^2}^{\min})$ *around* 0 means that there exists $\zeta \in C_c^\infty([0,\infty))$ with $\zeta = 1$ around 0 such that $g\zeta \in \mathcal{D}(L_{m^2}^{\min})$. Thus, for any $\kappa \in \mathbb{C}$, we define the family of operators $H_{m,\kappa}$ as

$$\begin{aligned}\mathcal{D}(H_{m,\kappa}) = \{f \in \mathcal{D}(L_{m^2}^{\max}) \mid \text{ for some } c \in \mathbb{C},\\ f(x) - c\,(\kappa x^{1/2-m} + x^{1/2+m}) \in \mathcal{D}(L_{m^2}^{\min}) \text{ around } 0\}.\end{aligned}$$

Note that for simplicity, the special case $\mathrm{Re}(m) = 0$ is not considered in the present manuscript.

Many properties of these operators have been exhibited in Ref. 3. For the spectral theory, let us simply mention that $H_{m,\kappa}$ is self-adjoint if and only if m and κ are real. The operator $H_{m,\kappa}$ has a finite number of eigenvalues which are located in $\mathbb{C} \setminus [0,\infty)$, and in addition, one has $[0,\infty) \subset \sigma(H_{m,\kappa})$. A limiting absorption principle has been shown for these operators on $(0,\infty)$, with a slight restriction if (m,κ) is an exceptional pair. We say that (m,κ) is *an exceptional pair* if $\kappa \neq 0$ and $\pm\pi \in \mathrm{Im}(\frac{1}{m}\mathrm{Ln}(\varsigma))$ with

$$\varsigma := \kappa\frac{\Gamma(-m)}{\Gamma(m)},$$

and where Γ is the usual Γ-function and Ln the multivalued logarithm.

Still in the non-exceptional case, an *incoming* and an *outgoing Hankel transformation* $\mathscr{F}_{m,\kappa}^{\mp}$ can be defined on $C_c^\infty(\mathbb{R}_+)$ by the kernels

$$\mathscr{F}_{m,\kappa}^{\mp}(x,y) := \mathrm{e}^{\mp \mathrm{i}\frac{\pi}{2}m}\sqrt{\frac{2}{\pi}}\,\frac{\mathcal{J}_m(xy) - \varsigma\,\mathcal{J}_{-m}(xy)\big(\frac{y^2}{4}\big)^m}{1 - \varsigma \mathrm{e}^{\mp\mathrm{i}\pi m}\big(\frac{y^2}{4}\big)^m}$$

with $\mathcal{J}_m(z) := \sqrt{\frac{\pi z}{2}}J_m(z)$ and J_m as the usual Bessel function. These operators extend then continuously to elements of $\mathcal{B}\big(L^2(\mathbb{R}_+)\big)$. Now, for any bounded operator B with integral kernel $B(x,y)$, let us set $B^\#$ for its transpose, i.e., for the operator satisfying $B^\#(x,y) = B(y,x)$. With this notation, the operators $\mathscr{F}_{m,\kappa}^{\pm}$ satisfy $\mathscr{F}_{m,\kappa}^{\pm\#}\mathscr{F}_{m,\kappa}^{\mp} = \mathbb{1}$. In addition, if we set

$$\mathbb{1}_{\mathbb{R}_+}(H_{m,\kappa}) := \mathscr{F}_{m,\kappa}^{+}\mathscr{F}_{m,\kappa}^{-\#} = \mathscr{F}_{m,\kappa}^{-}\mathscr{F}_{m,\kappa}^{+\#},$$

one observes that this operator is a projection. The following equalities have also been proved in Ref. 3: For any $k \in \mathbb{C}$ with $\mathrm{Re}(k) > 0$ and $-k^2 \notin \sigma_{\mathrm{p}}(H_{m,\kappa})$ one has

$$(H_{m,\kappa} + k^2)^{-1} \mathbb{1}_{\mathbb{R}_+}(H_{m,\kappa}) = \mathscr{F}^{\pm}_{m,\kappa}(L^2 + k^2)^{-1} \mathscr{F}^{\mp\#}_{m,\kappa} = \mathbb{1}_{\mathbb{R}_+}(H_{m,\kappa})(H_{m,\kappa} + k^2)^{-1},$$

where L is the usual operator of multiplication by the variable in $L^2(\mathbb{R}_+)$.

Finally, whenever (m, κ) and (m', κ') are not exceptional pairs, the wave operators for the pair of operators $(H_{m,\kappa}, H_{m',\kappa'})$ can be defined by

$$W^{\pm}_{m,\kappa;m',\kappa'} := \mathscr{F}^{\pm}_{m,\kappa} \mathscr{F}^{\mp\#}_{m',\kappa'}.$$

These operators satisfy the relations

$$W^{\mp\#}_{m,\kappa;m',\kappa'} W^{\pm}_{m,\kappa;m',\kappa'} = \mathbb{1}_{\mathbb{R}_+}(H_{m',\kappa'}) \tag{1}$$

and

$$W^{\pm}_{m,\kappa;m',\kappa'} W^{\mp\#}_{m,\kappa;m',\kappa'} = \mathbb{1}_{\mathbb{R}_+}(H_{m,\kappa}) \tag{2}$$

as well as the intertwining relation

$$W^{\pm}_{m,\kappa;m',\kappa'} H_{m',\kappa'} = H_{m,\kappa} W^{\pm}_{m,\kappa;m',\kappa'}.$$

The scattering operator is finally defined by

$$S_{m,\kappa;m',\kappa'} := W^{-\#}_{m,\kappa;m',\kappa'} W^{-}_{m,\kappa;m',\kappa'}.$$

III. THE MAIN RESULT

From now on, let us fix a pair (m, κ) with $|\mathrm{Re}(m)| \in (0, 1)$ and $\kappa \in \mathbb{C}$ which is not exceptional. For the reference system, we consider one of the simplest one, namely, the Dirichlet Laplacian H_{D} on the half-line. This operator is obtained from the general family for the indices $(\frac{1}{2}, 0)$. Note that by the chain-rule, any other pair (m', κ') can be easily used for the reference system.

In the following statement, we provide an alternative representation of the wave operator which is at the root of the algebraic framework presented in Ref. 10. For that purpose, note first that for the special choice $(\frac{1}{2}, 0)$, the transformation $\mathscr{F}^{+\#}_{\frac{1}{2},0} \equiv \mathscr{F}^{+}_{\frac{1}{2},0}$ corresponds to $\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} \mathscr{F}_{\mathrm{D}}$ with $\mathscr{F}_{\mathrm{D}}$, the usual Fourier sine transformation on $\mathbb{R}_+$. Let us also define for any $t \in \mathbb{R}$ the function Ξ_m given by

$$\Xi_m(t) := \mathrm{e}^{\mathrm{i}\ln(2)t} \frac{\Gamma(\frac{m+1+\mathrm{i}t}{2})}{\Gamma(\frac{m+1-\mathrm{i}t}{2})}.$$

We finally consider the unitary group $\{U_t\}_{t \in \mathbb{R}}$ acting on any $f \in L^2(\mathbb{R}_+)$ as

$$[U_t f](x) := \mathrm{e}^{t/2} f(\mathrm{e}^t x), \qquad \forall x \in \mathbb{R}_+,$$

which is usually called *the unitary group of dilations*. Its self-adjoint generator is denoted by A and is called *the generator of dilations*.

Lemma 1. If $m \in \mathbb{C}$ with $|\mathrm{Re}(m)| \in (0, 1)$ and if (m, κ) is not an exceptional pair then the operator $W^{-}_{m,\kappa;\frac{1}{2},0}$ is equal to

$$\mathrm{e}^{\mathrm{i}\frac{\pi}{4}} \Xi_{\frac{1}{2}}(A) \Big(\Xi_m(-A) - \varsigma \Xi_{-m}(-A) \big(\tfrac{H_{\mathrm{D}}}{4}\big)^m\Big) \frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}m}}{1 - \varsigma \mathrm{e}^{-\mathrm{i}\pi m} \big(\frac{H_{\mathrm{D}}}{4}\big)^m}. \tag{3}$$

Motivated by the formula obtained in the previous statement, let us now define the function of two variables: $\Gamma_{m,\kappa;\frac{1}{2},0} : \mathbb{R}_+ \times \mathbb{R}$ by

$$\Gamma_{m,\kappa;\frac{1}{2},0}(x, t) := \mathrm{e}^{\mathrm{i}\frac{\pi}{4}} \Xi_{\frac{1}{2}}(t) \Big(\Xi_m(-t) - \varsigma \Xi_{-m}(-t) \big(\tfrac{x^2}{4}\big)^m\Big) \frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}m}}{1 - \varsigma \mathrm{e}^{-\mathrm{i}\pi m} \big(\frac{x^2}{4}\big)^m}.$$

Note that the condition (m, κ) is not an exceptional pair precisely prevents the denominator in the last factor to vanish. In addition, it is easily observed that this function is continuous on the square $\blacksquare := [0,+\infty] \times [-\infty,+\infty]$, and therefore, its restriction on the boundary $\square$ of the square is also well defined and continuous. Note that this boundary is made of four parts: $\square = B_1 \cup B_2 \cup B_3 \cup B_4$ with $B_1 = \{0\} \times [-\infty,+\infty]$, $B_2 = [0,+\infty] \times \{+\infty\}$, $B_3 = \{+\infty\} \times [-\infty,+\infty]$, and $B_4 = [0,+\infty] \times \{-\infty\}$. Thus, the algebra $C(\square)$ of continuous functions on $\square$ can be viewed as a subalgebra of

$$C([-\infty,+\infty]) \oplus C([0,+\infty]) \oplus C([-\infty,+\infty]) \oplus C([0,+\infty]) \tag{4}$$

given by elements $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4)$ which coincide at the corresponding end points, that is, $\Gamma_1(+\infty) = \Gamma_2(0)$, $\Gamma_2(+\infty) = \Gamma_3(+\infty)$, $\Gamma_3(-\infty) = \Gamma_4(+\infty)$, and $\Gamma_4(0) = \Gamma_1(-\infty)$. As shown in Sec. IV, one gets for $\kappa \neq 0$

$$\Gamma_1(t) := \Gamma_{m,\kappa;\frac{1}{2},0}(0,t) = \begin{cases} e^{i\frac{\pi}{2}(\frac{1}{2}-m)}\Xi_{\frac{1}{2}}(t)\Xi_m(-t) & \text{if } \mathrm{Re}(m) > 0, \\ e^{i\frac{\pi}{2}(\frac{1}{2}+m)}\Xi_{\frac{1}{2}}(t)\Xi_{-m}(-t) & \text{if } \mathrm{Re}(m) < 0, \end{cases} \tag{5}$$

$$\Gamma_2(x) := \Gamma_{m,\kappa;\frac{1}{2},0}(x,+\infty) = e^{i\pi(\frac{1}{2}-m)}\frac{1-\varsigma e^{+i\pi m}\left(\frac{x^2}{4}\right)^m}{1-\varsigma e^{-i\pi m}\left(\frac{x^2}{4}\right)^m}, \tag{6}$$

$$\Gamma_3(t) := \Gamma_{m,\kappa;\frac{1}{2},0}(+\infty,t) = \begin{cases} e^{i\frac{\pi}{2}(\frac{1}{2}+m)}\Xi_{\frac{1}{2}}(t)\Xi_{-m}(-t) & \text{if } \mathrm{Re}(m) > 0, \\ e^{i\frac{\pi}{2}(\frac{1}{2}-m)}\Xi_{\frac{1}{2}}(t)\Xi_{m}(-t) & \text{if } \mathrm{Re}(m) < 0, \end{cases} \tag{7}$$

$$\Gamma_4(x) := \Gamma_{m,\kappa;\frac{1}{2},0}(x,-\infty) = 1. \tag{8}$$

In the special case, $\kappa = 0$, one has $\Gamma_1(t) = \Gamma_3(t) = e^{i\frac{\pi}{2}(\frac{1}{2}-m)}\Xi_{\frac{1}{2}}(t)\Xi_m(-t)$, $\Gamma_2(x) = e^{i\pi(\frac{1}{2}-m)}$, and $\Gamma_4(x) = 1$.

Let us now observe that the boundary $\square$ of $\blacksquare$ is homeomorphic to the circle $\mathbb{S}$. Observe in addition that the restriction $\Gamma^{\square}_{m,\kappa;\frac{1}{2},0}$ of the function $\Gamma_{m,\kappa;\frac{1}{2},0}$ to $\square$ takes its values in $\mathbb{C} \setminus \{0\}$. Then, since $\Gamma^{\square}_{m,\kappa;\frac{1}{2},0}$ is a continuous function on the closed curve $\square$ and takes non-zero values, its winding number $\mathrm{Wind}(\Gamma^{\square}_{m,\kappa;\frac{1}{2},0})$ is well defined. By convention, we shall turn around $\square$ clockwise and the increase in the winding number is also counted clockwise. Let us stress that the contribution on B_3 has to be computed from $+\infty$ to $-\infty$, and the contribution on B_4 from $+\infty$ to 0. Our main result now reads as follows.

Theorem 2. *If $m \in \mathbb{C}$ with $|\mathrm{Re}(m)| \in (0,1)$ and if (m, κ) is not an exceptional pair then*

$$\mathrm{Wind}(\Gamma^{\square}_{m,\kappa;\frac{1}{2},0}) = \textit{number of eigenvalues of } H_{m,\kappa}. \tag{9}$$

The above statement is a topological version of Levinson's theorem and corresponds to an index theorem. In order to make the link with the usual formulation, it is necessary to consider the 4 partial contributions Γ_j separately. First of all, we explain the meaning of *a partial contribution* with some simpler notations.

Let us identify the circle $\mathbb{S}$ with the interval $[0, 2\pi]$ and let us set $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$. We first consider a smooth function $\gamma : [0,2\pi] \to \mathbb{T}$ with $\gamma(0) = \gamma(2\pi)$. The winding number of this function can be computed by the expression $\mathrm{Wind}(\gamma) = -\frac{1}{2\pi i}\int_0^{2\pi}\gamma(\theta)^{-1}\gamma'(\theta)\mathrm{d}\theta$, with the minus sign coming from our conventions mentioned above. Clearly, if $a, b \in [0, 2\pi]$ with $a < b$, then $-\frac{1}{2\pi i}\int_a^b \gamma(\theta)^{-1}\gamma'(\theta)\mathrm{d}\theta$ is not an integer in general, but it provides the partial contribution to the total winding number of the function γ restricted on the interval $[a, b]$. If the function γ takes values in $\mathbb{C} \setminus \{0\}$, then the partial contribution can be computed either by considering the function $\gamma/|\gamma|$ in the previous computation or by keeping only the real part of $-\frac{1}{2\pi i}\int_a^b \gamma(\theta)^{-1}\gamma'(\theta)\mathrm{d}\theta$, and these two values coincide. This partial contribution of γ on the interval $[a, b]$ is denoted by $\mathrm{Wind}(\gamma|_{[a,b]})$.

Let us now come back to the function $\Gamma^{\square}_{m,\kappa;\frac{1}{2},0}$ defined on the square $\square$ and consider the restriction Γ_i of $\Gamma^{\square}_{m,\kappa;\frac{1}{2},0}$ to the part B_i of $\square$. Since $\Gamma_1(t) = \Gamma_{m,\kappa;\frac{1}{2},0}(0,t)$, the corresponding operator $\Gamma_1(A)$ can be understood as an operator related to the 0-energy of the scattering process. Similarly, $\Gamma_3(A)$ is an operator associated with the energy ∞ of the scattering process. On the other hand, expression (6) corresponds to the scattering operator, or more precisely one has $\Gamma_2(\sqrt{H_D}) = S_{m,\kappa;\frac{1}{2},0} \equiv S_{m,\kappa;\frac{1}{2},0}(H_D)$, where $S_{m,\kappa;\frac{1}{2},0}$ is the scattering operator for the pair $(H_{m,\kappa}, H_D)$, see Ref. 3, Sec. 6.5. Now, by a slight adaptation of the proof of Ref. 6, Lemma 4, the contribution to the winding number coming from Γ_1 and Γ_3 are, respectively, equal to $\frac{\mathrm{Re}(m)}{2} - \frac{1}{4}$ and $\frac{\mathrm{Re}(m)}{2} + \frac{1}{4}$ when $\mathrm{Re}(m) > 0$. Similarly, for $\mathrm{Re}(m) < 0$, these two contributions are respectively equal to $-\frac{\mathrm{Re}(m)}{2} - \frac{1}{4}$ and $-\frac{\mathrm{Re}(m)}{2} + \frac{1}{4}$. By collecting these information, one gets

Corollary 3. If $m \in \mathbb{C}$ with $|\mathrm{Re}(m)| \in (0,1)$, if $\kappa \neq 0$ and if (m,κ) is not an exceptional pair one has

$$\mathrm{Wind}\left(S_{m,\kappa;\frac{1}{2},0}(\cdot)\right) + |\mathrm{Re}(m)| = \textit{number of eigenvalues of } H_{m,\kappa}. \tag{10}$$

Note that the special case $\kappa = 0$ is less interesting since $S_{m,0;\frac{1}{2},0}$ is a constant and both sides of (9) in this case are equal to 0.

Remark 4. For general m and κ in the range mentioned above, the computation of the r.h.s. of (10) is rather involved and can take arbitrary large but finite values. Some useful information is provided in Ref. 3, Proposition 5.3, and Lemma 5.4. Equivalently, the computation of the winding number of the map $x \mapsto S_{m,\kappa;\frac{1}{2},0}(x)$ is fairly tricky for complex numbers m and κ. Anyway, we can directly see from Corollary 3 that the computation of this winding number does not provide enough information for deducing the number of bound states of $H_{m,\kappa}$, and that the correction $+|\mathrm{Re}(m)|$ can take arbitrary values in (0,1).

IV. THE PROOFS

Before introducing the necessary algebraic framework, let us check the analytical part of our investigations. More precisely let us check some properties of the wave operators and of the function $\Gamma_{m,\kappa;\frac{1}{2},0}$.

Proof of Lemma 1. We first look at the wave operator in the spectral representation of H_D, or more precisely let us consider the operator

$$\mathscr{F}_D^* W^-_{m,\kappa;\frac{1}{2},0} \mathscr{F}_D = \mathscr{F}_D^* \mathscr{F}^-_{m,\kappa} \mathscr{F}^{+\#}_{\frac{1}{2},0} \mathscr{F}_D = e^{i\frac{\pi}{4}} \mathscr{F}_D^* \mathscr{F}^-_{m,\kappa}$$

with

$$\mathscr{F}_D^* = \mathscr{F}_D = \Xi_{\frac{1}{2}}(-A)J \tag{11}$$

and

$$\mathscr{F}^-_{m,\kappa} = J\left(\Xi_m(A) - \varsigma\,\Xi_{-m}(A)\left(\tfrac{L^2}{4}\right)^m\right)\frac{e^{-i\frac{\pi}{2}m}}{1 - \varsigma e^{-i\pi m}\left(\frac{L^2}{4}\right)^m}\,. \tag{12}$$

The unitary and self-adjoint transformation $J : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is defined by the formula $(Jf)(x) = \frac{1}{x} f\left(\frac{1}{x}\right)$ for any $f \in L^2(\mathbb{R}_+)$ and $x \in \mathbb{R}_+$ Note that equalities (11) and (12) have been obtained in Ref. 3, Proposition 4.5 and Lemma 6.3. By taking into account the relations $\mathscr{F}_D^* A \mathscr{F}_D = -A$ and $\mathscr{F}_D^* L^2 \mathscr{F}_D = H_D$ one directly deduces formula (3). □

For showing that the function $\Gamma_{m,\kappa;\frac{1}{2},0}$ extends to the square $[0,+\infty] \times [-\infty,+\infty]$ let us compute its asymptotics. For the limits we recall from Ref. 3, Eq. (4.25) that

$$\Xi_{\frac{1}{2}}(\mp\infty)\Xi_{m'}(\pm\infty) = e^{\mp i\frac{\pi}{2}(\frac{1}{2} - m')}.$$

One thus infers that for $\kappa \neq 0$,

$$\Gamma_{m,\kappa;\frac{1}{2},0}(x,-\infty) = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\left(\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(\frac{1}{2}-m)} - \varsigma \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(\frac{1}{2}+m)}\left(\tfrac{x^2}{4}\right)^m\right)\frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}m}}{1-\varsigma\mathrm{e}^{-\mathrm{i}\pi m}\left(\frac{x^2}{4}\right)^m}$$
$$= \frac{1-\varsigma\mathrm{e}^{-\mathrm{i}\pi m}\left(\frac{x^2}{4}\right)^m}{1-\varsigma\mathrm{e}^{-\mathrm{i}\pi m}\left(\frac{x^2}{4}\right)^m}$$
$$= 1$$

while

$$\Gamma_{m,\kappa;\frac{1}{2},0}(x,+\infty) = \mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\left(\mathrm{e}^{+\mathrm{i}\frac{\pi}{2}(\frac{1}{2}-m)} - \varsigma \mathrm{e}^{+\mathrm{i}\frac{\pi}{2}(\frac{1}{2}+m)}\left(\tfrac{x^2}{4}\right)^m\right)\frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}m}}{1-\varsigma\mathrm{e}^{-\mathrm{i}\pi m}\left(\frac{x^2}{4}\right)^m}$$
$$= e^{\mathrm{i}\pi(\frac{1}{2}-m)}\frac{1-\varsigma\mathrm{e}^{+\mathrm{i}\pi m}\left(\frac{x^2}{4}\right)^m}{1-\varsigma\mathrm{e}^{-\mathrm{i}\pi m}\left(\frac{x^2}{4}\right)^m}.$$

For the other two limits, one gets if $\mathrm{Re}(m) > 0$

$$\Gamma_{m,\kappa;\frac{1}{2},0}(0,t) = \lim_{x\to 0}\Gamma_{m,\kappa;\frac{1}{2},0}(x,t) = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\frac{1}{2}-m)}\Xi_{\frac{1}{2}}(t)\Xi_m(-t),$$

while if $\mathrm{Re}(m) < 0$ one has $\Gamma_{m,\kappa;\frac{1}{2},0}(0,t) = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\frac{1}{2}+m)}\Xi_{\frac{1}{2}}(t)\Xi_{-m}(-t)$. On the other hand, if $\mathrm{Re}(m) > 0$ one gets

$$\Gamma_{m,\kappa;\frac{1}{2},0}(+\infty,t) = \lim_{x\to+\infty}\Gamma_{m,\kappa;\frac{1}{2},0}(x,t) = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\frac{1}{2}+m)}\Xi_{\frac{1}{2}}(t)\Xi_{-m}(-t)$$

while if $\mathrm{Re}(m) < 0$ one has $\Gamma_{m,\kappa;\frac{1}{2},0}(+\infty,t) = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(\frac{1}{2}-m)}\Xi_{\frac{1}{2}}(t)\Xi_m(-t)$.

The above expressions are summarized in Eqs. (5)–(8). The case $\kappa = 0$ can be obtained similarly and is simpler.

Let us now discuss the properties of the kernel and of the range of the wave operator $W^-_{m,\kappa;\frac{1}{2},0}$. By taking into account (Ref. 5, Theorem III.6.17) on the separation of the spectrum into two components, and by using an alternative definition of the projection $\mathbb{1}_{\mathbb{R}_+}(H_{m,\kappa})$ in terms of the spectral density integrated on $\mathbb{R}_+$, as provided in Ref. 3, Sec. 6.4, one infers that the subspace defined by $\mathbb{1}_{\mathbb{R}_+}(H_{m,\kappa})$ is complementary to the subspace generated by the eigenfunctions of the operator $H_{m,\kappa}$. Since these eigenvalues are always in finite number, the codimension of $\mathbb{1}_{\mathbb{R}_+}(H_{m,\kappa})L^2(\mathbb{R}_+)$ is always finite. Similarly, since H_{D} has no eigenvalue one gets $\mathbb{1}_{\mathbb{R}_+}(H_{\mathrm{D}}) = \mathbb{1}$. Then, by the properties (1) and (2), one infers that $W^-_{m,\kappa;\frac{1}{2},0}$ is a Fredholm operator. Summing up these information one gets:

Lemma 5. Let $m \in \mathbb{C}$ with $|\mathrm{Re}(m)| \in (0,1)$ and assume that (m,κ) is not an exceptional pair. Then $W^-_{m,\kappa;\frac{1}{2},0}$ is a Fredholm operator with

$$\dim\ker\left(W^-_{m,\kappa;\frac{1}{2},0}\right) - \dim\operatorname{coker}\left(W^-_{m,\kappa;\frac{1}{2},0}\right) = -\#\sigma_{\mathrm{p}}(H_{m,\kappa}).$$

The content of Theorem 2 can now be deduced either from Ref. 10, Theorem 4.4 (with $n = -1$) or from Ref. 2, Theorem 3. Let us however make some comments. In Ref. 10 statements similar to Theorem 2 are provided for various models related to quantum mechanics. The main difference with these models is that here the operator $H_{m,\kappa}$ is not always self-adjoint. It follows that the wave operator $W^-_{m,\kappa;\frac{1}{2},0}$ is not always an isometry but only a Fredholm operator, and accordingly the operator $\Gamma^\square_{m,\kappa;\frac{1}{2},0}$ is not always a unitary operator but is still an invertible operator. However, for the algebraic construction used for obtaining a topological version of Levinson's theorem, these differences are perfectly manageable and the construction works in this case as well, see for example Ref. 11, Rem. 8.1.7.

The main ingredient for the algebraic construction consists first in exhibiting a natural subalgebra of $\mathcal{B}(L^2(\mathbb{R}_+))$ which contains the wave operator $W^-_{m,\kappa;\frac{1}{2},0}$. By looking at the special representation obtained for the wave operator in (3) one deduces that $W^-_{m,\kappa;\frac{1}{2},0}$ belongs to the C^*-algebra $\mathcal{E}_{(H_{\mathrm{D}},A)}$ introduced in Ref. 10, Sec. 4.4 and inspired from the earlier paper.[4] This C^*-algebra is generated by product of the form $\psi(H_D)\eta(A)$ with $\psi \in C([0,\infty])$ and $\eta \in C([-\infty,\infty])$. Our interest in this algebra comes from its easily understandable quotient through the ideal $\mathcal{K}(L^2(\mathbb{R}_+))$ of compact operators on $L^2(\mathbb{R}_+)$. In fact, this quotient is isomorphic to the C^*-algebra $C(\square)$ which can be viewed as a subalgebra of the one introduced in (4). Thus, if we set $q: \mathcal{E}_{(H_D,A)} \to \mathcal{E}_{(H_D,A)}/\mathcal{K}(L^2(\mathbb{R}_+))$ for the quotient map, one gets that $q(W^-_{m,\kappa;\frac{1}{2},0}) = \Gamma^\square_{m,\kappa;\frac{1}{2},0} = (\Gamma_1,\Gamma_2,\Gamma_3,\Gamma_4)$ with Γ_j introduced in (5)–(8). The final argument consists in borrowing the information from Ref. 10, Theorem 4.4, that the winding number of $\Gamma^\square_{m,\kappa;\frac{1}{2},0}$ is equal to (minus) the Fredholm index of the operator $W^-_{m,\kappa;\frac{1}{2},0}$. By using Lemma 5 this leads directly to Theorem 2.

Alternatively, one can directly use the content of Theorem 3 of Ref. 2 once it is observed that the algebra presented in that paper corresponds to the algebra $\mathcal{E}_{(L,A)}$ introduced in Ref. 10, Sec. 4.4. This algebra is isomorphic to the C^*-algebra $\mathcal{E}_{(H_D,A)}$ by a conjugation with the unitary map $\mathscr{F}_D$. The difference of a minus sign between the content of Ref. 2, Theorem 3 and Theorem 2 comes from the equality $\mathscr{F}_D^* A \mathscr{F}_D = -A$ which reverses part of the construction. Let us also mention Theorem 4 in Ref. 1 which provides a similar abstract result but in a larger setting.

Remark 6. A more analytical approach involving Jost function could also be used for proving Corollary 3. However, the flavor of an index theorem would be lost, and the contributions of the operators at 0*-energy and at energy* $+\infty$ *would not appear so explicitly.*

ACKNOWLEDGMENTS

Supported by JSPS Grant-in-Aid for Young Scientists A No. 26707005.

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