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Spectral Analysis for Adjacency Operators on Graphs

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Abstract. We put into evidence graphs with adjacency operator whose singular subspace is prescribed by the kernel of an auxiliary operator. In particular, for a family of graphs called admissible, the singular continuous spectrum is absent and there is at most an eigenvalue located at the origin. Among other examples, the one-dimensional XY model of solid-state physics is covered. The proofs rely on commutators methods.

1. Introduction

Let $(X, \sim)$ be a graph. We write $x \sim y$ whenever the vertices (points) x and y of X are connected. For simplicity, we do not allow multiple edges or loops. In the Hilbert space $\mathcal{H} := \ell^2(X)$ we consider the *adjacency operator*

$$(Hf)(x) := \sum_{y \sim x} f(y), \quad f \in \mathcal{H}, \quad x \in X.$$

We denote by $\deg(x) := \#\{y \in X : y \sim x\}$ the degree of the vertex x. Under the assumption that $\deg(X) := \sup_{x \in X} \deg(x)$ is finite, H is a bounded selfadjoint operator in $\mathcal{H}$. We are interested in the nature of its spectral measure. Useful sources concerning operators acting on graphs are [3, 21, 22], see also the references therein.

Rather few adjacency operators on graphs are known to have purely absolutely continuous spectrum. This occurs for the lattice $\mathbb{Z}^n$ and for homogeneous trees. These and several other examples are presented briefly in [22]. Adjacency operators may also have non-void singular spectrum. In [26] the author exhibits families of ladder-type graphs for which the existence of singular continuous spectrum is generic. Percolation graphs with highly probable dense pure point spectrum are presented in [28], see also [15] and [6] for earlier works. Even Cayley graphs of infinite discrete groups can have adjacency operators with dense pure point spectrum, cf. [13] and [9].

In the sequel we use commutator methods to study the nature of the spectrum of adjacency operators. Mourre theory [2, 23], already applied to operators on trees [1, 11], may be a well-fitted tool, but it is not easy to use it in non-trivial situations. We use a simpler commutator method, introduced in [4, 5] and called "the method of the weakly conjugate operator". It is an unbounded version of the Kato–Putnam theorem [24], which will be presented briefly in Section 2.

The method of the weakly conjugate operator provides estimates on the behaviour of the resolvent $(H - z)^{-1}$ when z approaches the spectrum of H. These estimates are global, i.e., uniform in $\mathsf{Re}(z)$. They imply precise spectral properties for H. For the convenience of the reader, we are going to state now spectral results only in the particular case of "admissible graphs" introduced in Section 5. The general results, including boundary estimates for the resolvent and perturbations, are stated in Section 3 and proved in Section 4.

The notion of admissibility requires (among other things) the graph to be directed. Thus the family of neighbours $N(x) := \{y \in X : y \sim x\}$ is divided into two disjoint sets $N^-(x)$ (fathers) and $N^+(x)$ (sons), $N(x) = N^-(x) \sqcup N^+(x)$. We write $y < x$ if $y \in N^-(x)$ and $x < y$ if $y \in N^+(x)$. On drawings, we set an arrow from y to x ($x \leftarrow y$) if $x < y$, and say that the edge from y to x is positively oriented.

We assume that the subjacent directed graph, from now on denoted by $(X, <)$, is *admissible* with respect to these decompositions, i.e., (i) it admits a position function and (ii) it is uniform. A *position function* is a function $\Phi : X \to \mathbb{Z}$ such that $\Phi(y)+1 = \Phi(x)$ whenever $y < x$. It is easy to see that it exists if and only if all paths between two points have the same index (which is the difference between the number of positively and negatively oriented edges). Position functions and the number operator from [11, Section 2] present some common features. The directed graph $(X, <)$ is called *uniform* if for any $x, y \in X$ the number $\#[N^-(x) \cap N^-(y)]$ of common fathers of x and y equals the number $\#[N^+(x) \cap N^+(y)]$ of common sons of x and y. Thus the admissibility of a directed graph is an explicit property that can be checked directly, without making any choice. The graph $(X, \sim)$ is *admissible* if there exists an admissible directed graph subjacent to it.

Theorem 1.1. *The adjacency operator of an admissible graph $(X, \sim)$ is purely absolutely continuous, except at the origin, where it may have an eigenvalue with eigenspace*

$$\ker(H) = \left\{ f \in \mathcal{H} : \sum_{y<x} f(y) = 0 = \sum_{y>x} f(y) \quad \textit{for each } x \in X \right\} . \tag{1.1}$$

Theorem 3.3, which is more general, relies on the existence of a *function adapted to the graph*, a concept generalizing that of a position function. Examples of periodic graphs, both admissible and non-admissible, are presented in Section 6. It is explained that periodicity does not lead automatically to absolute continuity, especially (but not only) if the number of orbits is infinite, which actually occurs for some of our examples. In Section 7 we treat D-products of graphs. We show that adapted functions of the components can be added to form an adapted function of the more complicated D-product graph. Cayley graphs of non-Abelian, discrete groups can also be approached by the methods of the present article; we intend to treat this topic in an extended framework in a subsequent publication.

Our initial motivation in studying the nature of the spectrum of operators on graphs comes from spin models on lattices. We refer to [8] for some results on the essential spectrum and localization properties for the one-dimensional Heisenberg model and for more general Toeplitz-like operators. In the final section of the present article we show that our spectral analysis applies to the one-dimensional XY model (see Corollary 8.4 and Remark 8.5). This seems interesting, since it consists in showing that the non-trivial graph naturally associated with the XY Hamiltonian is admissible.

However it should be noted that Professor Colin de Verdière [7] has kindly informed us of an independent proof of the absolute continuity of the spectral measure for that model.

2. The method of the weakly conjugate operator

In this section we recall the basic characteristics of the method of the weakly conjugate operator. It was introduced and applied to partial differential operators in [4,5]. Several developments and applications may be found in [14,19,20,25]. The method works for unbounded operators, but for our purposes it will be enough to assume H bounded.

We start by introducing some notations. The symbol $\mathcal{H}$ stands for a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Given two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote by $\mathscr{B}(\mathcal{H}_1,\mathcal{H}_2)$ the set of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, and put $\mathscr{B}(\mathcal{H}) := \mathscr{B}(\mathcal{H},\mathcal{H})$. We assume that $\mathcal{H}$ is endowed with a strongly continuous unitary group $\{W_t\}_{t\in\mathbb{R}}$. Its selfadjoint generator is denoted by A and has domain $D(A)$. In most of the applications A is unbounded.

Definition 2.1. A bounded selfadjoint operator H in $\mathcal{H}$ belongs to $C^1(A;\mathcal{H})$ if one of the following equivalent condition is satisfied:

(i) the map $\mathbb{R}\ni t\mapsto W_{-t}HW_t\in\mathscr{B}(\mathcal{H})$ is strongly differentiable,

(ii) the sesquilinear form

$$D(A)\times D(A)\ni(f,g)\mapsto i\,\langle Hf,Ag\rangle - i\,\langle Af,Hg\rangle\in\mathbb{C}$$

is continuous when $D(A)$ is endowed with the topology of $\mathcal{H}$.

We denote by B the strong derivative in (i), or equivalently the bounded selfadjoint operator associated with the extension of the form in (ii). The operator B provides a rigorous meaning to the commutator $i[H,A]$. We shall write $B>0$ if B is positive and injective, namely if $\langle f,Bf\rangle>0$ for all $f\in\mathcal{H}\setminus\{0\}$.

Definition 2.2. The operator A is *weakly conjugate to* the bounded selfadjoint operator H if $H\in C^1(A;\mathcal{H})$ and $B\equiv i[H,A]>0$.

For $B > 0$ let us consider the completion $\mathcal{B}$ of $\mathcal{H}$ with respect to the norm $\|f\|_{\mathcal{B}} := \langle f, Bf \rangle^{1/2}$. The adjoint space $\mathcal{B}^*$ of $\mathcal{B}$ can be identified with the completion of $B\mathcal{H}$ with respect to the norm $\|g\|_{\mathcal{B}^*} := \langle g, B^{-1}g \rangle^{1/2}$. One has then the continuous dense embeddings $\mathcal{B}^* \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{B}$, and B extends to an isometric operator from $\mathcal{B}$ to $\mathcal{B}^*$. Due to these embeddings it makes sense to assume that $\{W_t\}_{t\in\mathbb{R}}$ restricts to a C_0-group in $\mathcal{B}^*$, or equivalently that it extends to a C_0-group in $\mathcal{B}$. Under this assumption (tacitly assumed in the sequel) we keep the same notation for these C_0-groups. The domain of the generator of the C_0-group in $\mathcal{B}$ (resp. $\mathcal{B}^*$) endowed with the graph norm is denoted by $D(A,\mathcal{B})$ (resp. $D(A,\mathcal{B}^*)$). In analogy with Definition 2.1 the requirement $B \in C^1(A;\mathcal{B},\mathcal{B}^*)$ means that the map $\mathbb{R} \ni t \mapsto W_{-t}BW_t \in \mathscr{B}(\mathcal{B},\mathcal{B}^*)$ is strongly differentiable, or equivalently that the sesquilinear form

$$D(A,\mathcal{B}) \times D(A,\mathcal{B}) \ni (f,g) \mapsto i\,\langle f, BAg \rangle - i\,\langle Af, Bg \rangle \in \mathbb{C}$$

is continuous when $D(A,\mathcal{B})$ is endowed with the topology of $\mathcal{B}$. Here, $\langle \cdot, \cdot \rangle$ denotes the duality between $\mathcal{B}$ and $\mathcal{B}^*$. Finally let $\mathcal{E}$ be the Banach space $(D(A,\mathcal{B}^*),\mathcal{B}^*)_{1/2,1}$ defined by real interpolation (see for example [2, Proposition 2.7.3]). One has then the natural continuous embeddings $\mathscr{B}(\mathcal{H}) \subset \mathscr{B}(\mathcal{B}^*,\mathcal{B}) \subset \mathscr{B}(\mathcal{E},\mathcal{E}^*)$ and the following results [5, Theorem 2.1]:

Theorem 2.3. *Assume that A is weakly conjugate to H and that $B \equiv i[H,A]$ belongs to $C^1(A;\mathcal{B},\mathcal{B}^*)$. Then there exists a constant $\mathrm{C} > 0$ such that*

$$\left|\left\langle f, (H - \lambda \mp i\mu)^{-1} f \right\rangle\right| \le \mathrm{C}\|f\|_{\mathcal{E}}^2 \tag{2.1}$$

for all $\lambda \in \mathbb{R}$, $\mu > 0$ and $f \in \mathcal{E}$. In particular the spectrum of H is purely absolutely continuous.

For readers not accustomed with real interpolation or with the results of [2], we mention that one can replace $\|f\|_{\mathcal{E}}$ by $\|f\|_{D(A,\mathcal{B}^*)}$ in formula (2.1), loosing part of its strength. In the applications it may even be useful to consider smaller, but more explicit, Banach spaces $\mathcal{F}$ continuously and densely embedded in $D(A,\mathcal{B}^*)$. In such a setting we state a corollary of Theorem 2.3, which follows by applying the theory of smooth operators [4, 24]. The adjoint space of $\mathcal{F}$ is denoted by $\mathcal{F}^*$.

Corollary 2.4.

(a) *If T belongs to $\mathscr{B}(\mathcal{F}^*,\mathcal{H})$, then T is an H-smooth operator.*
(b) *Let U be a bounded selfadjoint operator in $\mathcal{H}$ such that $|U|^{1/2}$ extends to an element of $\mathscr{B}(\mathcal{F}^*,\mathcal{H})$. For $\gamma \in \mathbb{R}$, let $H_\gamma := H + \gamma U$. Then there exists $\gamma_0 > 0$ such that for $\gamma \in (-\gamma_0,\gamma_0)$, $H_\gamma := H + \gamma U$ is purely absolutely continuous and unitarily equivalent to H through the wave operators $\Omega_\gamma^\pm :=$ $\text{s-}\lim_{t\to\pm\infty} \mathrm{e}^{itH_\gamma}\,\mathrm{e}^{-itH}$.*

3. Statement of the main result

Some preliminaries on graphs could be convenient, since notations and conventions do not seem commonly accepted in graph theory.

A *graph* is a couple $(X, \sim)$ formed of a non-void countable set X and a symmetric relation $\sim$ on X such that $x \sim y$ implies $x \neq y$. The points $x \in X$ are called *vertices* and couples $(x, y) \in X \times X$ such that $x \sim y$ are called *edges*. So, for simplicity, multiple edges and loops are forbidden in our definition of a graph. Occasionally $(X, \sim)$ is said to be a *simple* graph.

For any $x \in X$ we denote by $N(x) := \{y \in X : y \sim x\}$ the set of *neighbours* of x. We write $\deg(x) := \#N(x)$ for the *degree* or *valence* of the vertex x and $\deg(X) := \sup_{x\in X} \deg(x)$ for the degree of the graph. We also suppose that $(X, \sim)$ is *uniformly locally finite*, i.e., that $\deg(X) < \infty$. When the function $x \mapsto \deg(x)$ is constant, we say that the graph is *regular*.

A *path* from x to y is a sequence $p = (x_0, x_1, \ldots, x_n)$ of elements of X, usually denoted by $x_0x_1 \ldots x_n$, such that $x_0 = x$, $x_n = y$ and $x_{j-1} \sim x_j$ for each $j \in \{1, \ldots, n\}$. The *length* of the path p is the number n. If $x_0 = x_n$ we say that the path is *closed*. A graph is *connected* if there exists a path connecting any two vertices x and y. On any connected graph $(X, \sim)$ one may define the *distance function* d as follows: $d(x, x) := 0$ and $d(x, y)$ is equal to the length of the shortest path from x to y if $x \neq y$.

Throughout the paper we restrict ourselves tacitly to graphs $(X, \sim)$ which are simple, infinite countable and uniformly locally finite. Given such a graph we consider the *adjacency operator* H acting in the Hilbert space $\mathcal{H} := \ell^2(X)$ as

$$(Hf)(x) := \sum_{y\sim x} f(y)\,, \quad f \in \mathcal{H}\,, \quad x \in X\,.$$

Due to [22, Theorem 3.1], H is a bounded selfadjoint operator with $\|H\| \leq \deg(X)$ and spectrum $\sigma(H) \subset [-\deg(X), \deg(X)]$. If $(X, \sim)$ is not connected, H can be written as a direct sum in an obvious manner and each component can be treated separately. Most of the time $(X, \sim)$ will be assumed to be connected.

For further use, we also sketch some properties of a larger class of operators. Any element of $\mathscr{B}[\ell^2(X)]$ is an "integral" operator of the form $(I_af)(x) = \sum_{y\in X} a(x, y)f(y)$ for some matrix $a \equiv \{a(x, y)\}_{x,y\in X}$. Formally I_a is symmetric if and only if a is symmetric, i.e., $\overline{a(x, y)} = a(y, x)$, and I_a, I_b satisfy the multiplication rule $I_aI_b = I_{a\circ b}$ with $(a \circ b)(x, y) := \sum_{z\in X} a(x, z)b(z, y)$. A bound on the norm of I_a is given by the relation

$$\|I_a\| \leq \max\left\{\sup_{x\in X} \sum_{y\in X} |a(x, y)|\,,\ \sup_{y\in X} \sum_{x\in X} |a(x, y)|\right\}. \tag{3.1}$$

In the sequel we shall encounter only matrices $a \in \ell^\infty(X \times X)$ such that there exists a positive integer k with $\max\{\#[\operatorname{supp} a(x, \cdot)], \#[\operatorname{supp} a(\cdot, x)]\} \leq k$ for all $x \in X$. Then an easy calculation using formula (3.1) gives $\|I_a\| \leq k\,\|a\|_\infty$.

In particular we call *local* an operator I_a for which $a(x,y) \neq 0$ only if $x \sim y$. In this case, if a is symmetric and bounded, then I_a is selfadjoint and bounded, with $\|I_a\| \leq \deg(X)\, \|a\|_\infty$.

The methods of this article apply to the latter class of operators (commutator calculations involve operators I_a which are not local, but bounded since they satisfy $a(x,y) = 0$ if $d(x,y) \geq 3$). However we refrained from treating more general objects than adjacency operators for simplicity and because we have nothing remarkable to say about the general case.

We now introduce the key concept. Sums over the empty set are zero by convention.

Definition 3.1. A function $\Phi : X \to \mathbb{R}$ is *semi-adapted* to the graph $(X, \sim)$ if

(i) there exists $\mathrm{c} \geq 0$ such that $|\Phi(x) - \Phi(y)| \leq \mathrm{c}$ for all $x, y \in X$ with $x \sim y$,

(ii) for any $x, y \in X$ one has

$$\sum_{z \in N(x) \cap N(y)} \big[2\Phi(z) - \Phi(x) - \Phi(y)\big] = 0 \,. \tag{3.2}$$

If in addition for any $x, y \in X$ one has

$$\sum_{z \in N(x) \cap N(y)} \big[\Phi(z) - \Phi(x)\big]\big[\Phi(z) - \Phi(y)\big]\big[2\Phi(z) - \Phi(x) - \Phi(y)\big] = 0 \,, \tag{3.3}$$

then Φ is *adapted* to the graph $(X, \sim)$.

Let $M_Z(\Phi)$ be the mean of the function Φ over a finite subset Z of X, i.e., $M_Z(\Phi) := (\#Z)^{-1} \sum_{z \in Z} \Phi(z)$. One may then rephrase condition (3.2) as

$$M_{\{x,y\}}(\Phi) = M_{N(x) \cap N(y)}(\Phi) \quad \text{for any} \quad x, y \in X \,.$$

In particular, if $x = y$, one simply has to check that $\Phi(x) = [\deg(x)]^{-1} \sum_{y \sim x} \Phi(y)$ for all $x \in X$.

In order to formulate the main result we need a few more definitions. For a function Φ semi-adapted to the graph $(X, \sim)$ we consider in $\mathcal{H}$ the operator K given by

$$(Kf)(x) := i \sum_{y \sim x} \big[\Phi(y) - \Phi(x)\big] f(y) \,, \quad f \in \mathcal{H} \,, \quad x \in X \,.$$

The operator K is selfadjoint and bounded due to the condition (i) of Definition 3.1 and the discussion preceding it. It commutes with H, as a consequence of condition (3.2). We also decompose the Hilbert space $\mathcal{H}$ into the direct sum $\mathcal{H} = \mathcal{K} \oplus \mathcal{G}$, where $\mathcal{G}$ is the closure of the range $K\mathcal{H}$ of K, thus the orthogonal complement of the closed subspace

$$\mathcal{K} := \ker(K) = \left\{ f \in \mathcal{H} : \sum_{y \in N(x)} \Phi(y) f(y) = \Phi(x) \sum_{y \in N(x)} f(y) \ \forall\, x \in X \right\}.$$

It is easy to see that H and K are reduced by this decomposition. Their restrictions H_0 and K_0 to the Hilbert space $\mathcal{G}$ are bounded selfadjoint operators. The proofs of the following results are given in the next section.

Theorem 3.2. *Assume that Φ is a function semi-adapted to the graph $(X, \sim)$. Then H_0 has no point spectrum.*

In order to state a limiting absorption principle for H_0 in the presence of an adapted function, we introduce an auxiliary Banach space. We denote by $\mathcal{F}$ the completion of $K\mathcal{H} \cap D(\Phi)$ with respect to the norm $\|f\|_{\mathcal{F}} := \||K_0|^{-1}f\| + \|\Phi f\|$ and we write $\mathcal{F}^*$ for the adjoint space of $\mathcal{F}$. We shall prove subsequently the existence of the continuous dense embeddings $\mathcal{F} \hookrightarrow \mathcal{G} \hookrightarrow \mathcal{F}^*$ and the following result:

Theorem 3.3. *Let Φ be a function adapted to the graph $(X, \sim)$. Then*

(a) *There exists a constant $\mathrm{C} > 0$ such that $\left|\left\langle f, (H_0 - \lambda \mp i\mu)^{-1} f\right\rangle\right| \leq \mathrm{C}\, \|f\|_{\mathcal{F}}^2$ for all $\lambda \in \mathbb{R}$, $\mu > 0$ and $f \in \mathcal{F}$.*

(b) *The operator H_0 has a purely absolutely continuous spectrum.*

In the next section we introduce a larger space $\mathcal{E}$ obtained by real interpolation. The limiting absorption principle is then obtained between the space $\mathcal{E}$ and its adjoint $\mathcal{E}^*$. Of course, everything is trivial when $\mathcal{K} = \mathcal{H}$. This happens if and only if Φ is a constant function (obviously adapted to any graph). We shall avoid this trivial case in the examples. In many situations the subspace $\mathcal{K}$ can be calculated explicitly. On the other hand, if several adapted functions exist, one may use this to enlarge the space $\mathcal{G}$ on which H is proved to be purely absolutely continuous.

The following result on the stability of the nature of the spectrum of H_0 under small perturbations is a direct consequence of Corollary 2.4.

Corollary 3.4. *Let U_0 be a bounded selfadjoint operator in $\mathcal{G}$ such that $|U_0|^{1/2}$ extends to an element of $\mathscr{B}(\mathcal{F}^*, \mathcal{G})$. Then, for $|\gamma|$ small enough, the operator $H_0 + \gamma U_0$ ispurely absolutely continuous and is unitarily equivalent to H_0 through the wave operators.*

4. Proof of the main result

In this section we choose and fix a semi-adapted function Φ. As a consequence of condition (3.2), one checks easily that the bounded selfadjoint operators H and K commute. Aside H and K we also consider the operator L in $\mathcal{H}$ given by

$$(Lf)(x) := -\sum_{y \sim x} \left[\Phi(y) - \Phi(x)\right]^2 f(y)\,, \quad f \in \mathcal{H}\,, \quad x \in X\,.$$

Due to the discussion in Section 3, the operator L is selfadjoint and bounded. Furthermore one may verify that H, K and L leave invariant the domain $D(\Phi)$ of the operator of multiplication Φ and that one has on $D(\Phi)$ the relations

$$K = i[H, \Phi]\,, \qquad L = i[K, \Phi]\,.$$

These relations imply that H and K belong to $C^1(\Phi;\mathcal{H})$ (see Definition 2.1). If in addition Φ is adapted to the graph, formula (3.3) implies that $i[K,L]=0$.

The operators

$$\mathcal{A} := \frac{1}{2}(\Phi K + K\Phi) \qquad \text{and} \qquad \mathcal{A}' := \frac{1}{2}(\Phi L + L\Phi)$$

are well-defined and symmetric on $D(\Phi)$.

Lemma 4.1. *Let Φ be a function semi-adapted to the graph $(X,\sim)$.*

(a) *The operator $\mathcal{A}$ is essentially selfadjoint on $D(\Phi)$. The domain of its closure A is $D(A)=D(\Phi K)=\{f\in\mathcal{H} : \Phi K f\in\mathcal{H}\}$ and A acts on $D(A)$ as the operator $\Phi K - \frac{i}{2}L$.*

(b) *The operator $\mathcal{A}'$ is essentially selfadjoint on $D(\Phi)$. The domain of its closure A' is $D(A')=D(\Phi L)=\{f\in\mathcal{H} : \Phi L f\in\mathcal{H}\}$.*

Proof. One just has to reproduce the proof of [11, Lemma 3.1], replacing their couple (N,S) by (Φ,K) for the point (a) and by (Φ,L) for the point (b). □

In the next lemma we collect some results on commutators with A or A'.

Lemma 4.2. *Let Φ be a function semi-adapted to the graph $(X,\sim)$.*

(a) *The quadratic form $D(A)\ni f\mapsto i\langle Hf,Af\rangle - i\langle Af,Hf\rangle$ extends uniquely to the bounded form defined by the operator K^2.*

(b) *The quadratic form $D(A)\ni f\mapsto i\langle K^2f,Af\rangle - i\langle Af,K^2f\rangle$ extends uniquely to the bounded form defined by the operator $KLK+\frac{1}{2}(K^2L+LK^2)$ (which reduces to $2KLK$ if Φ is adapted).*

(c) *If Φ is adapted, the quadratic form $D(A')\ni f\mapsto i\langle Kf,A'f\rangle - i\langle A'f,Kf\rangle$ extends uniquely to the bounded form defined by the operator L^2.*

The proof is straightforward. Computations may be performed on the core $D(\Phi)$. These results imply that $H\in C^1(A;\mathcal{H})$, $K^2\in C^1(A;\mathcal{H})$ and (when Φ is adapted) $K\in C^1(A';\mathcal{H})$.

Using the results of Lemma 4.2 we shall now establish a relation between the kernels of the operators H, K and L. For any selfadjoint operator T in the Hilbert space $\mathcal{H}$ we write $\mathcal{H}_{\mathrm{p}}(T)$ for the closed subspace of $\mathcal{H}$ spanned by the eigenvectors of T.

Lemma 4.3. *For a function Φ semi-adapted to the graph $(X,\sim)$ one has*

$$\ker(H)\subset\mathcal{H}_{\mathrm{p}}(H)\subset\ker(K)\subset\mathcal{H}_{\mathrm{p}}(K)\,.$$

If Φ is adapted, one also has

$$\mathcal{H}_{\mathrm{p}}(K)\subset\ker(L)\subset\mathcal{H}_{\mathrm{p}}(L)\,.$$

Proof. Let f be an eigenvector of H. Due to the Virial theorem [2, Proposition 7.2.10] and the fact that H belongs to $C^1(A;\mathcal{H})$, one has $\langle f, i[H,A]f\rangle = 0$. It follows then by Lemma 4.2(a) that $0=\langle f,K^2f\rangle = \|Kf\|^2$, i.e., $f\in\ker(K)$.

The inclusion $\mathcal{H}_{\rm p}(H) \subset \ker(K)$ follows. Similarly, by using A' instead of A and Lemma 4.2(c) one gets the inclusion $\mathcal{H}_{\rm p}(K) \subset \ker(L)$ and the lemma is proved. □

We are finally in a position to prove all the statements of Section 3.

Proof of Theorem 3.2. Since H and K are commuting bounded selfadjoint operators, the invariance of $\mathcal{K}$ and $\mathcal{G}$ under H and K is obvious. Let us recall that H_0 and K_0 denote, respectively, the restrictions of the operators H and K to the subspace $\mathcal{G}$. By Lemma 4.3 one has $\mathcal{H}_{\rm p}(H) \subset \mathcal{K}$, thus H_0 has no point spectrum. □

Lemma 4.4. *If Φ is adapted to the graph $(X, \sim)$, then the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{G}$ reduces the operator A. The restriction of A to $\mathcal{G}$ defines a selfadjoint operator denoted by A_0.*

Proof. We already know that on $D(A) = D(\Phi K)$ one has $A = \Phi K - \frac{i}{2}L$. By using Lemma 4.3 it follows that $\mathcal{K} \subset \ker A \subset D(A)$. Then one trivially checks that (i) $A[\mathcal{K} \cap D(A)] \subset \mathcal{K}$, (ii) $A[\mathcal{G} \cap D(A)] \subset \mathcal{G}$ and (iii) $D(A) = [\mathcal{K} \cap D(A)] + [\mathcal{G} \cap D(A)]$, which means that A is reduced by the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{G}$. Thus by [29, Theorem 7.28] the restriction of A to $D(A_0) \equiv D(A) \cap \mathcal{G}$ is selfadjoint in $\mathcal{G}$. □

Proof of Theorem 3.3. We shall prove that the method of the weakly conjugate operator, presented in Section 2, applies to the operators H_0 and A_0 in the Hilbert space $\mathcal{G}$.

(i) Lemma 4.2(a) implies that $i(H_0A_0 - A_0H_0)$ is equal in the form sense to K_0^2 on $D(A_0) \equiv D(A) \cap \mathcal{G}$. Therefore the corresponding quadratic form extends uniquely to the bounded form defined by the operator K_0^2. This implies that H_0 belongs to $C^1(A_0; \mathcal{G})$.

(ii) Since $B_0 := i[H_0, A_0] \equiv K_0^2 > 0$ in $\mathcal{G}$, the operator A_0 is weakly conjugate to H_0. So we define the space $\mathcal{B}$ as the completion of $\mathcal{G}$ with respect to the norm $\|f\|_{\mathcal{B}} := \langle f, B_0 f\rangle^{1/2}$. The adjoint space of $\mathcal{B}$ is denoted by $\mathcal{B}^*$ and can be identified with the completion of $B_0\mathcal{G}$ with respect to the norm $\|f\|_{\mathcal{B}^*} := \left\langle f, B_0^{-1} f\right\rangle^{1/2}$. It can also be expressed as the closure of the subspace $K\mathcal{H} = K_0\mathcal{G}$ with respect to the same norm $\|f\|_{\mathcal{B}^*} = \left\||K_0|^{-1} f\right\|$. Due to Lemma 4.2(b) the quadratic form $D(A_0) \ni f \mapsto i\,\langle B_0 f A_0 f\rangle - i\,\langle A_0 f, B_0 f\rangle$ extends uniquely to the bounded form defined by the operator $2K_0L_0K_0$, where L_0 is the restriction of L to $\mathcal{G}$. We write $i[B_0, A_0]$ for this extension.

(iii) We check now that $\{W_t\}_{t\in\mathbb{R}}$ extends to a C_0-group in $\mathcal{B}$. This easily reduces to proving that for any $t \in \mathbb{R}$ there exists a constant $\mathrm{c}(t)$ such that $\|W_t f\|_{\mathcal{B}} \leq \mathrm{c}(t)\|f\|_{\mathcal{B}}$ for all $f \in D(A_0)$. Due to point (ii) one has for each $f \in D(A_0)$

$$\begin{aligned} \|W_t f\|_{\mathcal{B}}^2 &= \langle f, B_0 f\rangle + \int_0^t \mathrm{d}\tau \Big\langle \mathcal{W}_\tau f, i[B_0, A_0] W_\tau f\Big\rangle \\ &\leq \|f\|_{\mathcal{B}}^2 + 2\|L_0\| \int_0^{|t|} \mathrm{d}\tau\, \|W_\tau f\|_{\mathcal{B}}^2 \,. \end{aligned}$$

Since $\mathcal{G} \hookrightarrow \mathcal{B}$, the function $(0,|t|) \ni \tau \mapsto \|W_\tau f\|^2_{\mathcal{B}} \in \mathbb{R}$ is bounded. Thus we get the inequality $\|W_t f\|_{\mathcal{B}} \leq \mathrm{e}^{|t|\|L_0\|}\, \|f\|_{\mathcal{B}}$ by using a simple form of the Gronwall lemma. Therefore $\{W_t\}_{t\in\mathbb{R}}$ extends to a C_0-group in $\mathcal{B}$, and by duality $\{W_t\}_{t\in\mathbb{R}}$ also defines a C_0-group in $\mathcal{B}^*$. It follows immediately that the quadratic form $i[B_0, A_0]$ defines an element of $\mathscr{B}(\mathcal{B},\mathcal{B}^*)$. This concludes the proof of the fact that B_0 extends to an element of $C^1(A_0;\mathcal{B},\mathcal{B}^*)$.

Thus all hypotheses of Theorem 2.3 are satisfied and the limiting absorption principle (2.1) holds for H_0, with $\mathcal{E}$ given by $(D(A_0,\mathcal{B}^*),\mathcal{B}^*)_{1/2,1}$.

(iv) *A fortiori* the limiting absorption principle holds in the space $D(A_0,\mathcal{B}^*)$ endowed with its graph norm. Let us show that the space $\mathcal{F}$ introduced in Section 3 is even smaller, with a stronger topology. We recall that for $f \in D(A_0,\mathcal{B}^*) = \{f \in D(A_0)\cap\mathcal{B}^* : A_0 f \in \mathcal{B}^*\}$ (cf. [2, Eqation 6.3.3]) one has

$$\|f\|^2_{D(A_0,\mathcal{B}^*)} = \|f\|^2_{\mathcal{B}^*} + \|A_0 f\|^2_{\mathcal{B}^*} = \big\||K_0|^{-1} f\big\|^2 + \big\||K_0|^{-1} A_0 f\big\|^2 .$$

We first prove that $K\mathcal{H}\cap D(\Phi)$ is dense in $\mathcal{G}$ and that $K\mathcal{H}\cap D(\Phi) \subset D(A_0,\mathcal{B}^*)$. For the density it is enough to observe that $KD(\Phi) \subset K\mathcal{H}\cap D(\Phi)$ and that $KD(\Phi)$ is dense in $\mathcal{G} = \overline{K\mathcal{H}}$ since $D(\Phi)$ is dense in $\mathcal{H}$ and K is bounded. For the second statement, since $K\mathcal{H} = K_0\mathcal{G}$, any f in $K\mathcal{H}\cap D(\Phi)$ belongs to $\mathcal{B}^*$ and to $D(A_0) = D(\Phi K)\cap\mathcal{G}$. Furthermore, since $[K,L]=0$, we have $A_0 f = K\Phi f + \frac{i}{2}Lf \in K\mathcal{H} \subset \mathcal{B}^*$. This finishes to prove that $K\mathcal{H}\cap D(\Phi) \subset D(A_0,\mathcal{B}^*)$. We observe now that for f in $K\mathcal{H}\cap D(\Phi)$ one has

$$\begin{aligned}\big\||K_0|^{-1}A_0 f\big\| &= \big\||K_0|^{-1}\left(K\Phi + \frac{i}{2}L\right) f\big\| \\ &\leq \|\Phi f\| + \frac{1}{2}\|L\|\big\||K_0|^{-1} f\big\| \leq \mathrm{c}\|f\|_{\mathcal{F}}\end{aligned}$$

for some constant $\mathrm{c} > 0$ independent of f. It follows that $\|f\|_{D(A_0,\mathcal{B}^*)} \leq \mathrm{c}'\|f\|_{\mathcal{F}}$ for all $f \in K\mathcal{H}\cap D(\Phi)$ and a constant c' independent of f. Thus one has proved that $\mathcal{F} \hookrightarrow \mathcal{G}$, and the second continuous dense embedding $\mathcal{G} \hookrightarrow \mathcal{F}^*$ is obtained by duality. □

5. Admissible graphs

In this section we put into evidence a class of graphs for which very explicit (and essentially unique) adapted functions exist. For this class the spectral results are sharpened and simplified.

Assume that the graph $(X,\sim)$ is connected and deduced from a directed graph, i.e., some relation $<$ is given on X such that, for any $x,y \in X$, $x \sim y$ is equivalent to $x < y$ or $y < x$, and one cannot have both $y < x$ and $x < y$. We also write $y > x$ for $x < y$, and note that $x < x$ is forbidden.

Alternatively, one gets $(X,<)$ by decomposing for any $x \in X$ the set of neighbours of x into a disjoint union, $N(x) = N^-(x) \sqcup N^+(x)$, taking care that $y \in N^-(x)$ if and only if $x \in N^+(y)$. We call the elements of $N^-(x)$ *the fathers* of x and the elements of $N^+(x)$ *the sons* of x, although this often leads to shocking situations. Obviously, we set $x < y$ if and only if $x \in N^-(y)$, or equivalently, if and only if $y \in N^+(x)$. When using drawings, one has to choose a direction (an arrow) for any edge. By convention, we set $x \leftarrow y$ if $x < y$, i.e., any arrow goes from a son to a father. When directions have been fixed, we use the notation $(X, <)$ for the *directed graph* and say that $(X, <)$ is *subjacent* to $(X, \sim)$.

Let $p = x_0 x_1 \dots x_n$ be a path. Its *index* is the difference between the number of positively oriented edges and that of the negatively oriented ones, i.e., $\mathrm{ind}(p) := \#\{j : x_{j-1} < x_j\} - \#\{j : x_{j-1} > x_j\}$. The index is additive under juxtaposition of paths: if $p = x_0 x_1 \dots x_n$ and $q = y_0 y_1 \dots y_m$ with $x_n = y_0$, then the index of the path $pq := x_0 x_1 \dots x_{n-1} y_0 y_1 \dots y_m$ is the sum of the indices of the paths p and q.

Definition 5.1. A directed graph $(X, <)$ is called *admissible* if

(i) it is *univoque*, i.e., any closed path in X has index zero,
(ii) it is *uniform*, i.e., for any $x, y \in X$, $\#[N^-(x) \cap N^-(y)] = \#[N^+(x) \cap N^+(y)]$.

A graph $(X, \sim)$ is called *admissible* if there exists an admissible directed graph $(X, <)$ subjacent to $(X, \sim)$.

Definition 5.2. A *position function* on a directed graph $(X, <)$ is a function $\Phi : X \to \mathbb{Z}$ satisfying $\Phi(x) + 1 = \Phi(y)$ if $x < y$.

We give now some properties of the position function.

Lemma 5.3.

(a) *A directed graph $(X, <)$ is univoque if and only if it admits a position function.*
(b) *Any position function on an admissible graph $(X, \sim)$ is surjective.*
(c) *A position function on a directed graph $(X, <)$ is unique up to a constant.*

Proof. (a) Let Φ be a position function on $(X, <)$ and p a path from x to y. Then $\mathrm{ind}(p) = \Phi(y) - \Phi(x)$. Thus $\mathrm{ind}(p) = 0$ for any closed path. Conversely, assume univocity. It is equivalent to the fact that, for any $x, y \in X$, each path from x to y has the same index. Fix $z_0 \in X$ and for any $z \in X$ set $\Phi(z) := \mathrm{ind}(p)$ for some path $p = z_0 z_1 \dots z$. Then $\Phi(z)$ does not depend on the choice of p and is clearly a position function.

(b) Since $\#N^-(x) = \#N^+(x)$ for any $x \in X$, it follows that each point of X belongs to a path which can be extended indefinitely in both directions.

(c) If Φ_1 and Φ_2 are two position functions and p is a path from x to y (which exists since X is connected), then $\Phi_1(y) - \Phi_1(x) = \mathrm{ind}(p) = \Phi_2(y) - \Phi_2(x)$, thus $\Phi_1(y) - \Phi_2(y) = \Phi_1(x) - \Phi_2(x)$. □

Let us note that any univoque directed graph is bipartite, i.e., it can be decomposed into two disjoint subsets X_1, X_2 such that the edges connect only couples of the form $(x_1, x_2) \in X_1 \times X_2$. This is achieved simply by setting $X_1 = \Phi^{-1}(2\mathbb{Z} + 1)$ and $X_2 = \Phi^{-1}(2\mathbb{Z})$. It follows then by [22, Corollary 4.9] that the spectrum of H is symmetric with respect to the origin.

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We first show that for an admissible graph, any position function is adapted. Condition (i) from Definition 3.1 is obvious. In the two remaining conditions of Definition 3.1 one can decompose the sums over $N(x)\cap N(y)$ as sums over the four disjoint sets $N^-(x)\cap N^-(y)$, $N^+(x)\cap N^+(y)$, $N^-(x)\cap N^+(y)$ and $N^+(x)\cap N^-(y)$. In the last two cases the sums are zero and in the other two cases the sums give together $2(\#[N^+(x)\cap N^+(y)] - \#[N^-(x)\cap N^-(y)])$, which is also zero by the uniformity of the graph.

Therefore Theorem 3.3 can be applied. If Φ is a position function, one has $\Phi(y)-\Phi(x)=\pm 1$ if $x\sim y$ and thus $L=-H$. Consequently, Lemma 4.3 gives the equalities

$$\begin{aligned}\mathcal{H}_{\mathrm{p}}(H) &= \ker(K) = \mathcal{H}_{\mathrm{p}}(K) = \ker(H)\\ &= \left\{ f\in\mathcal{H} : \sum_{y>x} f(y) = 0 = \sum_{y<x} f(y) \text{ for each } x\in X\right\}\end{aligned}$$

which complete the proof. □

Note that even when $\mathcal{K}\neq\{0\}$ the singular continuous spectrum of H is empty. Indeed, in the canonical decomposition $\mathcal{H}=\mathcal{H}_{\mathrm{p}}(H)\oplus\mathcal{H}_{\mathrm{ac}}(H)\oplus\mathcal{H}_{\mathrm{sc}}(H)$, $\mathcal{H}_{\mathrm{p}}(H)$ is identified with $\mathcal{K}$, $\mathcal{H}_{\mathrm{ac}}(H)$ with $\mathcal{G}$, and $\mathcal{H}_{\mathrm{sc}}(H)$ is thus trivial. Furthermore, a look at the proof above shows that the results of Theorem 1.1 hold in fact for any graph with an adapted function Φ satisfying $\Phi(y)-\Phi(x)=\pm 1$ if $x\sim y$. We decided to insist on the particular case of admissible graphs because admissibility can be checked straightforwardly by inspecting the subjacent directed graph; in case of successful verification the function Φ is generated automatically.

Remark 5.4. For a directed graph $(X,<)$, define $(Uf)(x) := \sum_{y<x} f(y)$ for each $f\in\mathcal{H}$ and $x\in X$. The operator U is bounded and its adjoint is given by $(U^*f)(x) = \sum_{y>x} f(y)$. One has $H = 2\,\mathsf{Re}\,U$ and $K = 2\,\mathsf{Im}\,U$. Uniformity of $(X,<)$ is equivalent to the normality of U, thus to the fact that H and K commute. In [11] it is shown that the adjacency operator of a homogeneous rooted tree can be written as $H = 2\,\mathsf{Re}\,U$ for U a completely non unitary isometry (i.e., an isometry such that $U^{*n}\to 0$ strongly). This fact is used to prove the existence of an operator N (called *number operator*) satisfying $UNU^* = N-1$. N is used to construct an operator $A = N(\mathsf{Im}\,U) + (\mathsf{Im}\,U)N$, which is conjugate (in the sense of Mourre theory) to H and to some classes of perturbations of H. Note that N is not a multiplication operator. It would be interesting to find an approach unifying the present study with the work [11].

One can show that finite cartesian products of admissible directed graphs are admissible. Indeed uniformity follows rather easily from the definitions and, if Φ_j is the position function for $(X_j,<_j)$, then Φ defined by $\Phi(x_1,\dots,x_n) := \sum_{j=1}^n \Phi_j(x_j)$ is the natural position function for the cartesian product $\prod_j (X_j,<_j)$. As an example, $\mathbb{Z}^n$ is admissible, since $\mathbb{Z}$ is obviously admissible. We shall not give details here since these are simple facts, largely covered by Section 7.

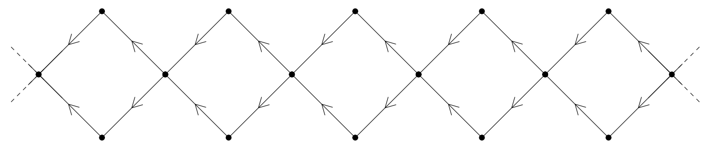

Figure 1. Example of an admissible, non-injective directed graph

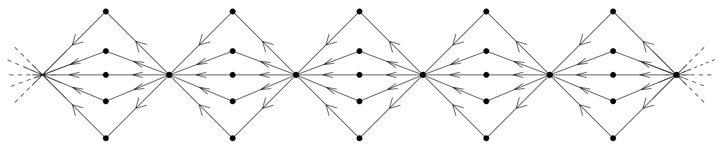

Figure 2. Example of an admissible, non-injective directed graph

6. Examples

We present some examples of graphs (admissible or not) with an adapted function which can be easily drawn in the plane. Although some of them might be subject to other treatments, we would like to stress the relative ease and unity of our approach, which also furnishes boundary estimates for the resolvent and applies to some classes of perturbations. In many situations we will be able to determine the kernel $\mathcal{K}$ of the operator K explicitly. In the case $\mathcal{K} = \{0\}$ the graph is said to be *injective*; the examples will show that this is quite a delicate matter. For admissible graphs, we recall that $\ker(K) = \ker(H)$ coincides with the singular subspace of H and that it is given by formula (1.1).

The directed graph X of Figure 1 is admissible, non-regular and not injective.

Indeed, $\mathcal{K}$ is composed of all $f \in \ell^2(X)$ taking the value 0 on the middle row and opposite values on the other two rows.

The same type of results are available for similar graphs (see for example Figure 2).

One can sometimes construct admissible graphs X by juxtaposing admissible graphs in some coherent manner. For instance the directed graph of Figure 3 is admissible and injective, so that its adjacency operator is purely absolutely continuous.

Writting the condition $\sum_{w<x} f(w) = 0$ for $f \in \ell^2(X)$ and x as in Figure 3, one gets $f(z) = 0$. But one has also $f(z) + f(z') = 0$ due to the same condition for the vertex y. Thus $f(z') = 0$, and the graph is injective since the same argument holds for each vertex of X. Extension of the graph in both vertical directions induces the standard Cayley graph of $\mathbb{Z}^2$, which is clearly admissible and injective. If we extend the graph only downwards, then we obtain the subgraph $\{(x_1, x_2) \in \mathbb{Z}^2 : x_1 < x_2\}$, which is also admissible and injective.

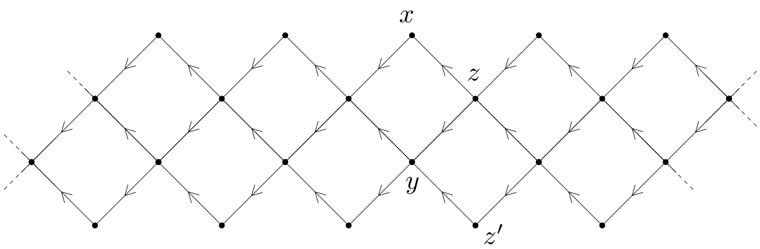

Figure 3. Example of an admissible, injective directed graph

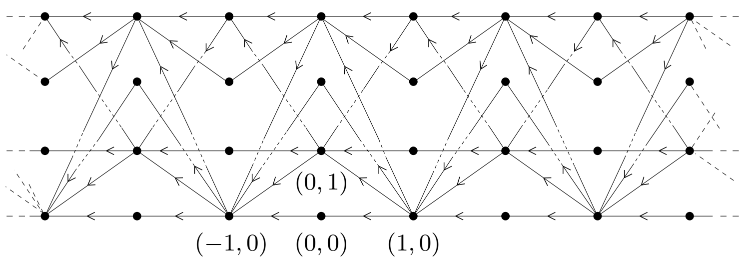

Figure 4. Example of an admissible directed graph

A general construction leading to admissible graphs is the following: Let p, q be two integers with $p \leq q$ (we allow $p = -\infty$ or $q = \infty$ or both). Set $\mathbb{Z}_{p,q} := \{x_2 \in \mathbb{Z} : p \leq x_2 \leq q\}$ and $X := \mathbb{Z} \times \mathbb{Z}_{p,q}$. Fix a function $x_2 \mapsto A(x_2)$, sending elements of $\mathbb{Z}_{p,q}$ to finite subsets of $\mathbb{Z}_{p,q}$ with cardinal smaller than a constant D. For any $x_1 \in \mathbb{Z}$ and $x_2 \in \mathbb{Z}_{p,q}$, set $N^{\pm}(2x_1, x_2) := \{2x_1 \pm 1\} \times A(x_2)$. This defines uniquely a directed graph $(X, <)$, namely one has automatically $N^-(2x_1 + 1, x_2) = \{(2x_1, y_2) : x_2 \in A(y_2)\}$ and $N^+(2x_1 + 1, x_2) = \{(2x_1 + 2, y_2) : x_2 \in A(y_2)\}$ and there are no other arrows than those already indicated. Even if it is not strictly necessary, we insure that the graph is connected by requiring that $A(x_2) \neq \varnothing$ for all $x_2 \in \mathbb{Z}_{p,q}$ and that $\bigcup_{x_2 \in \mathbb{Z}_{p,q}} A(x_2) = \mathbb{Z}_{p,q}$. We also impose the number of elements $\{x_2 : y_2 \in A(x_2)\}$ to be bounded by a constant D' not depending on $y_2 \in \mathbb{Z}_{p,q}$. As a consequence $(X, <)$ will be uniformly locally finite. The only sets of common fathers or sons which could be non-void are $N^{\pm}(2x_1, x_2) \cap N^{\pm}(2x_1, x_2') = \{(2x_1 \pm 1)\} \times [A(x_2) \cap A(x_2')]$, $N^-(2x_1 + 1, x_2) \cap N^-(2x_1 + 1, x_2') = \{(2x_1, y_2) : x_2, x_2' \in A(y_2)\}$ and $N^+(2x_1 + 1, x_2) \cap N^+(2x_1 + 1, x_2') = \{(2x_1 + 2, y_2) : x_2, x_2' \in A(y_2)\}$. Thus $(X, <)$ is admissible. As a consequence of Theorem 1.1, the corresponding adjacency operator is purely absolutely continuous outside the origin. Analogous constructions in higher dimensions are available.

In Figure 4 we present the (rather simple) case $X := \mathbb{Z} \times \mathbb{Z}_{0,3}$, with $A(0) = \{0\}$, $A(1) = \{0, 1, 3\}$, $A(2) = \{0\}$ and $A(3) = \{0, 2, 3\}$.

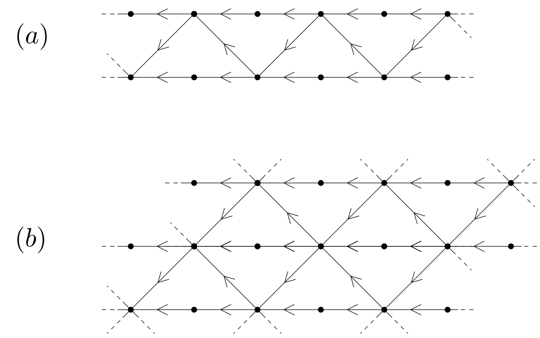

FIGURE 5. Examples of admissible, injective directed graphs

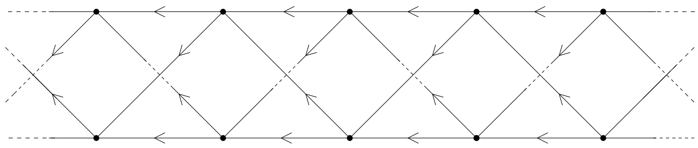

FIGURE 6. Example of an admissible, non-injective directed graph

For a general directed graph of this type it could be difficult to calculate the subspace $\mathcal{K}$ and, in particular, to decide upon injectivity. We are going to put into evidence situations where this is possible by a direct use of the explicit definition of $\mathcal{K}$ (this is an easy task, left to the reader).

The directed graph of Figure 5(a) is admissible and injective, so its adjacency operator has no singular continuous spectrum and no point spectrum. One shows easily that admissibility and injectivity are preserved under a finite or infinite number of vertical juxtapositions of the graph with itself (see Figure 5(b)). On the other hand, if one puts Figure 5(a) on top of itself, deletes all the arrows belonging to the middle row as well as the vertices left unconnected, one gets an admissible, non-injective directed graph.

The directed graph of Figure 6 is admissible, regular but not injective. The graph deduced from it is the Cayley graph of $\mathbb{Z} \times \mathbb{Z}_2$, with generating system $\{(\pm 1, 1), (\pm 1, -1)\}$, without being a cartesian product. The elements of $\mathcal{K}$ are all ℓ^2-functions which are anti-symmetric with respect to a vertical flip. If two copies of this graph are juxtaposed vertically, the resulting graph is still admissible, but also regular and injective. If one deletes some chosen arrows in the resulting graph, one obtains a nice admissible, non-injective graph with vertices of degree 2, 4 and 6 (see Figure 7).

Admissible graphs are of a very restricted type. For instance closed paths of odd length and vertices of odd degree are forbidden. We give now a few more

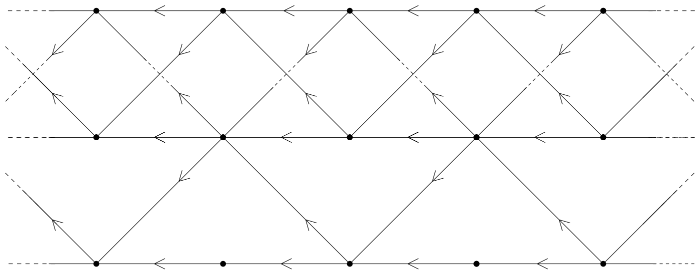

Figure 7. Example of an admissible, non-injective directed graph

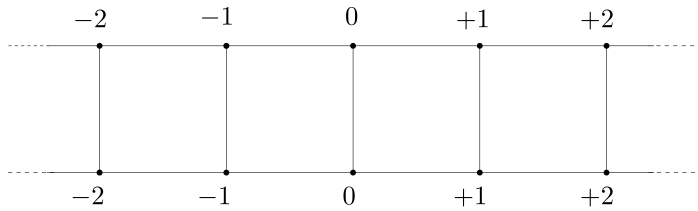

Figure 8. Example of a non-admissible, adapted, injective graph

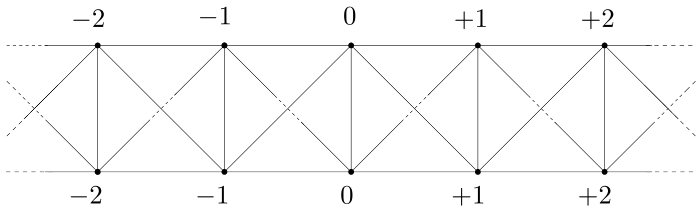

Figure 9. Example of a non-admissible, adapted, non-injective graph

examples of graphs, for which non-constant adapted functions Φ exist. At each vertex, the indicated number corresponds to the value of Φ.

Easy computations show that the function Φ associated with the non-admissible regular graph of Figure 8 is adapted. Furthermore, this graph is injective. This is not unexpected, since it is a very simple Cayley graph of the Abelian group $\mathbb{Z} \times \mathbb{Z}_2$. Deleting steps in this ladder graph leads generically to singular continuous spectrum as pointed out in [26].

The function Φ indicated for the non-admissible regular graph of Figure 9 is adapted. One shows easily that the space $\mathcal{K}$ coincides with the eigenspace of the adjacency operator H associated with the eigenvalue -1. The rest of the spectrum is purely absolutely continuous. The function Φ of the non-admissible non-regular graph of Figure 10 is adapted. However, we believe that this graph is not injective. More graphs with an adapted function will be indicated in the next sections.

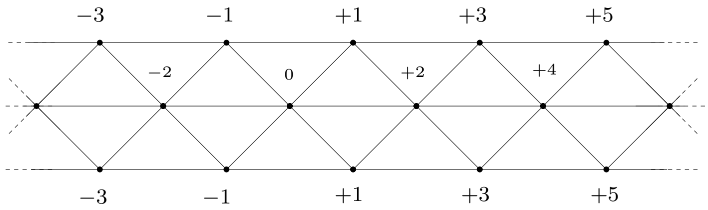

Figure 10. Example of a non-admissible, adapted graph

Remark 6.1. All of the examples presented here are $\mathbb{Z}$-periodic. More involved, $\mathbb{Z}^n$-periodic situations are also available. However, in general, periodicity of a graph is very far from excluding singular spectrum. First of all, part of the examples are not "co-compact", i.e., the set of orbits under the action of $\mathbb{Z}$ is infinite. In this situation, we are not aware of any general result relying on periodicity. If at least one of the integers in the generic example $X = \mathbb{Z} \times \mathbb{Z}_{p,q}$ above is infinite, we get a very precise result on the spectrum of a large class of periodic graphs with infinitely many orbits. This result does not seem to be within reach by other known methods. On the other hand, if only a finite number of orbits are present, it is known [10,12] that the singular continuous spectrum is empty. But eigenvalues are quite common [16,17] and this is related to the absence of a Unique Continuation Principle for operators on graphs. Thus our results on the point spectrum for the examples of this section seem to be non-trivial even in the co-compact case.

Remark 6.2. We also insist on the global Limiting Absorption Principle. In [12] a very general and abstract theory is developed for perturbations of direct integral operators with fibers that have a compact resolvent and depend analytically on the base parameter. Mourre theory is used and a Limiting Absorption Principle is proved. However the estimates are localized outside a set of thresholds, which is defined implicitly. Furthermore the results of [12] rely heavily on the compactness condition in the fibers.

Remark 6.3. It is also common for operators on periodic, co-compact graphs that local perturbations embed eigenvalues with compactly supported eigenfunctions in the continuum spectrum of the unperturbed operator [16,18]. Corollary 3.4 puts into evidence classes of perturbations for which this phenomenon does not occur, at least for small values of a coupling constant.

7. *D*-products

We recall now some properties of adjacency operators on the class of *D-products*. We call *D*-product what is referred as *non-complete extended p-sum with basis D* in [22].

Consider a family $\{(X_j, \sim_j)\}_{j=1}^n$ of simple graphs, which are all infinite countable and uniformly locally finite. Let D be a subset of $\{0,1\}^n$ not containing

$(0,0,\dots,0)$. Then we endow the product $X := \prod_{j=1}^n X_j$ with the following (D-product) graph structure: if $x, y \in X$ then $x \sim y$ if and only if there exists $d \in D$ such that $x_j \sim_j y_j$ if $d_j = 1$ and $x_j = y_j$ if $d_j = 0$. The resulting graph $(X, \sim)$ is again simple, infinite countable and uniformly locally finite. Note that the tensor product as well as the cartesian product are special cases of D-product. We shall not assume $(X_j, \sim_j)$ connected and even if we did, the D-product could fail to be so.

It is easy to see that the adjacency operator H of the D-product $(X, \sim)$ may be written as

$$H = \sum_{d \in D} H_1^{d_1} \otimes \cdots \otimes H_n^{d_n} ,$$

where H_j is the adjacency operator of $(X_j, \sim_j)$, $H_j^0 = 1$ and $H_j^1 = H_j$. The operator H acts in the Hilbert space $\ell^2(X) \simeq \bigotimes_{j=1}^n \ell^2(X_j)$.

Proposition 7.1. *For each $j \in \{1, \dots, n\}$, let Φ_j be a function adapted to the graph $(X_j, \sim_j)$ and $c_j \in \mathbb{R}$. Then $\Phi_c : X \to \mathbb{R}$, $(x_1, \dots, x_n) \mapsto \sum_{j=1}^n c_j \Phi_j(x_j)$ is a function adapted to $(X, \sim)$.*

Proof. Rather straightforward calculations show that Φ_c satisfies (3.2) and (3.3). It is simpler to indicate a simpler operatorial proof:

Define $K_j := i[H_j, \Phi_j]$ and $L_j := i[K_j, \Phi_j]$ in $\ell^2(X_j)$. Since Φ_j is adapted the three operators H_j, K_j and L_j commute (use the Jacobi identity for the triple H_j, K_j and Φ_j). Since the multiplication operator Φ_c can be written in $\otimes_j \ell^2(X_j)$ as $\Phi_c = \sum_{j=1}^n c_j \, 1 \otimes \cdots \otimes \Phi_j \otimes \cdots \otimes 1$, where Φ_j stands on the j'th position, one has

$$K := i[H, \Phi_c] = \sum_{d \in D} \sum_j c_j \, H_1^{d_1} \otimes \cdots \otimes K_j(d_j) \otimes \cdots \otimes H_n^{d_n} ,$$

where $K_j(d_j)$ stands on the j'th position and is equal to K_j if $d_j = 1$ and to 0 if $d_j = 0$. Analogously one has

$$\begin{aligned} L := i[K, \Phi_c] = & \sum_{d \in D} \sum_{j \neq k} c_j c_k \, H_1^{d_1} \otimes \cdots \otimes K_j(d_j) \otimes \cdots \otimes K_k(d_k) \otimes \cdots \otimes H_n^{d_n} \\ & + \sum_{d \in D} \sum_j c_j^2 \, H_1^{d_1} \otimes \cdots \otimes L_j(d_j) \otimes \cdots \otimes H_n^{d_n} , \end{aligned}$$

where $L_j(d_j)$ is equal to L_j if $d_j = 1$ and to 0 if $d_j = 0$. It is clear that $i[H, K] = 0 = i[K, L]$, which is equivalent to the statement of the proposition. □

Notice that we could very well have no valuable information on some of the graphs X_j and take $\Phi_j = 0$. As soon as Φ_c is not a constant, the space $\mathcal{G}$ on which we have a purely absolutely continuous restricted operator is non-trivial. So one can perform various D-products, including factors for which an adapted function has already been shown to exist (as those in the preceding section). But it is not clear how the space $\mathcal{K} = \ker(K)$ could be described in such a generality.

8. The one-dimensional XY model

In the sequel we apply the theory of Section 5 to the Hamiltonian of the one-dimensional XY model. We follow [8] for the brief and rather formal presentation of the model. Further details may be found in [27].

We consider the one-dimensional lattice $\mathbb{Z}$ with a spin-1/2 attached at each vertex. Let

$$\mathbb{F}(\mathbb{Z}) := \big\{\alpha : \mathbb{Z} \to \{0,1\} : \operatorname{supp}(\alpha) \text{ is finite}\big\}\,,$$

and write $\{e^0, e^1\} := \{(0,1),(1,0)\}$ for the canonical basis of the (spin-1/2) Hilbert space $\mathbb{C}^2$. For any $\alpha \in \mathbb{F}(\mathbb{Z})$ we denote by e^α the element $\{e^{\alpha(x)}\}_{x\in\mathbb{Z}}$ of the direct product $\prod_{x\in\mathbb{Z}} \mathbb{C}^2_x$. We distinguish the vector e^{α_0}, where $\alpha_0(x) := 0$ for all $x \in \mathbb{Z}$. Each element e^α is interpreted as a state of the system of spins, and e^{α_0} as its ground state with all spins pointing down. The Hilbert space $\mathcal{M}$ of the system (which is spanned by the states with all but finitely many spins pointing down) is the "incomplete tensor product" [27, Section 2]

$$\mathcal{M} := \bigotimes_{x\in\mathbb{Z}}^{\alpha_0} \mathbb{C}^2_x \equiv \text{closed span}\big\{e^\alpha : \alpha \in \mathbb{F}(\mathbb{Z})\big\}\,.$$

The dynamics of the spins is given by the nearest-neighbour XY Hamiltonian

$$M := -\frac{1}{2} \sum_{|x-y|=1} \Big(\sigma_1^{(x)}\sigma_1^{(y)} + \sigma_2^{(x)}\sigma_2^{(y)}\Big)\,.$$

The operator $\sigma_j^{(x)}$ acts in $\mathcal{M}$ as the identity operator on each factor $\mathbb{C}^2_y$, except on the component $\mathbb{C}^2_x$ where it acts as the Pauli matrix σ_j. To go further on, we need to introduce a new type of directed graphs.

Definition 8.1. Let $(X,<)$ be a directed graph. For $N \in \mathbb{N}$, we set $\mathbb{F}_N(X) := \{\alpha : X \to \{0,1\} : \#\operatorname{supp}(\alpha) = N\}$ and endow it with the natural directed graph structure defined as follows: if $\alpha, \beta \in \mathbb{F}_N(X)$ then $\alpha < \beta$ if and only if there exist $x \in \operatorname{supp}(\alpha)$, $y \in \operatorname{supp}(\beta)$ such that $x < y$ and $\operatorname{supp}(\alpha) \setminus \{x\} = \operatorname{supp}(\beta) \setminus \{y\}$.

From now on, we shall no longer make any distinction between an element $\alpha \in \mathbb{F}_N(X)$ and its support, which is a subset of X with N elements. We recall from [8, Section 2] that M is unitarily equivalent to a direct sum $\bigoplus_{N\in\mathbb{N}} H_N$, where H_N is the selfadjoint operator in $\mathcal{H}_N := \ell^2[\mathbb{F}_N(\mathbb{Z})]$ acting as

$$(H_N f)(\alpha) = -2 \sum_{\beta\sim\alpha} f(\beta)\,, \quad f \in \mathcal{H}_N\,, \quad \alpha \in \mathbb{F}_N(\mathbb{Z})\,.$$

Thus the spectral analysis of M reduces to determining the nature of the spectrum of the adjacency operators on $\mathcal{H}_N$. Moreover the graph $(\mathbb{F}_N(\mathbb{Z}),\sim)$ deduced from $(\mathbb{F}_N(\mathbb{Z}),<)$ satisfies

Lemma 8.2. $(\mathbb{F}_N(\mathbb{Z}),\sim)$ *is an admissible graph.*

Proof. Due to Definition 5.1 one simply has to prove that $(\mathbb{F}_N(\mathbb{Z}), <)$ is admissible. In point (i) we show that $(\mathbb{F}_N(\mathbb{Z}), <)$ is uniform. In point (ii) we give the (natural) position function for $(\mathbb{F}_N(\mathbb{Z}), <)$.

(i) Given $\alpha \in \mathbb{F}_N(\mathbb{Z})$ and $x \in \operatorname{supp}(\alpha)$, $y \notin \operatorname{supp}(\alpha)$, we write α_x^y for the function of $\mathbb{F}_N(\mathbb{Z})$ such that $\operatorname{supp}(\alpha_x^y) = \operatorname{supp}(\alpha) \sqcup \{y\} \setminus \{x\}$.

Thus one has

$$N^-(\alpha) \cap N^-(\beta) = \left\{\gamma : \exists x \in \alpha,\, x-1 \notin \alpha, \exists y \in \beta,\, y-1 \notin \beta,\ \gamma = \alpha_x^{x-1} = \beta_y^{y-1}\right\}$$

and

$$N^+(\alpha) \cap N^+(\beta) = \left\{\gamma : \exists x \in \alpha,\, x+1 \notin \alpha, \exists y \in \beta,\, y+1 \notin \beta,\ \gamma = \alpha_x^{x+1} = \beta_y^{y+1}\right\},$$

the couples (x, y) being unique for a given γ in both cases.

Suppose there exist $x \in \alpha$, $y \in \beta$ such that $x - 1 \notin \alpha$, $y - 1 \notin \beta$ and $\alpha_x^{x-1} = \beta_y^{y-1}$, so that $\alpha_x^{x-1} \in \{N^-(\alpha) \cap N^-(\beta)\}$. If $x = y$, then $\alpha = \beta$, and $\#N^-(\alpha)$, $\#N^+(\alpha)$ are both equal to the number of connected components of α. If $x \neq y$, then one has $x-1 \in \beta$, $x \notin \beta$, $y-1 \in \alpha$, $y \notin \alpha$ together with the equality $\alpha_{y-1}^y = \beta_{x-1}^x$. Therefore $\alpha_{y-1}^y \in \{N^+(\alpha) \cap N^+(\beta)\}$ and one has thus obtained a bijective map from $N^-(\alpha) \cap N^-(\beta)$ to $N^+(\alpha) \cap N^+(\beta)$.

(ii) If $\Phi_{\mathbb{Z}}$ is a position function for $\mathbb{Z}$ (for instance $\Phi_{\mathbb{Z}}(x) = x$), it is easily checked that Φ defined by $\Phi(\alpha) := \sum_{x\in\alpha} \Phi_{\mathbb{Z}}(x)$ is a position function for $\mathbb{F}_N(\mathbb{Z})$. □

Remark 8.3. One could presume that $(\mathbb{F}_N(\mathbb{Z}^2), <)$ is also an admissible directed graph. But this is wrong, as it can be seen from the following example. For $N = 2$, consider $\alpha := \{(1, 0), (1, 1)\}$ and $\beta := \{(0, 1), (1, 1)\}$. It can be easily checked that $N^-(\alpha) \cap N^-(\beta) = \{\{(0, 0), (1, 1)\}, \{(1, 0), (0, 1)\}\}$, whereas $N^+(\alpha) \cap N^+(\beta) = \varnothing$. This contradicts the uniformity hypothesis.

As a corollary of Theorem 1.1 and of the admissibility of $(\mathbb{F}_N(\mathbb{Z}), \sim)$, one obtains:

Corollary 8.4. *The spectrum of M is purely absolutely continuous, except maybe at the origin.*

Remark 8.5. We would obtain that the spectrum of M is purely absolutely continuous if we could show that $\ker(H_N) = \{0\}$ for any N. Unfortunately we have been able to obtain such a statement only for $N = 1, 2, 3$ and 4. Our proof consists in showing that if there exists $f \in \ker(H_N)$ such that $f(\alpha) \neq 0$ for some $\alpha \in \mathbb{F}_N(\mathbb{Z})$, then there exists an infinite number of elements $\alpha' \in \mathbb{F}_N(\mathbb{Z})$ such that $f(\alpha') = f(\alpha)$, which contradicts the requirement $f \in \ell^2[\mathbb{F}_N(\mathbb{Z})]$. In any case, even if we did not succeed in extending such an argument for $N > 4$, the kernel of H_N is trivial for any N. This follows from the fact that H_N can also be shown to be purely absolutely continuous using an approach similar to the image charge method in electrostatics [7].

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