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Spectral and propagation results for magnetic Schrödinger operators; A C^*-algebraic framework

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Abstract

We study generalized magnetic Schrödinger operators of the form $H_h(A,V) = h(\Pi^A) + V$, where h is an elliptic symbol, $\Pi^A = -i\nabla - A$, with A a vector potential defining a variable magnetic field B, and V is a scalar potential. We are mainly interested in anisotropic functions B and V. The first step is to show that these operators are affiliated to suitable C^*-algebras of (magnetic) pseudodifferential operators. A study of the quotient of these C^*-algebras by the ideal of compact operators leads to formulae for the essential spectrum of $H_h(A,V)$, expressed as a union of spectra of some asymptotic operators, supported by the quasi-orbits of a suitable dynamical system. The quotient of the same C^*-algebras by other ideals give localization results on the functional calculus of the operators $H_h(A,V)$, which can be interpreted as non-propagation properties of their unitary groups.

Keywords: Magnetic field; Pseudodifferential operators; Dynamical system; Twisted crossed-product; Essential spectrum; Schrödinger operators

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0. Introduction

Until rather recently, the nature of the essential spectrum of self-adjoint partial differential operators with anisotropic coefficients was poorly understood. It was clear that what counts is the long-scale behavior of these coefficients, but it was not clear how to express this in a general and unified manner.

In recent years a significant progress was achieved. We do not intend to trace the history of this topic here, but just quote [15,19,36–38] and [39] for some general results for continuous or discrete operators. A natural and elegant procedure makes use of the theory of C^*-algebras in conjunction with some configurational framework as dynamical systems or Lie groupoids. We refer for example to [1,4,6,7,11–14,20,22,23,32,43] and references therein. The list is by no means complete and we do not describe in detail the different but connected points of view of these works. We only say some words on the common part of the ideas involved (cf. also [13]). Let H be a self-adjoint operator in a Hilbert space $\mathcal{H}$. The central fact is that this operator is *affiliated* to some C^*-algebra $\mathfrak{C}$ of bounded operators in $\mathcal{H}$; this means that its resolvent family belongs to $\mathfrak{C}$. The essential spectrum of H can be calculated if we understand the image of H in the quotient C^*-algebra $\mathfrak{C}/\mathfrak{C}\cap K(\mathcal{H})$, where $K(\mathcal{H})$ is the ideal of compact operators in $\mathcal{H}$. Many choices for $\mathfrak{C}$ are possible; the skill consists in choosing one for which the quotient is comprehensible. This is usually done by keeping track of the subjacent configuration space X of the problem. Such a space is available because we study differential, or more generally, pseudodifferential operators. In [12–14,22,23] it is assumed that X is an abelian, locally compact group, and this leads to a dynamical system background for the C^*-algebras. In [20] and [32] the authors work in a smooth groupoid setting, which is very general.

In [3] the discrete case with quasi-periodic potentials is treated and a formula for the structure of the spectrum is proved. Let us point out in connection with this remark that formula (4.1.10) (in [4]), gives a decomposition of the complete spectrum of an operator H as a union of a family of spectra for the case of a Hamiltonian belonging to the C^*-algebra that is the twisted crossed-product associated to an abelian algebra of functions on a compact space on which acts a discrete group, the cocycle taking values in the group $U(1)$. Thus, the Hamiltonian is a bounded operator (element of the C^*-algebra), the group is discrete and the cocycle takes values in $U(1)$. In our Theorem 1.11, we treat unbounded Hamiltonians affiliated to a twisted crossed-product of an algebra of functions on a compactification of $\mathbb{R}^n$ on which $\mathbb{R}^n$ acts by translations and the cocycle takes values in the group $C(\mathbb{R}^n, U(1))$ (a non-locally compact, Polish group). Moreover, we obtain a decomposition of the essential spectrum of H as a union of spectra of asymptotic Hamiltonians. Let us strengthen that our main difficulties come exactly from the facts that the cocycle takes values in $C(\mathbb{R}^n, U(1))$ (this fact being specific for non-constant magnetic fields) and not in $U(1)$ and the Hamiltonians are unbounded and an essential technical fact that we prove is their affiliation to the twisted crossed-product C^*-algebra.

We are aware of only few general results concerning the structure of the essential spectrum of Hamiltonians with variable magnetic fields: [15] and the preprint [19] that appeared after a first version of this paper.

Spectral analysis for (pseudo-)differential operators with variable magnetic fields may be considered as a difficult matter; one of the reasons is gauge covariance: the vector potential A defining the magnetic field B by the relation $B = dA$ and appearing in the explicit expression of the operator is highly non-unique and largely irrelevant. What counts is the magnetic field, which is hidden. Obviously, good spectral results should be expressed with no reference to any vector potential (see also [8] and [31] for related results concerning the discrete spectrum). On the other hand, it is clear that the magnetic field plays a very different role than a scalar potential. Thus, one needs C^*-algebras incorporating naturally magnetic fields, in a manifestly invariant way.

Based on works done in [26,27], it was shown in [28] how to achieve this. The key concept is that of *twisted crossed product C^*-algebras*. These algebras have been developed in a much more general setting in [5,33] and [34]. They are a more sophisticated version of the wide-known notion of crossed product algebras, already used in connection with anisotropic operators (without magnetic fields) in [2,12–14] or [22]. We have shown in [28] that certain twisted crossed products are related to a twisted version of the Weyl pseudodifferential calculus, introduced in [17,18,27,30], which is the natural pseudodifferential calculus when twisted observables as magnetic momenta are present. This is also basic for a strict deformation quantization à la Rieffel for physical systems placed in magnetic fields, as explained in [29] and [25].

It will be shown in the present article that twisted crossed product algebras and their natural Hilbert space representations are the natural structures that lead to results on the essential spectrum of magnetic Schrödinger operators. To describe briefly the output, let us consider in $\mathbb{R}^N$ an elliptic symbol h, a magnetic field B, a vector potential A for the magnetic field and a scalar potential V. Let $H_h(A, V)$ denote the operator $h(\Pi^A) + V$. We shall prove that the essential spectrum $\sigma_{\mathrm{ess}}(H_h(A, V))$ of $H_h(A, V)$ is equal to $\overline{\bigcup_\nu \sigma[H_h(A_\nu, V_\nu)]}$, where A_ν is a vector potential for the magnetic field B_ν. Here B_ν and V_ν are defined, respectively, by the asymptotic behavior of the magnetic field B and of the scalar potential V at infinity. Actually this behavior is codified by a C^*-algebra of functions on $\mathbb{R}^N$. The Gelfand spectrum of this C^*-algebra is a compact dynamical system and the functions B_ν and V_ν are just restrictions of B and V to quasi-orbits of this dynamical system situated at infinity. We emphasize that our proofs are manifestly gauge-independent; the main result is formulated without any choice of a magnetic potential.

In a sense that will be discussed in Section 5.1, this result is a consistent extension, in the bounded case, of similar results of Helffer and Mohamed obtained by strictly analytical methods for a restricted class of perturbed magnetic Laplacians. However, in [15] the potential V and the magnetic field B are allowed to be unbounded (under suitable restrictions). Unbounded scalar potentials V are also considered in a great generality in [13] and [14], but we are mainly interested in the case when a magnetic field is also present. We stress that our results are valid for any elliptic symbol h, and not only for the usual magnetic Laplacian. There exist very few spectral results in such a general framework; even to define the right gauge-covariant Hilbert space operator $h(\Pi^A)$ for general elliptic symbols is a non-trivial matter.

Actually, a C^*-algebraic setting can support other problems in the spectral theory of self-adjoint operators than just calculating essential spectra. In [1] and [41], C^*-algebras are used in order to get a Mourre estimate, which is basic for obtaining useful resolvent estimates, finer spectral properties and scattering theory. Such developments require usually more detailed informations about the models under study and cannot be done for magnetic operators in the very general setting in which we will be placed below. But there is still a spectral topic that is available in the present generality, that of *localization properties*. Such results say roughly that if the support of a continuous function η does not intersect the spectrum of an asymptotic operator $H_h(A_\nu, V_\nu)$, then the operator $\eta[H_h(A, V)]$ will be small when localized in the neighborhood of the quasi-orbit that defines $H_h(A_\nu, V_\nu)$. This has as an immediate consequence a non-propagation statement for the unitary group generated by $H_h(A, V)$: if a state has a spectral support with respect to $H_h(A, V)$ which do not intersect the spectrum of the asymptotic operator $H_h(A_\nu, V_\nu)$, then this state cannot evolve under the unitary evolution generated by $H_h(A, V)$ towards the corresponding quasi-orbit. We refer to Section 1.4 for a precise statement, to Section 4 for the proof and to [2] and [23] for more explanations in the case $B = 0$. Some particular examples can also be found in [9].

Let us describe the content of this article. In Section 1 we introduce the framework, recall some useful formulae and state precisely all the results mentioned above. A powerful affiliation criterion is exposed in Theorem 1.8 and Corollary 1.10; such a result is crucial in using C^*-algebraic techniques in spectral analysis. The essential spectrum is calculated in Theorem 1.11, and propagation results are contained in Theorem 1.12. Section 2 is mainly devoted to the proof of the affiliation criterion. It is the most technical part of this paper; the analysis of the resolvent cannot be performed by using only a Neumann series and treating the magnetic field as a perturbation. Some new ideas are involved, treating the magnetic Moyal product as a deformation of the usual one. The ingredients used for the description of the essential spectrum are explained in Section 3, and an abstract version of Theorem 1.11 is presented and proved. The short proof of the propagation property is given in Section 4. And the last section is dedicated to examples and to a comparison with the results of [15].

Our affiliation result is a main technical step for a number of developments we have in view. By some extra technical effort we could have obtained certain minor ameliorations of the results. Sometimes this will be rather evident to the attentive reader. For sake of simplicity we stick to the present version. The main goal of an improved subsequent work would be to allow singular, unbounded functions B_{jk} and V.

In our opinion [14] proposes a rather complete and very natural treatment for the structure of the essential spectrum of Hamiltonians with position anisotropy in the absence of a magnetic field. Let us mention that our results cannot be obtained by simple generalizations of the techniques in [12] and [14], but it is possible that some of the ideas in [14] might be useful to continue our work towards unbounded perturbations.

Notations. We briefly set together some conventions and notations. X denotes the euclidean space $\mathbb{R}^N$, with $N \in \mathbb{N}$, and $X^\star$ denotes its dual space, commonly identified with $\mathbb{R}^N$. The Lebesgue measures on X and $X^\star$ are normalized in such a way that the Fourier transform $\mathcal{F} : L^1(X) \to C_0(X^\star)$, with $(\mathcal{F}f)(p) = \int_X \mathrm{d}x\, e^{ip\cdot x} f(x)$, induces a unitary map from $L^2(X)$ to $L^2(X^\star)$. $BC(X)$, $BC_\mathrm{u}(X)$ and $C_0(X)$ are, respectively, the algebra of bounded and continuous functions on X, the algebra of bounded and uniformly continuous functions on X and the ideal of continuous functions on X that converge to 0 at infinity. We denote by $BC^\infty(X)$ the space of smooth complex functions on X with bounded derivatives of any order. Except in some specified and well-defined context, $\mathcal{H}$ is the Hilbert space $L^2(X)$, $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded operators in $\mathcal{H}$, and $K(\mathcal{H})$ the ideal of compact operators in $\mathcal{H}$.

1. Main results

1.1. The framework

In recent papers a pseudodifferential calculus [17,18,27] and an algebraic framework [26,28] where introduced in order to deal with the quantization problem for a particle in a magnetic field. We start by recalling very briefly some aspects of each construction. These approaches are complementary and both are relevant for generalized magnetic Schrödinger operators and, specifically, for the statement of our main results. The relations between these approaches, rigorously investigated in [28, Sections 3.1, 3.2], will be outlined at the end of the section. We refer to the publications quoted above for more informations and details.

1.1.1. The magnetic Weyl calculus

We recall the magnetic version of the usual Weyl calculus and the associated magnetic symbolic calculus. The corresponding magnetic Moyal algebra is also introduced with a brief review of some of its properties.

Assume that B is a continuous magnetic field on X, i.e. a closed 2-form on X, and let A be a continuous vector potential that generates the magnetic field, i.e. A is a 1-form on X that satisfies $\partial_j A_k - \partial_k A_j = B_{jk}$. In the Hilbert space $\mathcal{H}$, Q_j denotes the operator of multiplication by the jth coordinate, and $\Pi_j^A := -i\partial_j - A_j$ is the jth component of the usual magnetic momentum. The magnetic Weyl calculus is a gauge covariant prescription that assigns to suitable symbols $f : X \times X^\star \to \mathbb{C}$ an operator $\mathfrak{Op}^A(f) \equiv f(Q, \Pi^A)$ acting in $\mathcal{H}$. More precisely, if ρ is a scalar function on X and $A' := A + \nabla\rho$ is another vector potential that generates the same magnetic field, then the relation $e^{i\rho}\mathfrak{Op}^A(f)e^{-i\rho} = \mathfrak{Op}^{A'}(f)$ holds. The prescription is formally given, for any $u \in \mathcal{H}$, by

$$\big[\mathfrak{Op}^A(f)u\big](x) := \int_X \mathrm{d}y \int_{X^\star} \mathrm{d}p\, e^{ip\cdot(x-y)} \lambda^A(x; y-x) f\left(\frac{1}{2}(x+y), p\right) u(y), \tag{1.1}$$

where

$$\lambda^A(q; x) := \exp\big(-i\Gamma^A[q, q+x]\big) \tag{1.2}$$

and $\Gamma^A[q, q+x]$ is the circulation of A along the segment of ends q and $q+x$.

For brevity let us denote by Ξ the phase space $X \times X^\star$. The magnetic symbolic calculus is a non-commutative composition law $\circ$ acting on functions $f, g : \Xi \to \mathbb{C}$ such that the relation $\mathfrak{Op}^A(f \circ g) = \mathfrak{Op}^A(f)\mathfrak{Op}^A(g)$ is satisfied. This operation, called *the magnetic Moyal product*, is formally defined, for $\xi = (q, p)$, $\eta = (x, k)$ and $\zeta = (y, l)$ in Ξ, by

$$[f \circ g](\xi) := 4^N \int_\Xi \mathrm{d}\eta \int_\Xi \mathrm{d}\zeta\, e^{-2i\sigma(\eta,\zeta)} \omega^B\big(q - x - y; 2x, 2(y-x)\big) f(\xi - \eta) g(\xi - \zeta), \tag{1.3}$$

where

$$\omega^B(q; x, y) := \exp\big(-i\Gamma^B\langle q, q+x, q+x+y\rangle\big) \tag{1.4}$$

and $\Gamma^B\langle q, q+x, q+x+y\rangle$ is the flux of the magnetic field through the triangle defined by the points q, $q+x$ and $q+x+y$. An explicit parametrized formula for $\omega^B(q; x, y)$ is given in Eq. (3.3). The expression $\sigma(\eta, \zeta)$ in (1.3) is equal to $k \cdot y - l \cdot x$. Let us mention that an involution can also be defined by $f^\circ(\xi) := \overline{f(\xi)}$ and satisfies $\mathfrak{Op}^A(f^\circ) = \mathfrak{Op}^A(f)^*$.

The integrals defining $f \circ g$ are absolutely convergent only for restricted classes of symbols. In order to deal with more general distributions, an extension by duality was proposed in [27] under an additional smoothness condition on the magnetic field. So let us assume that the components of the magnetic field are $C^\infty_{\mathrm{pol}}(X)$-functions, i.e. they are indefinitely differentiable and each derivative is polynomially bounded. The duality approach is based on the observation [27, Lemma 14]: for any f, g in the Schwartz space $\mathcal{S}(\Xi)$, we have

$$\int_{\Xi} d\xi\,[f\circ g](\xi)=\int_{\Xi} d\xi\,[g\circ f](\xi)=\int_{\Xi} d\xi\, f(\xi)g(\xi)=\langle\overline{f},g\rangle\equiv(f,g).$$

As a consequence, if f, g and h belong to $\mathcal{S}(\Xi)$, the equalities $(f\circ g,h)=(f,g\circ h)=(g,h\circ f)$ hold.

Definition 1.1. For any distribution $F\in\mathcal{S}'(\Xi)$ and any function $f\in\mathcal{S}(\Xi)$ we define

$$(F\circ f,h):=(F,f\circ h),\qquad (f\circ F,h):=(F,h\circ f)\quad\text{for all } h\in\mathcal{S}(\Xi).$$

The expressions $F\circ f$ and $f\circ F$ are a priori tempered distributions. The Moyal algebra is precisely the set of elements of $\mathcal{S}'(\Xi)$ that preserves regularity by composition.

Definition 1.2. The Moyal algebra $\mathcal{M}(\Xi)$ is defined by

$$\mathcal{M}(\Xi):=\bigl\{F\in\mathcal{S}'(\Xi)\bigm| F\circ f\in\mathcal{S}(\Xi)\text{ and } f\circ F\in\mathcal{S}(\Xi)\text{ for all } f\in\mathcal{S}(\Xi)\bigr\}.$$

For two distributions F and G in $\mathcal{M}(\Xi)$, the Moyal product can be extended by

$$(F\circ G,h):=(F,G\circ h)\quad\text{for all } h\in\mathcal{S}(\Xi).$$

Remark 1.3. The set $\mathcal{M}(\Xi)$ with this composition law and the complex conjugation $F\mapsto F^{\circ}$ is a unital *-algebra. Actually, this extension by duality also gives compositions $\mathcal{M}(\Xi)\circ\mathcal{S}'(\Xi)\subset\mathcal{S}'(\Xi)$ and $\mathcal{S}'(\Xi)\circ\mathcal{M}(\Xi)\subset\mathcal{S}'(\Xi)$. One checks plainly that associativity holds for any three factors product with two factors belonging to $\mathcal{M}(\Xi)$ and one in $\mathcal{S}'(\Xi)$.

An important result [27, Proposition 23] concerning the Moyal algebra is that it contains $C^{\infty}_{\mathrm{pol,u}}(\Xi)$, the space of infinitely differentiable complex functions on Ξ having uniform polynomial growth at infinity. Finally let us quote a result linking $\mathcal{M}(\Xi)$ with the functional calculus $\mathfrak{Op}^A$ [27, Proposition 21]: for any vector potential A belonging to $C^{\infty}_{\mathrm{pol}}(X)$, $\mathfrak{Op}^A$ is an isomorphism of *-algebras between $\mathcal{M}(\Xi)$ and $\mathcal{L}[\mathcal{S}(X)]\cap\mathcal{L}[\mathcal{S}'(X)]$, where $\mathcal{L}[\mathcal{S}(X)]$ and $\mathcal{L}[\mathcal{S}'(X)]$ are, respectively, the spaces of linear continuous operators on $\mathcal{S}(X)$ and $\mathcal{S}'(X)$.

Remark 1.4. We note for further use that very often it is easier to work with regularized expressions. For instance, if f and g belong to $C^{\infty}_{\mathrm{pol,u}}(\Xi)$, we can interpret $f\circ g$ as the limit $\lim_{m,n\to\infty}(\chi_n f)\circ(\chi_m g)$, where $\chi\in C^{\infty}_c(\Xi)$ with $\chi(0)=1$ and $\chi_n(\xi):=\chi(\xi/n)$. Then $\chi_n f$ is a sequence approximating f in $\mathcal{S}'(\Xi)$ (for example) and $(\chi_n f)\circ(\chi_m g)$ is given by the explicit formula (1.3) of the composition law.

1.1.2. Twisted crossed product algebras

Now we recall the definitions of *magnetic* twisted C^*-dynamical systems, of the corresponding twisted C^*-algebras, and the construction of some of their representations in the Hilbert space $\mathcal{H}$. These algebras are particular instances of the concept of twisted groupoïd C^*-algebra [40] and of the twisted C^*-algebras extensively studied in [5,33] and [34] (see also references therein).

For this purpose, let $\mathcal{A}$ be a unital C^*-subalgebra of $BC_{\mathrm{u}}(X)$. We shall always assume that $\mathcal{A}$ contains the ideal $C_0(X)$ and is stable by translations, i.e. $\theta_x(a) := a(\cdot + x) \in \mathcal{A}$ for all $a \in \mathcal{A}$ and $x \in X$. In the references cited above and in [28] $\mathcal{A}$ was also assumed to be separable but this is not needed for our developments. This algebra can be thought of as a way to encode the anisotropic behavior of the magnetic fields and of the scalar potentials. Thus we consider a magnetic field B on X whose components B_{jk} belong to $\mathcal{A}$. The expression ω^B defined in (1.4) has then some special properties: for fixed x and y, the function $\omega^B(\cdot; x, y) \equiv \omega^B(x, y)$ belongs to the unitary group $\mathcal{U}(\mathcal{A})$ of $\mathcal{A}$. Moreover, the mapping $X \times X \ni (x, y) \mapsto \omega^B(x, y) \in \mathcal{U}(\mathcal{A})$ is a 2-cocycle on X with values in $\mathcal{U}(\mathcal{A})$.

The quadruplet $(\mathcal{A}, \theta, \omega^B, X)$ is a *magnetic* example of *an abelian twisted C^*-dynamical system* $(\mathcal{A}, \theta, \omega, X)$. In the general case X is an abelian second countable locally compact group, $\mathcal{A}$ is an abelian C^*-algebra, θ is a continuous morphism from X to the group of automorphisms of $\mathcal{A}$, and ω is a strictly continuous 2-cocycle with values in the unitary group of the multiplier algebra of $\mathcal{A}$. We refer to [28, Definition 2.1] for more explanations.

Given any abelian twisted C^*-dynamical system, a natural C^*-algebra can be defined. We recall its construction. Let $L^1(X; \mathcal{A})$ be the set of Bochner integrable functions on X with values in $\mathcal{A}$, with the L^1-norm $\|\phi\|_1 := \int_X \mathrm{d}x \,\|\phi(x)\|_{\mathcal{A}}$. For any $\phi, \psi \in L^1(X; \mathcal{A})$ and $x \in X$, we define the product

$$(\phi \diamond \psi)(x) := \int_X \mathrm{d}y \, \theta_{\frac{y-x}{2}}\big[\phi(y)\big]\theta_{\frac{y}{2}}\big[\psi(x-y)\big]\theta_{-\frac{x}{2}}\big[\omega(y, x-y)\big]$$

and the involution

$$\phi^{\diamond}(x) := \theta_{-\frac{x}{2}}\big[\omega(x, -x)^{-1}\big]\phi(-x)^*.$$

Note that in the magnetic case $\omega^B(x, -x) = 1$.

Definition 1.5. The enveloping C^*-algebra of $L^1(X, \mathcal{A})$ is called *the twisted crossed product* and is denoted by $\mathcal{A} \rtimes_\theta^\omega X$.

Let us now consider a continuous vector potential A that generates the magnetic field, i.e. A is a continuous 1-form on X that satisfies $\partial_j A_k - \partial_k A_j = B_{jk}$. The relation between λ^A defined in Eq. (1.2) and ω^B reads (by Stokes theorem)

$$\lambda^A(q; x)\lambda^A(q + x; y)\big[\lambda^A(q; x + y)\big]^{-1} = \omega^B(q; x, y). \tag{1.5}$$

If λ^A were a map $X \ni x \mapsto \lambda^A(\cdot; x) \in \mathcal{U}(\mathcal{A})$, this relation would have said that ω^B is a 2-coboundary, or equivalently that ω^B is a trivial 2-cocycle. But most the time this map has only image in $C(X; \mathbb{T})$, the set of continuous functions on X with values in the complex numbers of modulus 1. For that reason, one says that λ^A is a *pseudo-trivialization* of ω^B.

Based on relation (1.5), one can construct a faithful and irreducible representation of the algebra $\mathcal{A} \rtimes_\theta^{\omega^B} X$ in $\mathcal{B}(\mathcal{H})$, that we denote by $\mathfrak{Rep}^A$. Equivalently, this corresponds to a covariant representation of the associated abelian twisted C^*-dynamical system. For each $\phi \in L^1(X; \mathcal{A})$ and $u \in \mathcal{H}$, the representation is given by

$$\left[\mathfrak{Rep}^A(\phi)u\right](x) = \int_X \mathrm{d}y\, \lambda^A(x; y-x)\phi\left(\frac{1}{2}(x+y); y-x\right)u(y). \tag{1.6}$$

Let us mention that the choice of another vector potential generating the same magnetic field would lead to a unitarily equivalent representation of $\mathcal{A} \rtimes_\theta^{\omega^B} X$ in $\mathcal{B}(\mathcal{H})$ (gauge covariance).

By formally comparing (1.1) and (1.6), one sees that $\mathfrak{Op}^A$ and $\mathfrak{Rep}^A$ are connected by a partial Fourier transformation: $\mathfrak{Op}^A(f) = \mathfrak{Rep}^A[\mathfrak{F}^{-1}(f)]$, with

$$\left[\mathfrak{F}^{-1}(f)\right](x, y) := \int_{X^\star} \mathrm{d}p\, e^{-ip\cdot y} f(x, p),$$

for all $x, y \in X$ and suitable f. Then obviously the composition laws $\circ$ and $\diamond$ have to be intertwined by $\mathfrak{F}$, i.e. $f \circ g = \mathfrak{F}[\mathfrak{F}^{-1}(f) \diamond \mathfrak{F}^{-1}(g)]$, as it can be checked by a direct computation. The enveloping C^*-algebra $\mathfrak{B}_\mathcal{A}^B$ of $\mathfrak{F}(L^1(X; \mathcal{A}))$, endowed with the multiplication $\circ$ and the complex conjugation, is thus isomorphic to $\mathcal{A} \rtimes_\theta^{\omega^B} X$ via the canonical extension of $\mathfrak{F}$. Moreover, one has $\mathfrak{Op}^A(\mathfrak{B}_\mathcal{A}^B) = \mathfrak{Rep}^A(\mathfrak{C}_\mathcal{A}^B)$, where $\mathfrak{C}_\mathcal{A}^B$ denotes for shortness the C^*-algebra $\mathcal{A} \rtimes_\theta^{\omega^B} X$.

It might be here the right place to mention that untwisted crossed products are particular cases of groupoids. We suspect that by using *twisted* groupoids (see [40]) one could get more general results, unifying the present framework with the approach of [20] and [32].

1.2. Affiliation

In this section we start by recalling the meaning of affiliation, borrowed from [1] (the related concept from [44] is not directly relevant for our situation). This key concept will then be applied to generalized Schrödinger operators with magnetic fields.

Definition 1.6. An observable affiliated to a C^*-algebra $\mathfrak{C}$ is a morphism $\Phi : C_0(\mathbb{R}) \to \mathfrak{C}$.

If $\mathcal{H}$ is a Hilbert space and $\mathfrak{C}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, then a self-adjoint operator H in $\mathcal{H}$ defines an observable Φ_H affiliated to $\mathfrak{C}$ if and only if $\Phi_H(\eta) := \eta(H)$ belongs to $\mathfrak{C}$ for all $\eta \in C_0(\mathbb{R})$. A sufficient condition is that $(H-z)^{-1} \in \mathfrak{C}$ for some $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$. Thus an observable affiliated to a C^*-algebra is the abstract version of the functional calculus of a self-adjoint operator.

Given a magnetic field B whose components belong to $\mathcal{A}$, a continuous vector potential A that generates B and a suitable symbol $h : X^\star \to \mathbb{R}$, our aim is to show that the C_0-functional calculus of the magnetic Schrödinger operator $h(\Pi^A)$ (which needs to be carefully defined) belongs to the C^*-algebra $\mathfrak{Op}^A(\mathfrak{B}_\mathcal{A}^B) \subset \mathcal{B}(\mathcal{H})$. The proof of such a statement is rather difficult and we shall do it under some smoothness conditions on the magnetic field B and on the symbol h. We point out that we prove in fact a stronger result, Theorem 1.8, that does not depend on the choice of any particular vector potential.

Definition 1.7.

(a) For $s \in \mathbb{R}$, a function $h \in C^\infty(X^\star)$ is *a symbol of type s* if the following condition is satisfied:

$$\forall \alpha \in \mathbb{N}^N, \quad \exists c_\alpha > 0 \quad \text{such that} \quad \left|\left(\partial^\alpha h\right)(p)\right| \leqslant c_\alpha \langle p\rangle^{s-|\alpha|} \quad \text{for all } p \in X^\star,$$

where $\langle p\rangle := \sqrt{1+p^2}$.

(b) The symbol h is called *elliptic* if there exist $R > 0$ and $c > 0$ such that

$$c\langle p\rangle^s \leqslant h(p) \quad \text{for all } p \in X^\star \text{ and } |p| \geqslant R.$$

We denote by $S^s_{\mathrm{el}}(X^\star)$ the family of elliptic symbols of type s, and set $S^\infty_{\mathrm{el}}(X^\star) := \bigcup_s S^s_{\mathrm{el}}(X^\star)$. Note that all the classes $S^s(X^\star)$ are naturally contained in $C^\infty_{\mathrm{pol,u}}(\Xi)$, thus in $\mathcal{M}(\Xi)$. For any $z \notin \mathbb{R}$, we also set $r_z : \mathbb{R} \to \mathbb{C}$ by $r_z(\cdot) := (\cdot - z)^{-1}$.

We are in a position to state the results about affiliation.

Theorem 1.8. *Assume that B is a magnetic field whose components belong to $\mathcal{A} \cap BC^\infty(X)$. Then each real $h \in S^\infty_{\mathrm{el}}(X^\star)$ defines an observable Φ^B_h affiliated to $\mathfrak{B}^B_{\mathcal{A}}$, such that for any $z \notin \mathbb{R}$ one has*

$$(h - z) \circ \Phi^B_h(r_z) = 1 = \Phi^B_h(r_z) \circ (h - z). \tag{1.7}$$

In fact one even has $\Phi^B_h(r_z) \in \mathfrak{F}(L^1(X;\mathcal{A})) \subset \mathcal{S}'(\Xi)$, so the compositions can be interpreted as $\mathcal{M}(\Xi) \times \mathcal{S}'(\Xi) \to \mathcal{S}'(\Xi)$ and $\mathcal{S}'(\Xi) \times \mathcal{M}(\Xi) \to \mathcal{S}'(\Xi)$.

We shall now consider a scalar potential $V \in \mathcal{A}$. It is a standard fact that $\mathcal{A}$ consists of multipliers of the algebra $\mathfrak{F}(L^1(X;\mathcal{A}))$. A straightforward reformulation of the arguments in [1, pp. 365–366] allows then to define the observable $\Phi^B_{h,V} := \Phi^B_{h+V}$. Considering now $h + V \in \mathcal{S}'(\Xi)$ we remark that we can compute the Moyal product $(h + V - z) \circ \Phi^B_{h,V}(r_z) = (h - z) \circ \Phi^B_{h,V}(r_z) + V \circ \Phi^B_{h,V}(r_z) = 1$ (by the explicit formula of $\Phi^B_{h,V}$ given in [1]). This leads to the following statement:

Corollary 1.9. *We are in the framework of Theorem* 1.8. *Let also V be a real function in $\mathcal{A}$. Then $\Phi^B_{h,V}$ is an observable affiliated to $\mathfrak{B}^B_{\mathcal{A}}$, such that for any $z \notin \mathbb{R}$ one has*

$$(h + V - z) \circ \Phi^B_{h,V}(r_z) = 1 = \Phi^B_{h,V}(r_z) \circ (h + V - z).$$

These statements are elegant, being abstract, but in applications one also needs the represented version.

Corollary 1.10. *We are in the framework of Corollary* 1.9. *Let A be a continuous vector potential that generates B. Then $\mathfrak{Op}^A(h) + V(Q)$ defines a self-adjoint operator $H_h(A, V)$ in $\mathcal{H}$ with domain given by the image of the operator $\mathfrak{Op}^A[(h - z)^{-1}]$ (which do not depend on $z \notin \mathbb{R}$). This operator is affiliated to $\mathfrak{Op}^A(\mathfrak{B}^B_{\mathcal{A}}) = \mathfrak{Rep}^A(\mathfrak{C}^B_{\mathcal{A}})$.*

In [26] we have given an affiliation result for $h(p) = |p|^2$ and $\mathcal{A} = BC_{\mathrm{u}}(X)$. In this case we only need that the derivatives $\partial^\alpha B_{jk}$ are bounded for $|\alpha| \leqslant 2$.

1.3. The essential spectrum

We shall give now a description of the essential spectrum of any observable affiliated to the C^*-algebra $\mathfrak{C}^B_{\mathcal{A}}$. For the generalized magnetic Schrödinger operators of Theorem 1.8, this is expressed in terms of the spectra of so-called *asymptotic operators*. The affiliation criterion and the algebraic formalism introduced above play an essential role in the proof of this result; see Section 3. We start by recalling some definitions in relation with topological dynamical systems.

By Gelfand theory, the abelian C^*-algebra $\mathcal{A}$ is isomorphic to the C^*-algebra $C_0(S_{\mathcal{A}})$, where $S_{\mathcal{A}}$ is the spectrum of $\mathcal{A}$ (also the 'Hull' appearing in [3,4]). Since $\mathcal{A}$ was assumed unital and contains $C_0(X)$, $S_{\mathcal{A}}$ is a compactification of X. We shall therefore identify X with a dense open subset of $S_{\mathcal{A}}$. The group law $\theta : X \times X \to X$ extends then to a continuous map $\tilde{\theta} : X \times S_{\mathcal{A}} \to S_{\mathcal{A}}$, because $\mathcal{A}$ was also assumed to be stable under translations. Thus the complement $F_{\mathcal{A}}$ of X in $S_{\mathcal{A}}$ is closed and invariant; it is the space of a compact topological dynamical system. For any $\mathfrak{z} \in F_{\mathcal{A}}$, let us call the set $\{\tilde{\theta}(x, \mathfrak{z}) \mid x \in X\}$ *the orbit generated by* $\mathfrak{z}$, and its closure a *quasi-orbit*. Usually there exist many elements of $F_{\mathcal{A}}$ that generate the same quasi-orbit. In the sequel, we shall often encounter the restriction a_F of an element $a \in \mathcal{A} \equiv C(S_{\mathcal{A}})$ to a quasi-orbit F. Naturally a_F is an element of $C(F)$, but we shall show in Section 3 that this algebra can be realized as a subalgebra of $BC_{\mathrm{u}}(X)$. By a slight abuse of notation, we shall identify a_F with a function defined on X, thus inducing a multiplication operator in $\mathcal{H}$.

The calculation of the essential spectrum may be performed at an abstract level, i.e. without using any representation, as shown in Section 3.1. In the next statement we present for convenience a represented version.

Theorem 1.11. *Let B be a magnetic field whose components belong to $\mathcal{A} \cap BC^\infty(X)$ and let $V \in \mathcal{A}$ be a real function. Assume that $\{F_\nu\}_\nu$ is a covering of $F_{\mathcal{A}}$ by quasi-orbits. Then for each real $h \in S^\infty_{\mathrm{el}}(X^\star)$ one has*

$$\sigma_{\mathrm{ess}}\big[H_h(A, V)\big] = \overline{\bigcup_\nu \sigma\big[H_h(A_\nu, V_\nu)\big]}, \tag{1.8}$$

where A, A_ν are continuous vector potentials for B, $B_\nu \equiv B_{F_\nu}$, and $V_\nu \equiv V_{F_\nu}$.

The operators $H_h(A_\nu, V_\nu) \equiv h(\Pi^{A_\nu}) + V_\nu$ are the asymptotic operators mentioned earlier. We shall show in Section 3 that these operators are affiliated to faithful representations in $\mathcal{B}(\mathcal{H})$ of quotients of $\mathfrak{C}^B_{\mathcal{A}}$ by corresponding natural ideals. All the spectra appearing in (1.8) are only depending on the respective magnetic fields, by gauge covariance. This will be strengthened in Section 3.1 in which a manifestly invariant result will be given in an abstract framework.

1.4. A non-propagation result

We finally describe how the localization results proved in [2] in the case of Schrödinger operators without magnetic field can be extended to the situation where a magnetic field is present. Once again, the algebraic formalism and the affiliation criterion introduced above play an essential role in the proofs: see Section 4. We first introduce the trace on X of a base of neighborhoods of an arbitrary quasi-orbit in $S_{\mathcal{A}}$.

For any quasi-orbit F, let $\mathcal{N}_F$ be the family of sets of the form $W = \mathcal{W} \cap X$, where $\mathcal{W}$ is any element of a base of neighborhoods of F in $S_{\mathcal{A}}$. We write χ_W for the characteristic function of W.

Theorem 1.12. *Let B be a magnetic field whose components belong to $\mathcal{A} \cap BC^\infty(X)$, let V be a real scalar potential that belongs to $\mathcal{A}$ and let h be a real element of $S^\infty_{\mathrm{el}}(X^\star)$. Assume that*

$F \subset F_{\mathcal{A}}$ is a quasi-orbit. Let A, A_F be continuous vector potentials for B and B_F. If $\eta \in C_0(\mathbb{R})$ with $\operatorname{supp}(\eta) \cap \sigma[H_h(A_F, V_F)] = \emptyset$, then for any $\varepsilon > 0$ there exists $W \in \mathcal{N}_F$ such that

$$\left\| \chi_W(Q)\eta\left[H_h(A, V)\right]\right\| \leqslant \varepsilon.$$

In particular, the inequality

$$\left\| \chi_W(Q)e^{-itH_h(A,V)}\eta\left[H_h(A, V)\right]u\right\| \leqslant \varepsilon\|u\|$$

holds, uniformly in $t \in \mathbb{R}$ and $u \in \mathcal{H}$.

The last statement of this theorem gives a precise meaning to the notion of non-propagation. Heuristically, if the spectral support of $u \in \mathcal{H}$ with respect to the operator $H_h(A, V)$ does not meet the spectrum of the asymptotic operator corresponding to a quasi-orbit, then the state u cannot propagate under the evolution given by $e^{-itH_h(A,V)}$ in the direction of this quasi-orbit. We refer to the remark on p. 1223 of [2] for physical explanations and interpretations of this result.

2. Affiliation

In this section we derive our affiliation criterion. In Section 2.1 we indicate the main steps of the proof of Theorem 1.8. Some technical details are included in Appendix A. Corollary 1.10 is obtained in Section 2.2, as a direct consequence of the theorem. We assume tacitly all the hypotheses of Theorem 1.8.

2.1. The proof of the affiliation criterion

The proof of Theorem 1.8 will be based on the following strategy inspired by the 'parametrix' construction: let $\mathcal{M}$ be an associative algebra with a composition law denoted by $\circ$ and let $\mathfrak{h}$ be an element of $\mathcal{M}$. Our aim is to find the inverse for $\mathfrak{h}$. Assume that $\mathfrak{h}'$ is another element such that $\mathfrak{h} \circ \mathfrak{h}'$ and $\mathfrak{h}' \circ \mathfrak{h}$ are invertible. These inverses are written $(\mathfrak{h} \circ \mathfrak{h}')^{(-1)}$ and $(\mathfrak{h}' \circ \mathfrak{h})^{(-1)}$, respectively. Then, the element $\mathfrak{h}' \circ (\mathfrak{h} \circ \mathfrak{h}')^{(-1)}$ is obviously a right inverse for $\mathfrak{h}$ and the element $(\mathfrak{h}' \circ \mathfrak{h})^{(-1)} \circ \mathfrak{h}'$ a left inverse for $\mathfrak{h}$. Both expressions are thus equal to $\mathfrak{h}^{(-1)}$.

In the sequel, we shall take for $\mathfrak{h}$ the strictly positive symbol $h + a$, with a large enough, and for $\mathfrak{h}'$ its pointwise inverse $(h + a)^{-1}$. Finding an inverse $(h + a)^{(-1)}$ for $h + a$ with respect to the composition law $\circ$ will lead rather easily to an observable. In the calculations below we shall use tacitly the approximation procedure described in Remark 1.4. For several arguments we will be forced to get out of the algebra $\mathcal{M} = \mathcal{M}(\Xi)$. This will be easily dealt with, by a suitable use of elements of $\mathcal{S}'(\Xi)$.

Proof of Theorem 1.8. (i) Let us consider an elliptic symbol h of order s and fix some real number $a \geqslant -\inf h + 1$. We set $h_a := h + a$, and denote by h_a^{-1} its inverse with respect to pointwise multiplication, i.e. $h_a^{-1}(p) := (h(p) + a)^{-1}$ for all $p \in X^\star$. It is clear that h_a^{-1} is a symbol of type $-s$. Since both functions h_a and h_a^{-1} belong to $C^\infty_{\mathrm{pol,u}}(\Xi)$, and thus to the Moyal algebra $\mathcal{M}(\Xi)$, one can calculate their product. By using (1.3) we obtain

$$\big(h_a \circ h_a^{-1}\big)(q,p) = 4^N \int\limits_X \mathrm{d}x \int\limits_{X^\star} \mathrm{d}k \int\limits_X \mathrm{d}y \int\limits_{X^\star} \mathrm{d}l\, e^{-2i(k\cdot y - l\cdot x)} \gamma^B(q;2x,2y) \frac{h_a(p-k)}{h_a(p-l)}, \tag{2.1}$$

with $\gamma^B(q;2x,2y) := \omega^B(q-x-y;2x,2(y-x))$. The last factor in the integral does not depend on x and y; it can be developed:

$$\frac{h_a(p-k)}{h_a(p-l)} = 1 + \sum_{j=1}^N (l_j - k_j) \frac{\int_0^1 \mathrm{d}t\, (\partial_j h)(p-l+t(l-k))}{h(p-l)+a} =: 1 + \sum_{j=1}^N F_{a,j}(p;k,l). \tag{2.2}$$

Moreover, let $\tilde\gamma^B(q;k,l) \equiv (\mathbb{F}\gamma^B)(q;k,l) := \int_X \mathrm{d}x \int_X \mathrm{d}y\, e^{-ik\cdot y} e^{il\cdot x} \gamma^B(q;x,y)$. Then the following equality holds (in the sense of distributions, by using Remark 1.4):

$$\int\limits_{X^\star} \mathrm{d}k \int\limits_{X^\star} \mathrm{d}l\, \tilde\gamma^B(q;k,l) = \gamma^B(q;0,0) = 1. \tag{2.3}$$

Thus, by inserting (2.2) and (2.3) into (2.1), we obtain

$$h_a \circ h_a^{-1} = 1 + \sum_{j=1}^N f_{a,j},$$

with

$$f_{a,j}(q;p) := \int\limits_{X^\star} \mathrm{d}k \int\limits_{X^\star} \mathrm{d}l\, \tilde\gamma^B(q;k,l) F_{a,j}(p;k,l) = \big\langle \big(\mathbb{F}\gamma^B\big)(q;\cdot,\cdot), F_{a,j}(p;\cdot,\cdot)\big\rangle. \tag{2.4}$$

The last notation is used in order to emphasize the duality between $C^\infty_{\mathrm{pol,u}}(X^\star \times X^\star)$ and its dual. Indeed, for q,p fixed, Lemma A.2 proves that $F_{a,j}(p;\cdot,\cdot) \in C^\infty_{\mathrm{pol,u}}(X^\star \times X^\star)$, and Lemma A.1 proves that $\gamma^B(q,\cdot,\cdot) \in C^\infty_{\mathrm{pol}}(X \times X)$, so that $(\mathbb{F}\gamma^B)(q;\cdot,\cdot) \in [C^\infty_{\mathrm{pol,u}}(X^\star \times X^\star)]'$ [42, Chapter VII, Theorem XV].

(ii) We are now going to deduce some useful estimates on $f_{a,j}$. We set $\langle P_x\rangle \equiv \langle -i\partial_x\rangle$. For α, j fixed and m, n integers that we shall choose below, one has

$$\begin{aligned} &\big|\big(\partial_p^\alpha f_{a,j}\big)(q;p)\big| \\ &\quad\leqslant \sup_{x,y\in X} \big|\langle x\rangle^{-n}\langle y\rangle^{-n}\langle P_x\rangle^m\langle P_y\rangle^m \gamma^B(q;x,y)\big| \\ &\qquad\times \big\|\langle x\rangle^{-N}\langle y\rangle^{-N}\big\|_{L^2(X\times X)} \big\|\langle P_k\rangle^{n+N}\langle P_l\rangle^{n+N}\langle k\rangle^{-m}\langle l\rangle^{-m}\big(\partial_p^\alpha F_{a,j}\big)(p;\cdot,\cdot)\big\|_{L^2(X^\star\times X^\star)}. \end{aligned} \tag{2.5}$$

By taking into account (A.2), subject of Lemma A.2, and by some simple calculations, one can fix m such that the last factor of (2.5) is dominated by $c_n a^{-1/\mu}\langle p\rangle^{s/\mu-1-|\alpha|}$, with $\mu > \max\{1,s\}$. Then, by using Lemma A.1, one can choose n (depending on m) such that the first factor on the right-hand side term of (2.5) is bounded. Altogether, one obtains

$$\left|\left(\partial_p^\alpha f_{a,j}\right)(q;p)\right| \leqslant c a^{-1/\mu} \langle p\rangle^{s/\mu-1-|\alpha|}, \tag{2.6}$$

where c depends on α and j but not on p, q or a.

(iii) Let us now show that for each j, $\mathfrak{F}^{-1}(f_{a,j})$ is an element of $L^1(X;\mathcal{A})$, and thus belongs to the C^*-algebra $\mathfrak{C}^B_{\mathcal{A}}$. The partial Fourier transform $\mathfrak{F}$ was defined at the end of Section 1.1.

By taking into account Lemma A.1, the right-hand side of Eq. (2.4) can be rewritten as $\langle\gamma^B(q;\cdot,\cdot),(\mathbb{F}^*F_{a,j})(p,\cdot,\cdot)\rangle$, the duality between $C^\infty_{\mathrm{pol}}(X\times X)$ and $(C^\infty_{\mathrm{pol}}(X\times X))' = \mathbb{F}^*C^\infty_{\mathrm{pol,u}}(X^\star\times X^\star)$. As γ^B defines a function from $X\times X$ to $\mathcal{A}$ (see Lemma A.1) that is of class $C^\infty_{\mathrm{pol}}(X\times X)$, we can easily prove that $f_{a,j}(\cdot;p)$ belongs to $\mathcal{A}$, for all $p\in X^\star$ (by using partitions of unity on $X\times X$ and by approximating the duality pairing with finite linear combinations of elements in $\mathcal{A}$).

This observation together with (2.6) imply that the hypotheses of Lemma A.4 are fulfilled for each $f_{a,j}$, with $t=-(1-s/\mu)<0$. It follows that $\mathfrak{F}^{-1}(f_{a,j})$ belongs to $L^1(X;\mathcal{A})$ and that there exists $C>0$ such that

$$\left\|\mathfrak{F}^{-1}(f_{a,j})\right\|_1 \leqslant C a^{-1/\mu}.$$

Thus, for a large enough, the strict inequality $\|\sum_{j=1}^N \mathfrak{F}^{-1}(f_{a,j})\|_1 < 1$ holds. It follows that $\mathfrak{F}^{-1}(1+\sum_{j=1}^N f_{a,j})$ is invertible in $\widetilde{L^1}$, the minimal unitization of $L^1(X;\mathcal{A})$. Equivalently, $h_a\circ h_a^{-1}\equiv 1+\sum_{j=1}^N f_{a,j}$ is invertible in $\widetilde{\mathfrak{F}(L^1)}$, the minimal unitization of $\mathfrak{F}(L^1(X;\mathcal{A}))$. Its inverse will be denoted by $(h_a\circ h_a^{-1})^{(-1)}$.

(iv) We recall that $h_a^{-1}\in S^{-s}(X^\star)$. Then, by Lemma A.4 we get that $h_a^{-1}\in\mathfrak{F}(L^1(X))\subset\mathfrak{F}(L^1(X;\mathcal{A}))$. Thus $h_a^{-1}\circ(h_a\circ h_a^{-1})^{(-1)}$ is a well-defined element of $\mathfrak{F}(L^1(X;\mathcal{A}))$. Moreover, one readily gets $h_a\circ[h_a^{-1}\circ(h_a\circ h_a^{-1})^{(-1)}]=1$. For this, just think of h_a and h_a^{-1} as elements of the Moyal algebra $\mathcal{M}(\Xi)$ and interpret $(h_a\circ h_a^{-1})^{(-1)}\in\widetilde{\mathfrak{F}(L^1)}$ as an element of $\mathcal{S}'(\Xi)$. The needed associativity follows easily from the definition by duality of the composition law as stated in Remark 1.3. In the same way one obtains $[(h_a^{-1}\circ h_a)^{(-1)}\circ h_a^{-1}]\circ h_a=1$ in $\mathcal{M}(\Xi)$. In conclusion, there exists $a_0\geqslant-\inf h+1$ such that for any $a>a_0$ the symbol h_a possess an inverse with respect to the Moyal product

$$h_a^{(-1)} := h_a^{-1}\circ\left(h_a\circ h_a^{-1}\right)^{(-1)} = \left(h_a^{-1}\circ h_a\right)^{(-1)}\circ h_a^{-1}\in\mathcal{S}'(\Xi)$$

that also belongs to $\mathfrak{F}(L^1(X;\mathcal{A}))\subset\mathfrak{B}^B_{\mathcal{A}}$. The second equality follows from Remark 1.3 or Remark 1.4 by straightforward arguments.

(v) We define $\Phi^B_h(r_x):=h^{(-1)}_{-x}$ for $x<-a_0$. Then $\Phi^B_h(r_x)\in\mathfrak{F}(L^1(X;\mathcal{A}))\subset\mathfrak{B}^B_{\mathcal{A}}\cap\mathcal{S}'(\Xi)$, its norm is uniformly bounded for x in the given domain and $(h-x)\circ\Phi^B_h(r_x)=\Phi^B_h(r_x)\circ(h-x)=1$, as shown above. This allows us to obtain an extension to the half-strip $\{z=x+iy\mid x<-a_0,\ |y|<\delta\}$ for some $\delta>0$ by setting

$$\Phi^B_h(r_z) := \Phi^B_h(r_x)\circ\left\{1+(x-z)\Phi^B_h(r_x)\right\}^{(-1)}. \tag{2.7}$$

It follows that

$$(h-z)\circ\Phi^B_h(r_z)=\left\{(h-x)\circ\Phi^B_h(r_x)+(x-z)\Phi^B_h(r_x)\right\}\circ\left\{1+(x-z)\Phi^B_h(r_x)\right\}^{(-1)}=1.$$

We now prove that the map

$$\{z = x + iy \mid x < -a_0,\ |y| < \delta\} \ni z \mapsto \Phi_h^B(r_z) \in \mathfrak{F}\big(L^1(X;\mathcal{A})\big)$$

satisfies the resolvent equation. Let us choose two complex numbers z and z' in this domain and subtract the two equations

$$(h - z) \circ \Phi_h^B(r_z) = 1, \qquad (h - z') \circ \Phi_h^B(r_{z'}) = 1 \tag{2.8}$$

in order to get $(h - z) \circ \{\Phi_h^B(r_z) - \Phi_h^B(r_{z'})\} + (z' - z)\Phi_h^B(r_{z'}) = 0$. By multiplying at the left with $\Phi_h^B(r_z)$ and by using the associativity, we obtain the resolvent equation

$$\Phi_h^B(r_z) - \Phi_h^B(r_{z'}) = (z - z')\Phi_h^B(r_z) \circ \Phi_h^B(r_{z'}).$$

Now, setting $z' = \bar{z} = x - iy$ with $y > 0$ and taking norms we get

$$\big\|\Phi_h^B(r_z)\big\|_{\mathfrak{B}_\mathcal{A}^B} \leqslant y^{-1}.$$

With this estimate and formula (2.7), the function $z \mapsto \Phi_h^B(r_z)$ can be extended to the domain $\mathbb{C} \setminus [-a_0, +\infty)$, preserving the relations (2.8). The resolvent equation may be proved in a similar way to hold on the entire domain $\mathbb{C} \setminus [-a_0, +\infty)$ and analyticity of the defined function follows in an evident way.

(vi) Thus we have got an analytic map $\mathbb{C} \setminus [-a_0, +\infty) \ni z \to \Phi_h^B(r_z) \in \mathfrak{B}_\mathcal{A}^B$ satisfying the resolvent equation and the symmetry condition. A general argument presented in [1, p. 364] allows now to extend in a unique way the map Φ_h^B to a C^*-algebra morphism $C_0(\mathbb{R}) \to \mathfrak{B}_\mathcal{A}^B$. □

2.2. The represented version

This subsection consists only in the proof of the represented version on Theorem 1.8.

Proof of Corollary 1.10. We shall first consider the case $V = 0$ and then add V as a bounded perturbation.

Let us denote by $\mathcal{D}_z$ the range of the operator $\mathfrak{Op}^A[\Phi_h^B(r_z)] \in \mathcal{B}(\mathcal{H})$. By the resolvent identity it follows immediately that it is a subspace of $\mathcal{H}$ that does not depend on $z \in \mathbb{C} \setminus \mathbb{R}$. Thus we set $\mathcal{D}_z \equiv \mathcal{D}$. Since $h \in \mathcal{M}(\Xi)$, one has $\mathfrak{Op}^A(h) \in \mathcal{L}[\mathcal{S}(X)] \cap \mathcal{L}[\mathcal{S}'(X)]$. We interpret it as a linear operator in $\mathcal{S}'(X)$ and set $H_h(A, 0) := \mathfrak{Op}^A(h)|_\mathcal{D}$.

Now, by applying $\mathfrak{Op}^A$ to (1.7) we get

$$\big\{H_h(A, 0) - z\mathbf{1}\big\}\mathfrak{Op}^A\big[\Phi_h^B(r_z)\big] = \mathbf{1}$$

and

$$\mathfrak{Op}^A\big[\Phi_h^B(r_z)\big]\big\{\mathfrak{Op}^A(h) - z\mathbf{1}_{\mathcal{S}(X)}\big\} = \mathbf{1}_{\mathcal{S}(X)}.$$

The first identity shows that $H_h(A, 0)\mathcal{D} \subset \mathcal{H}$. Straightforwardly it is hermitian. The second equality implies that $\mathcal{S}(X) \subset \mathcal{D}$ and thus $\mathcal{D}$ is dense in $\mathcal{H}$. By the first equality above the ranges of $H_h(A,0) \pm i$ both coincide with $\mathcal{H}$. Thus, by the fundamental criterion of self-adjointness, $H_h(A,0)$ is self-adjoint.

By construction, $\{\mathfrak{Op}^A[\Phi_h^B(r_z)] \mid z \in \mathbb{C} \setminus \mathbb{R}\}$ is the resolvent family of $H_h(A,0)$, which is therefore affiliated to $\mathfrak{Op}^A(\mathfrak{B}_{\mathcal{A}}^B)$.

Then we define the standard operator sum $H_h(A,V) := H_h(A,0) + V : \mathcal{D} \to \mathcal{H}$. Using the second resolvent equation and the Neumann series the conclusion of the corollary follows easily using [28, Proposition 2.6] as in [26]. A different proof could start from the result of Corollary 1.9. □

3. The essential spectrum

In this section, we shall consider certain abelian twisted C^*-dynamical system $(\mathcal{A},\theta,\omega,X)$ and explain how to calculate the essential spectrum of any observable affiliated to the twisted crossed product algebra $\mathcal{A} \rtimes_\theta^\omega X$. This result is contained in Proposition 3.1. Then, by using the concrete affiliation criterion obtained in Section 1.2, we shall particularize the result to the case of magnetic Schrödinger operators and prove Theorem 1.11.

We start by recalling some definitions in relation with spectral analysis in a C^*-algebraic framework. Let $\pi : \mathfrak{C} \to \mathfrak{C}'$ be a morphism between two C^*-algebras and Φ an observable affiliated to $\mathfrak{C}$. Then $\pi[\Phi] : C_0(\mathbb{R}) \to \mathfrak{C}'$ given by $(\pi[\Phi])(\eta) := \pi[\Phi(\eta)]$ is an observable affiliated to $\mathfrak{C}'$, called *the image of Φ through π*. If $\mathfrak{K}$ is an ideal of $\mathfrak{C}$, *the $\mathfrak{K}$-essential spectrum of Φ* is

$$\sigma_{\mathfrak{K}}(\Phi) := \bigl\{\lambda \in \mathbb{R} \bigm| \text{if } \eta \in C_0(\mathbb{R}) \text{ and } \eta(\lambda) \neq 0, \text{ then } \Phi(\eta) \notin \mathfrak{K}\bigr\}.$$

If π denotes the canonical morphism $\mathfrak{C} \to \mathfrak{C}/\mathfrak{K}$, one has $\sigma_{\mathfrak{K}}(\Phi) = \sigma_{\{0\}}(\pi[\Phi])$.

In the particular situation when $\mathfrak{C}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, any self-adjoint operator H in $\mathcal{H}$ defines an observable Φ_H affiliated to $\mathfrak{C}$ by its functional calculus $C_0(\mathbb{R}) \ni \eta \mapsto \eta(H) \equiv \Phi_H(\eta)$ if and only if $\Phi_H(r_z) \in \mathfrak{C}$ for some $z \notin \mathbb{R}$. Then $\sigma_{\{0\}}(\Phi_H)$ is the usual spectrum $\sigma(H)$ of H. Moreover, if $\mathfrak{C}$ contains the ideal $\mathcal{K}(\mathcal{H})$ of compact operators on $\mathcal{H}$, then $\sigma_{\mathcal{K}(\mathcal{H})}(\Phi_H)$ is equal to the essential spectrum $\sigma_{\mathrm{ess}}(H)$ of H. Here we shall be mainly interested in the usual spectrum and in the essential spectrum. The need for the $\mathfrak{K}$-essential spectrum with $\mathfrak{K}$ different from $\{0\}$ or $\mathcal{K}(\mathcal{H})$ will appear only in Section 4.

3.1. *The abstract construction*

In this subsection $(\mathcal{A},\theta,\omega,X)$ will be an abelian twisted C^*-dynamical system. Thus X is an abelian, second countable locally compact group and $\mathcal{A}$ an abelian, unital C^*-subalgebra of $BC_{\mathrm{u}}(X)$ stable under translations and containing $C_0(X)$. We recall that the spectrum $S_{\mathcal{A}}$ of $\mathcal{A}$ is a compactification of X, endowed with an action $\tilde{\theta}$ of X by homeomorphisms. For any quasi-orbit F we define

$$\mathcal{A}^F := \bigl\{a \in C(S_{\mathcal{A}}) \bigm| a|_F = 0\bigr\}.$$

By identifying $\mathcal{A}$ with $C(S_{\mathcal{A}})$, $\mathcal{A}^F$ will be an invariant ideal of $\mathcal{A}$. Obviously the unitary group $\mathcal{U}(\mathcal{A}^F)$ of the multiplier algebra of $\mathcal{A}^F$ contains the unitary group $\mathcal{U}(\mathcal{A})$ of $\mathcal{A}$. Consequently, the abelian twisted dynamical system $(\mathcal{A}^F,\theta,\omega,X)$ obtained by replacing $\mathcal{A}$ with $\mathcal{A}^F$ and performing suitable restrictions is well defined. Furthermore, the twisted crossed product $\mathcal{A}^F \rtimes_\theta^\omega X$ may be identified with an ideal of $\mathcal{A} \rtimes_\theta^\omega X$ [34, Proposition 2.2].

In order to have an explicit description of the quotient, let us first note that $\mathcal{A}/\mathcal{A}^F$ is canonically isomorphic to the unital C^*-algebra $C(F)$ of all continuous functions on F. The natural action of X on $b \in C(F)$ is given by $(\theta_x b)(\mathfrak{z}) = b[\tilde{\theta}(x, \mathfrak{z})]$ for each $x \in X$ and $\mathfrak{z} \in F$. Now, for each $x, y \in X$, the restriction of $\omega(x, y) \in \mathcal{U}(\mathcal{A})$ to F gives rise to a 2-cocycle $\omega_F : X \times X \to \mathcal{U}(C(F))$. Thus $(C(F), \theta, \omega_F, X)$ is a well-defined abelian twisted C^*-dynamical system. Moreover the quotient $\mathcal{A} \rtimes_\theta^\omega X / \mathcal{A}^F \rtimes_\theta^\omega X$ may be identified with the corresponding twisted crossed product $C(F) \rtimes_\theta^{\omega_F} X$. This follows from [34, Proposition 2.2] if $\mathcal{A}$ is separable. For the non-separable case, just perform obvious modifications in the proof of [13, Theorem 2.10] to accommodate the 2-cocycle. Let us recall that a_F denotes the restriction of $a \in \mathcal{A} \equiv C(S_{\mathcal{A}})$ to F. Then the image of $\phi \in L^1(X; \mathcal{A})$ through the canonical morphism $\pi_F : \mathcal{A} \rtimes_\theta^\omega X \to C(F) \rtimes_\theta^{\omega_F} X$ is the element of $L^1(X; C(F))$ given by $(\pi_F[\phi])(x) = [\phi(x)]_F$ for all $x \in X$.

Let us consider a covering $\{F_\nu\}_\nu$ of $F_{\mathcal{A}}$ by quasi-orbits. At the algebraic level, the covering requirement reads $\bigcap_\nu \mathcal{A}^{F_\nu} = C_0(X)$. It implies the equality

$$\bigcap_\nu \big(\mathcal{A}^{F_\nu} \rtimes_\theta^\omega X\big) = C_0(X) \rtimes_\theta^\omega X.$$

By putting all these together one obtains, cf. [22, Proposition 1.5], the following proposition.

Proposition 3.1. *Let $\{F_\nu\}_\nu$ be a covering of $F_{\mathcal{A}}$ by quasi-orbits.*

(i) *There exists an injective morphism*

$$\mathcal{A} \rtimes_\theta^\omega X / C_0(X) \rtimes_\theta^\omega X \hookrightarrow \prod_\nu C(F_\nu) \rtimes_\theta^{\omega_{F_\nu}} X.$$

(ii) *If Φ is an observable affiliated to $\mathcal{A} \rtimes_\theta^\omega X$ and π_{F_ν} denotes the canonical surjective morphism $\mathcal{A} \rtimes_\theta^\omega X \to C(F_\nu) \rtimes_\theta^{\omega_{F_\nu}} X$, then, with $\mathfrak{K} := C_0(X) \rtimes_\theta^\omega X$, we have*

$$\sigma_{\mathfrak{K}}(\Phi) = \overline{\bigcup_\nu \sigma\big(\pi_{F_\nu}[\Phi]\big)}. \tag{3.1}$$

We now introduce a represented version of this proposition in the Hilbert space $\mathcal{H}$. Let $\lambda \in C(X; C(X; \mathbb{T}))$ be a 1-cochain satisfying the relation

$$\lambda(x)\theta_x\big[\lambda(y)\big]\lambda(x+y)^{-1} = \omega(x, y) \quad \text{for all } x, y \in X. \tag{3.2}$$

It was proved in [28, Proposition 2.14] that such a pseudo-trivialization function λ always exists. The associated representation of $\mathcal{A} \rtimes_\theta^\omega X$ in $\mathcal{B}(\mathcal{H})$ defined by (1.6), but with λ^A replaced by λ, is denoted by $\mathfrak{Rep}^\lambda$. We recall from [28, Proposition 2.17] that $\mathfrak{Rep}^\lambda$ is irreducible and faithful and that $\mathfrak{Rep}^\lambda(C_0(X) \rtimes_\theta^\omega X)$ is equal to $\mathcal{K}(\mathcal{H})$. If Φ is an observable affiliated to $\mathcal{A} \rtimes_\theta^\omega X$, then the left-hand side term of (3.1) is equal to $\sigma_{\text{ess}}(\mathfrak{Rep}^\lambda(\Phi))$, and it does not depend on a particular choice of λ.

In order to construct a faithful representation of $C(F_\nu) \rtimes_\theta^{\omega_{F_\nu}} X$ in $\mathcal{H}$, we rely on the natural realization of the restriction of $\mathcal{A}$ to a quasi-orbit mentioned in Section 1.3. Let F be a quasi-orbit and $\mathfrak{z}$ an element of $F_{\mathcal{A}}$ that generates it. Then, for any $b \in C(F)$ and $x \in X$, set $b_{\mathfrak{z}}(x) :=$

$b[\tilde{\theta}(x,\mathfrak{z})]$. By taking into account the surjectivity of the morphism $\mathcal{A}\to C(F)$ and the continuity of translations in $\mathcal{A}\subset BC_{\mathrm{u}}(X)$, one easily sees that $b_{\mathfrak{z}}:X\to\mathbb{C}$ belongs to $BC_{\mathrm{u}}(X)$. Furthermore, the induced action of X on $b_{\mathfrak{z}}$ coincides with the natural action of X on $BC_{\mathrm{u}}(X)$. One has thus obtained an embedding of $C(F)$ in $BC_{\mathrm{u}}(X)$. By an abuse of notation, we shall keep writing b for $b_{\mathfrak{z}}$, and $C(F)$ for the corresponding C^*-subalgebra of $BC_{\mathrm{u}}(X)$.

Now, by choosing any 1-cochain $\lambda^F\in C(X;C(X;\mathbb{T}))$ satisfying the pseudo-triviality relation (3.2) with $\lambda=\lambda^F$ and $\omega=\omega_F$, one can construct the faithful Schrödinger representation $\mathfrak{Rep}^{\lambda^F}$ of the algebra $C(F)\rtimes_\theta^{\omega_F}X$. Thus, if Φ is the observable affiliated to $\mathcal{A}\rtimes_\theta^{\omega}X$ of Proposition 3.1, then each observable $\pi_{F_\nu}(\Phi)$ can be represented as an observable affiliated to a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ and having the same spectrum. This remark makes the calculation of the right-hand side terms in (3.1) more concrete. The particular case treated in Theorem 1.11 is proved now.

3.2. Application to magnetic Schrödinger operators

We particularize the above construction to the case of a magnetic 2-cocycle ω^B. So, we consider a magnetic field B whose components belong to $\mathcal{A}$. We shall need the following parametrized formula: for $q,x,y\in X$

$$\omega^B(q;x,y)=\exp\left\{-i\sum_{j,k=1}^{N}x_jy_k\int_0^1\mathrm{d}s\int_0^1\mathrm{d}t\,sB_{jk}(q+sx+sty)\right\}. \tag{3.3}$$

We are now in a position to prove Theorem 1.11. It consists essentially in an application of Proposition 3.1 together with a partial Fourier transformation.

Proof of Theorem 1.11. Let us fix a quasi-orbit F_ν; obviously $\omega^B|_{F_\nu}=\omega^{B_{F_\nu}}$ with natural identifications. Then the morphism

$$\mathfrak{F}\big(L^1(X;\mathcal{A})\big)\ni f\mapsto\mathfrak{F}\big(\pi_{F_\nu}\big[\mathfrak{F}^{-1}(f)\big]\big)\in\mathfrak{F}\big(L^1\big(X;C(F)\big)\big)$$

extends to a surjective morphism $\tilde{\pi}_{F_\nu}:\mathfrak{B}_{\mathcal{A}}^B\to\mathfrak{B}_{C(F_\nu)}^{B_{F_\nu}}$. The equality (3.1) can then be rewritten in the framework of $\mathfrak{B}_{\mathcal{A}}^B$ and for the observable $\Phi_{h,V}^B$:

$$\sigma_{\mathrm{ess}}\big(\Phi_{h,V}^B\big)=\overline{\bigcup_\nu\sigma\big(\tilde{\pi}_{F_\nu}\big[\Phi_{h,V}^B\big]\big)}.$$

The result follows now from the central observation that $\tilde{\pi}_{F_\nu}[\Phi_{h,V}^B]$ is equal to $\Phi_{h,V_{F_\nu}}^{B_{F_\nu}}$, by considering faithful representations (i) of $\mathfrak{B}_{\mathcal{A}}^B$ through $\mathfrak{Op}^A$ and (ii) of $\mathfrak{B}_{C(F_\nu)}^{B_{F_\nu}}$ through $\mathfrak{Op}^{A_\nu}$ and by applying Corollary 1.10. □

4. Non-propagation properties

As mentioned earlier, the result of non-propagation is mainly an adaptation of [2] in the presence of a magnetic field together with the use of an approximate unit introduced in [23]. Since all notations and concepts have already been introduced, it only remains to prove Theorem 1.12. We start by recalling an easy result of [2, Lemma 1].

Lemma 4.1. *Let $\mathfrak{K}$ be an ideal in a C^*-algebra $\mathfrak{C}$ and Φ an observable affiliated to $\mathfrak{C}$. If $\eta \in C_0(\mathbb{R})$ and $\eta(\lambda) = 0$ for all $\lambda \in \sigma_{\mathfrak{K}}(\Phi)$, then $\Phi(\eta) \in \mathfrak{K}$.*

Proof of Theorem 1.12. Let $\mathfrak{K} := \mathfrak{B}^B_{\mathcal{A}^F} \equiv \mathfrak{F}(\mathfrak{C}^B_{\mathcal{A}^F})$, the ideal of $\mathfrak{B}^B_{\mathcal{A}}$ related to the quasi-orbit F, and let $\tilde{\pi}_F : \mathfrak{B}^B_{\mathcal{A}} \to \mathfrak{B}^{B_F}_{C(F)}$ be the corresponding morphism of kernel $\mathfrak{K}$. We consider the observable $\Phi^B_{h,V}$ that is affiliated to $\mathfrak{B}^B_{\mathcal{A}}$ by Theorem 1.8. Then, by taking into account the equality $\sigma_{\mathfrak{K}}(\Phi^B_{h,V}) = \sigma(\tilde{\pi}_F[\Phi^B_{h,V}]) = \sigma(\Phi^{B_F}_{h,V_F})$, the hypothesis on η and Lemma 4.1, we see that $\Phi^B_{h,V}(\eta)$ belongs to $\mathfrak{K}$.

By representing faithfully $\mathfrak{B}^B_{\mathcal{A}}$ in $\mathcal{B}(\mathcal{H})$ through $\mathfrak{Op}^A$ one has that $\mathfrak{Op}^A(\Phi^B_{h,V}(\eta))$ belongs to the ideal $\mathfrak{Op}^A(\mathfrak{K})$. For the final step of the proof, one only has to remark that the family $\{\mathbf{1} - \chi_W(Q)\}_{W \in \mathcal{N}_F}$ is an approximate unit in $\mathcal{B}(\mathcal{H})$ for $\mathfrak{Op}^A(\mathfrak{B}^B_{\mathcal{A}^F}) \equiv \mathfrak{Rep}^A(\mathfrak{C}^B_{\mathcal{A}^F})$, which is straightforward by the description of this type of algebras given in [28, Proposition 2.6]. □

5. Examples

In this last section, we illustrate Theorem 1.11 on the essential spectrum by choosing concrete examples of algebras $\mathcal{A}$. A similar transcription of Theorem 1.12 on propagation for these concrete situations could also be performed. Since an adaptation for the magnetic case of the examples given in [2] is rather straightforward, we leave this to the reader.

It is always assumed in the sequel that the components of the magnetic field B belong to $\mathcal{A} \cap BC^\infty(X)$ and that the scalar potential V belongs to $\mathcal{A}$. It is convenient to write $\sigma[H_h(B,V)]$ for $\sigma[H_h(A,V)]$ and $\sigma_{\mathrm{ess}}[H_h(B,V)]$ for $\sigma_{\mathrm{ess}}[H_h(A,V)]$ if $B = dA$. This is justified by the independence of these sets on a choice of a vector potential and, especially, by the abstract approach of Section 3.1.

The easiest and best-known situation is certainly when the algebra $\mathcal{A}$ is equal to $\mathbb{C} + C_0(X)$. In this situation $\mathcal{A}/C_0(X) \cong \mathbb{C}$ and one has $\sigma_{\mathrm{ess}}[H_h(B,V)] = \sigma[H_h(B_\infty, V_\infty)] = \sigma[H_h(B_\infty, 0)] + V_\infty$, where B_∞, V_∞ are, respectively, the limits of B and V at infinity. For instance, if $h(p) = |p|^2$ (giving the usual magnetic Schrödinger operator) in $X = \mathbb{R}^2$, we have for $B_\infty \neq 0$: $\sigma_{\mathrm{ess}}[H(B,V)] = (2\mathbb{N} + 1)B_\infty + V_\infty$, a translation by V_∞ of the familiar Landau levels. For $B_\infty = 0$ we clearly obtain $\sigma_{\mathrm{ess}}[H(B,V)] = [V_\infty, \infty)$. Some related results may be found in [35].

We shall now consider more complicated examples.

5.1. Vanishing oscillation

We take $\mathcal{A}$ to be the algebra $VO(X)$ of *vanishing oscillations functions*.

Definition 5.1. A bounded and uniformly continuous function a belongs to $VO(X)$ if for any $x \in X$, the difference $\theta_x[a] - a$ belongs to $C_0(X)$.

Obviously, $VO(X)$ is a unital C^*-algebra containing $C_0(X)$ and stable by translations. It contains also $C^{\mathrm{rad}}(X)$, the algebra of continuous functions that can be extended continuously to the radial compactification of X obtained by adding a sphere at infinity. But $VO(X)$ is in fact much larger than $C^{\mathrm{rad}}(X)$. For example, it also contains the set of all bounded C^1-functions with derivatives in $C_0(X)$. A simple typical example is $a(x) := f((1+|x|)^s)$ (suitably regularized at the origin), where f is a periodic C^1-function of one variable and s is a real number strictly smaller than 1.

To understand what the asymptotic operators should be, let us introduce the notion of *asymptotic range* of a real, bounded and continuous function φ defined on X. We write $\lambda \in \varphi(X)_{\mathrm{asy}}$ if and only if for any $\varepsilon > 0$, $\varphi^{-1}[(\lambda-\varepsilon, \lambda+\varepsilon)]$ is not relatively compact in X. Equivalently, $\lambda \in [\liminf_{x\to\infty}\varphi(x), \limsup_{x\to\infty}\varphi(x)]$, or there exists a divergent sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ such that $\varphi(x_n) \to \lambda$ when $n \to \infty$. We recall that a divergent sequence $\{x_n\}_{n\in\mathbb{N}}$ consists in a sequence of $x_n \in X$ such that $x_n \to \infty$ as $n \to \infty$. The interest in the set $\varphi(X)_{\mathrm{asy}}$ lies in the fact that for any $\mathcal{A} \equiv C(S_{\mathcal{A}})$ containing φ, the range of the restriction to $F_{\mathcal{A}}$ is exactly $\varphi(X)_{\mathrm{asy}}$, i.e. $\varphi(F_{\mathcal{A}}) = \varphi(X)_{\mathrm{asy}}$ with a loose notation.

A nice feature of $VO(X)$ is that it is the largest unital translational invariant C^*-subalgebra of $BC_{\mathrm{u}}(X)$ such that all quasi-orbits situated at infinity are reduced to points. This means that $R \equiv F_{VO(X)}$ admits the partition $R = \bigsqcup_{\mathfrak{z}\in R}\{\mathfrak{z}\}$ in (quasi-)orbits and $\varphi \mapsto (\varphi(\mathfrak{z}))_{\mathfrak{z}\in R}$ determines the embedding of $VO(X)/C_0(X)$ into $\prod_{\mathfrak{z}\in R}\mathbb{C}$. Using all these in conjunction with Theorem 1.11 leads to

$$\sigma_{\mathrm{ess}}\big[H_h(B,V)\big] = \overline{\bigcup_{\mathfrak{z}\in R}\sigma\big[H_h\big(B(\mathfrak{z}), V(\mathfrak{z})\big)\big]} = \overline{\bigcup_{\mathbf{x}\in\mathcal{R}}\sigma\big[H_h(B_{\mathbf{x}}, V_{\mathbf{x}})\big]},$$

where the second union is performed over the set $\mathcal{R}$ of all divergent sequences $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ such that there exist a constant magnetic field $B_{\mathbf{x}}$ and a number $V_{\mathbf{x}}$ satisfying $\sup_{j,k}|(B(x_n) - B_{\mathbf{x}})_{jk}| \to 0$ and $|V(x_n) - V_{\mathbf{x}}| \to 0$ as $n \to \infty$. We say that $B_{\mathbf{x}}$ and $V_{\mathbf{x}}$ are *asymptotic values* for B and V, respectively. Various particularizations are available.

5.2. Comparison with the results of [15]

The results of [15] are very interesting because large classes of unbounded potentials and magnetic fields are admitted. In the bounded case, however, they are entirely confined to the vanishing oscillation type of anisotropy, as we now argue.

For the comparison with the results of [15], we need

Lemma 5.2. *Let $r \in \mathbb{N}$ and $f \in BC(X) \cap C^r(X)$. Assume that $\partial^\alpha f \in C_0(X)$ for all $\alpha \in \mathbb{N}^N$ with $|\alpha| = r$. Then f belongs to $VO(X)$ and $\partial^\beta f \in C_0(X)$ for all $\beta \in \mathbb{N}^N$ with $1 \leqslant |\beta| \leqslant r-1$.*

Proof. Since we were not able to locate this result in the literature, we sketch its proof. Let us first state three remarks which are easily proved.

(i) Under the hypotheses on f, one has $\partial^\beta f \in BC(X)$ for all $\beta \in \mathbb{N}^N$ with $1 \leqslant |\beta| \leqslant r-1$, cf. for example [21].
(ii) If $g \in BC^1(X)$ and $\partial_j g \in C_0(X)$ for all $j \in \{1,\dots,N\}$, then $g \in VO(X)$.
(iii) If $h \in BC^1(X)$ and $\partial_j h \in VO(X)$, then $\partial_j h \in C_0(X)$.

Now, if $r = 1$, the result is obtained by (ii). If $r \geqslant 2$, let $\beta \in \mathbb{N}^N$ with $|\beta| = r-2$ and set $h := \partial^\beta f \in BC^2(X)$ by (i). For each $k \in \{1,\dots,N\}$, $\partial_k h$ belongs to $VO(X)$ by (ii), and then to

$C_0(X)$ by (iii). By varying β and k, one obtains $\partial^\gamma f \in C_0(X)$ for all $\gamma \in \mathbb{N}^N$ with $|\gamma| = N-1$. A bootstrap argument leads to the result. □

We describe now the results of [15] with slightly modified notations. They consider magnetic Schrödinger operators $H_h(A, V)$ for the particular case $h(p) \equiv h_0(p) := |p|^2$. The data V and $B = dA$ are subject to the following assumptions. There exist $q, r \in \mathbb{N}$ and some smooth function $\rho : X \to [1, \infty)$ with $\rho(x) \to \infty$ when $|x| \to \infty$, which is also tempered in a sense that is not important here [15, Eq. (1.14)], such that

(i) $V = V_0 + \sum_{l=1}^q V_l^2$,
(ii) $V_0 \geqslant -C_1$, $V_0 \in C^1(X)$ and $V_l \in C^{r+2}(X)$ for $l = 1, \dots, q$,
(iii) $\sum_{|\alpha|=1} |\partial^\alpha V_0| + \sum_{|\alpha|=r+2} \sum_{l=1}^q |\partial^\alpha V_l| \leqslant C_2 \rho^{-1}$,
(iv) for all j, k, $B_{jk} \in C^{r+3}(X)$ and $\sum_{|\alpha|=r+1}^{r+3} \rho^{|\alpha|-r-1} |\partial^\alpha B_{jk}| \leqslant C_3 \rho^{-1}$.

Under these assumptions it is proved that

$$\sigma_{\text{ess}}\big[H_{h_0}(B, V)\big] = \overline{\bigcup_{\mathbf{x} \in \mathcal{R}} \sigma\big[H_{h_0}(A_{\mathbf{x}}, V_{\mathbf{x}})\big]}, \tag{5.1}$$

where $\mathcal{R}$ is the set of divergent sequences $\mathbf{x} = \{x_n\}_{n \in \mathbb{N}}$ such that the following limits exist:

(a) $v_0 = \lim_n V_0(x_n)$ and $v_l^\alpha = \lim_n (\partial^\alpha V_l)(x_n)$ for $l = 1, \dots, q$ and $|\alpha| \leqslant r+1$,
(b) $B^\alpha = \lim_n (\partial^\alpha B)(x_n)$ for $|\alpha| \leqslant r$.

Then the asymptotic operators $H_{h_0}(A_{\mathbf{x}}, V_{\mathbf{x}})$ are constructed with the scalar potential

$$V_{\mathbf{x}}(x) := v_0 + \sum_{l=1}^q \bigg(\sum_{|\alpha| \leqslant r+1} \frac{v_l^\alpha}{\alpha!} x^\alpha \bigg)^2$$

and the magnetic potential

$$A_{\mathbf{x}}(x) := \sum_{|\alpha| \leqslant r} \frac{B^\alpha \cdot x}{\alpha! (2 + |\alpha|)} x^\alpha.$$

Let us see how the hypotheses and the conclusion look like when V and B are bounded. We ignore the temperedness condition; the fact that ρ diverges at infinity implies that the left-hand sides of the conditions (iii) and (iv) belong to $C_0(X)$. Lemma 5.2 can be applied and thus V and B_{jk} belong to $VO(X)$. In the bounded case the anisotropy covered by [15] is surely of the vanishing oscillation type.

To understand the conclusion under the extra condition that V and B are bounded, note that the same Lemma 5.2 says that all the derivatives of strictly positive order are in $C_0(X)$, thus the only non-null constant coefficients in (a) and (b) are those corresponding to $\alpha = 0$. Then (5.1) coincides with our result described above.

5.3. Mixed algebras

In the examples developed above, the quasi-orbits are reduced to singletons. We shall introduce some algebras with more complicated quasi-orbits, leading to non-trivial asymptotic operators with variable coefficients. There is also a very nice type of anisotropy studied in [12] and [13] under the name *potentials belonging to the bumps algebra*. It would be interesting to work out the magnetic counterpart.

We first reconsider an example introduced in [22], to which we refer for details and comments. Let us introduce the algebra $AP(X)$ of all continuous, almost periodic functions on X [10, 16.2.1].

Definition 5.3. A bounded and continuous function a on X belongs to $AP(X)$ if and only if it satisfies one of the following equivalent condition:

(a) The set $\{\theta_x[a] \mid x \in X\}$ is relatively compact in $BC(X)$.
(b) For any $\varepsilon > 0$ there is a trigonometric polynomial b on X such that $\|a - b\|_{L^\infty} \leqslant \varepsilon$.

The set $AP(X)$ is a translational invariant unital C^*-subalgebra of $BC_{\mathrm{u}}(X)$ whose Gelfand spectrum is called *the Bohr group* (denoted by T). All continuous functions on X which are periodic with respect to some closed subgroup Γ of X with compact quotient X/Γ lie in $AP(X)$, but there are many others.

We can consider the algebra $\mathcal{A} := \langle VO(X) \cdot AP(X)\rangle$ generated by $VO(X)$ and $AP(X)$. It is obviously a unital C^*-subalgebra of $BC_{\mathrm{u}}(X)$ containing $C_0(X)$ and stable by translations. Its Gelfand spectrum is the disjoint union $S_{\mathcal{A}} = X \sqcup (R \times T)$, where R is the part at infinity of the Gelfand spectrum of $VO(X)$. The relevant quasi-orbits are $\{\{\mathfrak{z}\} \times T \equiv T\}_{\mathfrak{z} \in R}$. This is by no means a general result; it expresses the fact that $VO(X)$ and $AP(X)$ are *asymptotically independent*, see [24] and references therein. Actually $AP(X)$ could be replaced by any C^*-algebra of *minimal functions* [23,24].

Instead of considering arbitrary elements of this algebra $\mathcal{A}$, let us concentrate on a simple example. Assume for simplicity that $V = 0$ and that each component of the magnetic field is a product of an element of $VO(X)$ and of an element of $AP(X)$, i.e. $B_{jk} = C_{jk} D_{jk}$, with $C_{jk} \in VO(X)$ and $D_{jk} \in AP(X)$. Let us once again invoke the asymptotic values of the matrix valued function $C := \{C_{jk}\}_{j,k=1}^{N}$: $C_{\mathbf{x}}$ is an asymptotic value if and only if there exists a divergent sequence $\mathbf{x} = \{x_n\}_{n \in \mathbb{N}}$ such that $\sup_{j,k} |(C(x_n) - C_{\mathbf{x}})_{jk}| \to 0$ as $n \to \infty$. Then one has

$$\sigma_{\mathrm{ess}}\big[H_h(B,0)\big] = \overline{\bigcup_{\mathbf{x}} \sigma\big[H_h(B_{\mathbf{x}},0)\big]},$$

where $B_{\mathbf{x}}$ is the magnetic field whose components are given by $(B_{\mathbf{x}})_{jk} := (C_{\mathbf{x}})_{jk} D_{jk} \in AP(X)$. The asymptotic values taken by C at infinity serve as coupling constants for the magnetic fields of the asymptotic operators, a phenomenon already observed in [22] for Schrödinger operators without magnetic field. Similarly, if $B_{jk} = C_{jk} + D_{jk}$, with the same assumptions on the functions C_{jk} and D_{jk}, the asymptotic operators are constructed with the almost periodic magnetic fields $\{C_{\mathbf{x}} + D\}_{\mathbf{x} \in R}$.

5.4. Cartesian anisotropy

In this paragraph we consider another type of spacial anisotropy, which is called *Cartesian*. The algebra $C^{\mathrm{cart}}(X)$ consists in the set of all continuous functions on X that can be extended to a hypercube compactifying X. We refer to [41] for a precise definition of this algebra and for an extensive study of Schrödinger operators related to this anisotropy (in the absence of magnetic field). Let us simply mention that the quasi-orbits are hypercubes of lower dimensions. We shall restrict here our investigation to a single example in the space $\mathbb{R}^2$. In this situation, the set of quasi-orbits consists of 4 closed segments and 4 points (corners).

For $N = 2$, the magnetic field has only one component B orthogonal to the space $\mathbb{R}^2$. Let us assume for simplicity that $B(x_1, x_2) = B_1(x_1)B_2(x_2) + B_0(x_1, x_2)$, where B_0 belongs to $C_0(\mathbb{R}^2)$ and $B_j(x_j) \to b_j^{\pm} \in \mathbb{R}$ as $x_j \to \pm\infty$. Let also V be of the form $V(x_1, x_2) = V_1(x_1)V_2(x_2) + V_0(x_1, x_2)$, where V_0 belongs to $C_0(\mathbb{R}^2)$ and $V_j(x_j) \to v_j^{\pm} \in \mathbb{R}$ as $x_j \to \pm\infty$. Then one has

$$\begin{aligned}\sigma_{\mathrm{ess}}\big[H_h(B, V)\big] = {}& \sigma\big[H_h\big(b_2^- B_1, v_2^- V_1\big)\big] \cup \sigma\big[H_h\big(b_2^+ B_1, v_2^+ V_1\big)\big] \\ & \cup \sigma\big[H_h\big(b_1^- B_2, v_1^- V_2\big)\big] \cup \sigma\big[H_h\big(b_1^+ B_2, v_1^+ V_2\big)\big].\end{aligned}$$

We stress that each asymptotic operator has a magnetic field that depends only on one variable. This kind of two-dimensional magnetic Schrödinger operators was studied in [16] and [25] and exhibits a band spectrum.

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Appendix A. Some technical results

Let us now state and prove the auxiliary technical results used in the proof of the affiliation criterion.

Lemma A.1. *Assume that the components of the magnetic field B belong to $\mathcal{A} \cap BC^\infty(X)$. Then γ^B belongs to $C^\infty_{\mathrm{pol}}(X \times X; \mathcal{A})$, or more precisely*:

(a) *for each $x, y \in X$, $\gamma^B(\cdot; x, y) \in \mathcal{A}$,*
(b) *for each $\alpha, \beta \in \mathbb{N}^N$, there exist $c > 0$, $s_1 \geqslant 0$ and $s_2 \geqslant 0$ such that for all $q, x, y \in X$*:

$$\big|\partial_x^\alpha \partial_y^\beta \gamma^B(q; x, y)\big| \leqslant c\langle x\rangle^{s_1} \langle y\rangle^{s_2}.$$

Proof. We use the explicit parametrized form of γ^B

$$\gamma^B(q; x, y) = \exp\left\{-i \sum_{j,k=1}^{N} x_j y_k \int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}s\, s B_{jk}\left(q - \frac{1}{2}x - \frac{1}{2}y + sx + st(y - x)\right)\right\}. \quad \text{(A.1)}$$

A careful examination of (A.1) leads directly to the results (a) and (b). See also the proof of Lemma 4.2 in [28]. □

Lemma A.2. *For each $j \in \{1, \dots, N\}$, each $\alpha, \beta, \gamma \in \mathbb{N}^N$ and each $\mu > \max\{1, s\}$ there exists $c > 0$ such that*

$$\left|\partial_p^\alpha \partial_k^\beta \partial_l^\gamma F_{a,j}(p; k, l)\right| \leqslant c a^{-1/\mu} \langle p\rangle^{s/\mu - 1 - |\alpha|} \langle k\rangle^s \langle l\rangle^{2s} \tag{A.2}$$

for all $p, k, l \in X^\star$ and $a \geqslant -\inf h + 1$.

Proof. It is enough to show that the expression

$$\sup_{t\in[0,1]} \left|\partial_p^\alpha \partial_k^\beta \partial_l^\gamma \left[(l_j - k_j)(\partial_j h)\big(p + (t-1)l - tk\big) h_a^{-1}(p - l)\right]\right| \tag{A.3}$$

is dominated by the right-hand side term of (A.2) with a constant c not depending on p, k, l and a.

It is easy to see that for any $\delta \in \mathbb{N}^N$, we have $\partial^\delta h_a^{-1} = h_a^{-1} u_{a,\delta}$, where $u_{a,\delta} \in S^{-|\delta|}(X^\star)$ uniformly in a. By using this, the Leibnitz formula and the inequality $\langle x + y\rangle \leqslant \sqrt{2}\langle x\rangle\langle y\rangle$, it follows straightforwardly that (A.3) is dominated by $c_1 h_a^{-1}(p - l)\langle p\rangle^{s-1-|\alpha|}\langle k\rangle^s \langle l\rangle^s$ for some $c_1 > 0$ independent of p, k, l and a. Furthermore, by using the ellipticity of h, we see that there exist $c_2 > 0$ and $c_3 > 0$ independent of p, l and a such that $h_a^{-1}(p - l) \leqslant c_2\langle l\rangle^s [a + c_3\langle p\rangle^s]^{-1}$ for all $p, l \in X^\star$. The final step consists in taking into account the inequality $a + c_3\langle p\rangle^s \geqslant \mu^{1/\mu}(\nu c_3)^{1/\nu} a^{1/\mu}\langle p\rangle^{s/\nu}$, valid for any $\mu \geqslant 1$, $\nu \geqslant 1$ with $\mu^{-1} + \nu^{-1} = 1$. □

In order to state the next lemma in its full generality, we need the definition.

Definition A.3. For $s \in \mathbb{R}$, $S^s(X^\star; \mathcal{A})$ denotes the set of all functions $f : X \times X^\star \to \mathbb{C}$ that satisfy:

(i) $f(\cdot; p) \in \mathcal{A}$ for all $p \in X^\star$,
(ii) $f(q; \cdot) \in C^\infty(X^\star)$, $\forall q \in X$, and for each $\alpha \in \mathbb{N}^N$

$$\sup_{q\in X} \left\| f(q; \cdot)\right\|_{s,\alpha} := \sup_{q\in X} \sup_{p\in X^\star} \left[\langle p\rangle^{-s+|\alpha|} \left|\partial_p^\alpha f(q; p)\right|\right] < \infty.$$

It is easily seen that the algebraic tensor product $\mathcal{A} \odot S^s(X^\star)$ is contained in $S^s(X^\star; \mathcal{A})$.

Lemma A.4. *Let f be an element of $S^t(X^\star; \mathcal{A})$ with $t < 0$. Then its partial Fourier transform $\mathfrak{F}^{-1}(f)$ is an element of $L^1(X; \mathcal{A})$ that satisfies for a suitable large integer m*

$$\left\|\mathfrak{F}^{-1}(f)\right\|_{L^1(X;\mathcal{A})} \leqslant c \max_{|\alpha|\leqslant m} \sup_{q\in X} \left\| f(q; \cdot)\right\|_{t,\alpha}. \tag{A.4}$$

Proof. This is a straightforward adaptation of the proof of [1, Proposition 1.3.3] (see also [1, Proposition 1.3.6]). We decided to present it in order to put into evidence the explicit bound (A.4). Actually, the arguments needed to control the behavior in the variable q are easy and we leave them to the reader; we take simply $f \in S^t(X^\star)$.

Since the case $t \leqslant -N$ is rather simple, we shall concentrate on the more difficult one: $-N < t < 0$. Let us first choose a cutoff function $\chi \in C_c^\infty(X)$ that is 1 in a neighborhood of 0. One has the estimates:

$$\begin{aligned}
&\left\|(1-\chi)\mathcal{F}^{-1}(f)\right\|_{L^1}\\
&\quad\leqslant C\sum_{|\alpha|=m}\left\||Q|^{-2m}(1-\chi)\mathcal{F}^{-1}\big(\partial^{2\alpha}f\big)\right\|_{L^1}\\
&\quad\leqslant C\bigg(\int_X \mathrm{d}x\,\big(1-\chi(x)\big)^2|x|^{-4m}\bigg)^{1/2}\sum_{|\alpha|=m}\left\|\partial^{2\alpha}f\right\|_{L^2}\\
&\quad\leqslant C'\bigg(\int_X \mathrm{d}x\,\big(1-\chi(x)\big)^2|x|^{-4m}\bigg)^{1/2}\bigg(\int_{X^\star}\mathrm{d}p\,\langle p\rangle^{2(t-2m)}\bigg)^{1/2}\max_{|\alpha|=2m}\|f\|_{t,\alpha},
\end{aligned}$$

where we take $m\in\mathbb{N}$ with $4m>N$ to make the integrals convergent.

We study now the behavior of $\mathcal{F}^{-1}(f)$ near the origin, a more difficult matter. Let us fix a second cutoff function $\varphi\in C^\infty(X^\star)$ such that $0\leqslant\varphi\leqslant 1$, $\varphi(p)=0$ for $|p|\leqslant 1$ and $\varphi(p)=1$ for $|p|\geqslant 2$. For $b>0$ we set $\varphi_b(p):=\varphi(bp)$. We have:

$$\left|\left\{\mathcal{F}^{-1}\big((1-\varphi_b)f\big)\right\}(y)\right|\leqslant\int_{|p|\leqslant 2/b}\mathrm{d}p\,\big|f(p)\big|\leqslant\|f\|_{t,0}\int_{|p|<2/b}\mathrm{d}p\,|p|^t\leqslant C\|f\|_{t,0}b^{-N-t}.$$

Moreover, if $m\in 2\mathbb{N}$ with $m\geqslant N+1$, then one has:

$$\begin{aligned}
&|y|^m\left|\big[\mathcal{F}^{-1}(\varphi_b f)\big](y)\right|\\
&\quad\leqslant C\sum_{|\alpha|=m}\left|\big[\mathcal{F}^{-1}\big(\partial^\alpha(\varphi_b f)\big)\big](y)\right|\\
&\quad\leqslant C\sum_{|\alpha|=m}\sum_{\beta\leqslant\alpha}C_\alpha^\beta b^{|\alpha-\beta|}\int_{X^\star}\mathrm{d}p\,\big|\big(\partial^{\alpha-\beta}\varphi\big)(bp)\big|\big|\big(\partial^\beta f\big)(p)\big|\\
&\quad\leqslant C'\max_{|\alpha|\leqslant m}\|f\|_{t,\alpha}\bigg\{\int_{|p|\geqslant 1/b}\mathrm{d}p\,|p|^{t-m}+\sum_{|\beta|<m}b^{m-|\beta|}\int_{1/b<|p|<2/b}\mathrm{d}p\,|p|^{t-|\beta|}\bigg\}\\
&\quad=C''\max_{|\alpha|\leqslant m}\|f\|_{t,\alpha}b^{m-N-t}.
\end{aligned}$$

By fixing $b:=|y|$, we get

$$\left|\big[\mathcal{F}^{-1}(\varphi_{|y|}f)\big](y)\right|\leqslant C''\max_{|\alpha|\leqslant m}\|f\|_{t,\alpha}|y|^{-N-t}.$$

The singularity at the origin is integrable, and putting all the inequalities together we obtain (A.4). □

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