



# Decay Estimates for Unitary Representations with Applications to Continuous- and Discrete-Time Models

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**Abstract.** We present a new technique to obtain polynomial decay estimates for the matrix coefficients of unitary operators. Our approach, based on commutator methods, applies to nets of unitary operators, unitary representations of topological groups, and unitary operators given by the evolution group of a self-adjoint operator or by powers of a unitary operator. Our results are illustrated with a wide range of examples in quantum mechanics and dynamical systems, as for instance Schrödinger operators, Dirac operators, quantum waveguides, horocycle flows, adjacency matrices, Jacobi matrices, quantum walks or skew products.

**Mathematics Subject Classification.** 22D10, 35Q40, 58J51, 81Q10.

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S. Richard is supported by the grant *Topological invariants through scattering theory and noncommutative geometry* from Nagoya University, and by JSPS Grant-in-Aid for scientific research C no 18K03328 & 21K03292, and on leave of absence from Univ. Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France.

R. Tiedra de Aldecoa is partially supported by the Chilean Fondecyt Grant 1210003.

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## 1. Introduction and Main Results

In recent papers [5, 27, 30, 34–39], it has been shown that one can combine certain tools from dynamical systems (averaging along the dynamics, ergodic theorems) and quantum mechanics (commutator methods) to determine spectral properties of various classes of continuous- and discrete-time models. In particular, a new criterion for strong mixing has been put into evidence in [27, 35]. Formally, it reads as follows: Let  $(U_j)$  be a family of unitary operators in a Hilbert space  $\mathcal{H}$ , let  $\ell_j$  be positive numbers such that  $\ell_j \rightarrow \infty$ , let  $A$  be a self-adjoint operator in  $\mathcal{H}$ , set  $D_j := \frac{1}{\ell_j}[A, U_j]U_j^{-1}$  and assume that the strong limit  $D := s\text{-}\lim_j D_j$  exists. Then

$$\lim_j |\langle \varphi, U_j \psi \rangle_{\mathcal{H}}| = 0 \quad \text{for all } \varphi \in \ker(D)^\perp \text{ and } \psi \in \mathcal{H}. \quad (1.1)$$

The operator  $D$  can be interpreted as a topological degree of the map  $j \mapsto U_j$ . Indeed, if one considers  $[A, \cdot]$  as a derivation along the set  $(U_j)$ , then  $D$  corresponds to a renormalised, operator-valued, winding number for the map  $j \mapsto U_j$  (the logarithmic derivative  $\frac{dz}{z}$  in the usual definition of winding number is replaced by the “logarithmic derivative”  $[A, U_j]U_j^{-1}$  associated to  $[A, \cdot]$ ). See [11–13, 16–18, 27, 36, 37, 39] for more details and examples.

In concrete situations, one usually seeks to get an explicit rate of decay in estimates like (1.1) in order to quantify the time propagation of the wave functions. This problem is a very broad and active field of research, with numerous results in a variety of setups. Decay of correlations, local decay estimates, pointwise decay estimates,  $L^p$  decay estimates, Strichartz estimates, microlocal estimates, propagation estimates, Morawetz estimates,... all are families of results related to this problem. In this paper, we pursue the study initiated

in [27, 35] and determine conditions that guarantee a polynomial rate of decay in (1.1). Our results are general, in the sense that they are stated first for general nets of unitary operators, then for unitary representations of topological groups, and finally for unitary operators given by the evolution group  $(e^{-itH})_{t \in \mathbb{R}}$  of a self-adjoint operator  $H$  or by the powers  $(U^n)_{n \in \mathbb{Z}}$  of a unitary operator  $U$ . Moreover, they apply to a wide range of models both in quantum mechanics and dynamical systems. And finally, they generalise to some extent the results of [15] (see also [19]) where the authors use commutator methods to establish abstract pointwise decay estimates for certain classes of self-adjoint operators. We refer to [2, 6, 20] for related results about pointwise decay estimates for self-adjoint and unitary operators.

Let us give a more detailed description of our results. In Sect. 2.1, we introduce our framework and determine sufficient conditions that guarantee a polynomial decay estimate

$$|\langle \varphi, U_j \psi \rangle_{\mathcal{H}}| \leq \frac{1}{\ell_j^n} c_{\varphi, \psi}, \quad \ell_j > 0, \quad n \in \mathbb{N}^*,$$

with  $c_{\varphi, \psi}$  a constant depending on  $\varphi, \psi$  (and  $n$ ) but not on  $j$ . A first set of conditions on  $D_j$  and  $\varphi, \psi$  leads to this estimate in the case  $n = 1$ , while more restrictive sets of conditions lead to this estimate for any fixed  $n \geq 1$  (Theorem 2.1). In addition, when the unitary operators  $U_j$  are given by a unitary representation  $\mathcal{U}$  of a topological group  $X$  and the scalars  $\ell_j$  are given by a proper length function on  $X$ , then we provide conditions ensuring that  $D$  commutes with  $\mathcal{U}$  and that  $\mathcal{U}$  has no nontrivial finite-dimensional unitary subrepresentation in  $\ker(D)^\perp$  (Proposition 2.3).

These general results are then applied in Sects. 2.2 and 2.3 to the case of a representation of  $\mathbb{R}$  given by an evolution group  $(e^{-itH})_{t \in \mathbb{R}}$  with self-adjoint generator  $H$  and to the case of a representation of  $\mathbb{Z}$  given by the powers  $(U^n)_{n \in \mathbb{Z}}$  of a unitary operator  $U$ . In the former case, the operators  $D_j$  can be written as Cesaro means

$$D_t := \frac{1}{t} \int_0^t d\tau e^{-i\tau H} (H + i)^{-1} [iH, A] (H - i)^{-1} e^{i\tau H}, \quad t > 0,$$

and our new results are the fact that  $D$  is decomposable in the spectral representation of  $H$  and criteria for the continuity or absolute continuity of the spectrum of  $H$  in  $\ker(D)^\perp$  (Lemma 2.4). In the latter case, the operators  $D_j$  can be written as Cesaro means

$$D_n = \frac{1}{n} \sum_{m=0}^{n-1} U^m ([A, U] U^{-1}) U^{-m}, \quad n \in \mathbb{N}^*,$$

and our new results are the fact that  $D$  is decomposable in the spectral representation of  $U$  and criteria for the continuity or absolute continuity of the spectrum of  $U$  in  $\ker(D)^\perp$  (Lemma 2.7). Furthermore, in Propositions 2.5 and 2.8 we pay a special attention to the particular cases  $[iH, A] = f(H)$  and  $[A, U] = \gamma(U)$  (with  $f$  and  $\gamma$  functions) which are important for applications.

In Sect. 3, we illustrate these abstract results with numerous examples. For some of them, the decay estimates we obtain are known, while for others

they are new. A similar dichotomy holds for the spectral results we obtain when we deal with representations admitting a self-adjoint or a unitary generator. However, the most striking feature of our approach does not really rely on any new result for a given example, but on its broad applicability. The whole variety of examples introduced in Sect. 3 is conveniently covered with the same philosophy and toolkit.

Since Sect. 3 contains a detailed presentation of each example, we just highlight here a few noticeable facts. First, we note that several important models of quantum mechanics and dynamical systems are discussed in Sect. 3. This is for example the case of Schrödinger operators, Dirac operators, quantum waveguides, horocycle flows, adjacency matrices, Jacobi matrices, quantum walks and skew products. Next, as mentioned at the beginning of the introduction, the operator  $D$  can sometimes be interpreted as a topological degree. This occurs for instance in the case of skew products, see Sect. 3.16. In other instances, the operator  $D$  can be expressed in terms of the square of an asymptotic velocity operator (a kinetic energy) for the unitary group under study. This occurs for instance in the case of quantum walks on  $\mathbb{Z}$ , see Sect. 3.14. Finally, in Sect. 3.1 we discuss the case of the left regular representation of a  $\sigma$ -compact locally compact Hausdorff group  $X$  with left Haar measure  $\mu$  and proper length function  $\ell$ . In that case, we obtain for any net  $(x_j)$  in  $X$  with  $x_j \rightarrow \infty$  and suitable  $\varphi, \psi \in L^2(X, \mu)$  the decay estimate

$$|\langle \varphi, \mathcal{U}(x_j)\psi \rangle_{\mathcal{H}}| \leq \frac{1}{\ell(x_j)} c_{\varphi, \psi}.$$

This estimate is similar to others in this paper, but with the interesting difference that in general the representation  $\mathcal{U}$  doesn't have either a self-adjoint generator or a unitary generator. It thus illustrates once again the fact that our approach is general, and not only applicable to families of unitary operators admitting a self-adjoint generator or unitary generator.

Finally, in “Appendices A and B”, we collect some technical results on commutators and regularity classes and on RAGE-type theorems for unitary operators.

**Notations:**  $\mathbb{N} := \{0, 1, 2, \dots\}$  is the set of natural numbers,  $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ ,  $\mathbb{R}_+ := (0, \infty)$ ,  $\mathbb{S}^1$  the complex unit circle,  $U(n)$  the group of  $n \times n$  unitary matrices, and  $\langle \cdot, \cdot \rangle := \sqrt{1 + |\cdot|^2}$ . Given a Hilbert space  $\mathcal{H}$ , we write  $\|\cdot\|_{\mathcal{H}}$  for its norm,  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$  for its scalar product (linear in the first argument),  $\mathcal{D}(A)$  for the domain of an operator in  $\mathcal{H}$ , and  $U(\mathcal{H})$  for the set of unitary operators on  $\mathcal{H}$ . Given two Hilbert spaces  $\mathcal{H}_1, \mathcal{H}_2$ , we write  $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$  (resp.  $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ ) for the set of bounded (resp. compact) operators from  $\mathcal{H}_1$  to  $\mathcal{H}_2$ . We also write  $\|\cdot\|_{\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)}$  for the norm of  $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ , and use the shorthand notations  $\mathcal{B}(\mathcal{H}_1) := \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$  and  $\mathcal{K}(\mathcal{H}_1) := \mathcal{K}(\mathcal{H}_1, \mathcal{H}_1)$ .

## 2. Decay Estimates for Unitary Representations

### 2.1. General Unitary Representations

We start with a general theorem on decay estimates for the matrix coefficients of unitary operators  $U_j$  in a Hilbert space  $\mathcal{H}$ . In the proof, we use standard results about commutators of operators recalled in “Appendix A”.

**Theorem 2.1** (Decay estimates). *Let  $(U_j)_{j \in J}$  be a net in  $U(\mathcal{H})$ , let  $(\ell_j)_{j \in J} \subset [0, \infty)$  satisfy  $\ell_j \rightarrow \infty$ , assume there exists a self-adjoint operator  $A$  in  $\mathcal{H}$  such that  $U_j \in C^1(A)$  for each  $j \in J$ , and suppose that the strong limit*

$$D := \text{s-lim}_j D_j \quad \text{with} \quad D_j := \frac{1}{\ell_j} [A, U_j] U_j^{-1}$$

*exists. Then*

- (a) *For each  $\varphi = D\tilde{\varphi} \in DD(A)$  and  $\psi \in \mathcal{D}(A)$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that*

$$|\langle \varphi, U_j \psi \rangle_{\mathcal{H}}| \leq \|(D - D_j)\tilde{\varphi}\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \frac{1}{\ell_j} c_{\varphi, \psi}, \quad \ell_j > 0.$$

*In particular,  $\lim_j \langle \xi, U_j \zeta \rangle_{\mathcal{H}} = 0$  for all  $\xi \in \ker(D)^\perp$  and  $\zeta \in \mathcal{H}$ .*

- (b) *Assume that  $D = D_j$  for all  $j \in J$ . Then for each  $\varphi \in DD(A)$  and  $\psi \in \mathcal{D}(A)$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that*

$$|\langle \varphi, U_j \psi \rangle_{\mathcal{H}}| \leq \frac{1}{\ell_j} c_{\varphi, \psi}, \quad \ell_j > 0.$$

- (c) *Assume that  $D = D_j$  for all  $j \in J$ , that  $D \in C^1(A)$ , and that  $[A, D] = DB$  with  $B \in C^{(n-1)}(A)$  ( $n \in \mathbb{N}^*$ ) and  $[D, B] = 0$ . Then for each  $\varphi \in D^n \mathcal{D}(A^n)$  and  $\psi \in \mathcal{D}(A^n)$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that*

$$|\langle \varphi, U_j \psi \rangle_{\mathcal{H}}| \leq \frac{1}{\ell_j^n} c_{\varphi, \psi}, \quad \ell_j > 0.$$

*Proof.* (a) Take  $\varphi = D\tilde{\varphi} \in DD(A)$ ,  $\psi \in \mathcal{D}(A)$ , and  $j \in J$  such that  $\ell_j > 0$ . Then we have

$$\begin{aligned} |\langle \varphi, U_j \psi \rangle_{\mathcal{H}}| &= | \langle (D - D_j)\tilde{\varphi}, U_j \psi \rangle_{\mathcal{H}} + \langle D_j \tilde{\varphi}, U_j \psi \rangle_{\mathcal{H}} | \\ &\leq \|(D - D_j)\tilde{\varphi}\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \frac{1}{\ell_j} | \langle [A, U_j] U_j^{-1} \tilde{\varphi}, U_j \psi \rangle_{\mathcal{H}} | \\ &\leq \|(D - D_j)\tilde{\varphi}\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \frac{1}{\ell_j} | \langle A\tilde{\varphi}, U_j \psi \rangle_{\mathcal{H}} | + \frac{1}{\ell_j} | \langle \tilde{\varphi}, U_j A\psi \rangle_{\mathcal{H}} | \\ &\leq \|(D - D_j)\tilde{\varphi}\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \frac{1}{\ell_j} c_{\varphi, \psi} \end{aligned}$$

with  $c_{\varphi, \psi} := \|A\tilde{\varphi}\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \|\tilde{\varphi}\|_{\mathcal{H}} \|A\psi\|_{\mathcal{H}}$ . This proves the first part of the claim. Since  $D = \text{s-lim}_j D_j$  and  $\ell_j \rightarrow \infty$ , we infer that  $\lim_j \langle \varphi, U_j \psi \rangle = 0$ , and thus the second part of the claim follows by the density of  $DD(A)$  in  $\overline{D\mathcal{H}} = \ker(D)^\perp$  and the density of  $\mathcal{D}(A)$  in  $\mathcal{H}$ .

(b) The claim is a direct consequence of point (a) in the case  $D = D_j$  for all  $j \in J$ .

(c) We prove the claim by induction on  $n \in \mathbb{N}^*$ . For  $n = 1$ , the claim is true due to point (b). For  $n - 1 \geq 1$ , we make the induction hypothesis that the claim is true. For  $n$ , we take  $\varphi \in D^n \mathcal{D}(A^n)$ ,  $\psi \in \mathcal{D}(A^n)$  and  $j \in J$  such that  $\ell_j > 0$ . Then, since  $D = D_j$  and  $\varphi = D\tilde{\varphi}$  with  $\tilde{\varphi} \in D^{n-1} \mathcal{D}(A^n)$ , we get

$$\langle \varphi, U_j \psi \rangle_{\mathcal{H}} = \langle D_j \tilde{\varphi}, U_j \psi \rangle_{\mathcal{H}} = \frac{1}{\ell_j} \langle A\tilde{\varphi}, U_j \psi \rangle_{\mathcal{H}} - \frac{1}{\ell_j} \langle \tilde{\varphi}, U_j A\psi \rangle_{\mathcal{H}}. \quad (2.1)$$

The induction hypothesis applies to the second term in (2.1) since  $\tilde{\varphi} \in D^{n-1}\mathcal{D}(A^n) \subset D^{n-1}\mathcal{D}(A^{n-1})$  and  $A\psi \in \mathcal{D}(A^{n-1})$ . So there exists  $c_{\tilde{\varphi}, A\psi} \geq 0$  such that

$$|\langle \tilde{\varphi}, U_j A\psi \rangle_{\mathcal{H}}| \leq \frac{1}{\ell_j^{n-1}} c_{\tilde{\varphi}, A\psi}. \quad (2.2)$$

For the first term in (2.1), we have  $\tilde{\varphi} = D^{n-1}\tilde{\tilde{\varphi}}$  with  $\tilde{\tilde{\varphi}} \in \mathcal{D}(A^n)$ . So, using the relations  $[A, D] = DB$  and  $[D, B] = 0$ , we get that

$$\begin{aligned} A\tilde{\varphi} &= (D^{n-1}A + [A, D^{n-1}])\tilde{\tilde{\varphi}} \\ &= (D^{n-1}A + \sum_{m=0}^{n-2} D^{n-2-m}[A, D]D^m)\tilde{\tilde{\varphi}} \\ &= D^{n-1}(A + (n-1)B)\tilde{\tilde{\varphi}} \end{aligned}$$

with  $(A + (n-1)B)\tilde{\tilde{\varphi}} \in \mathcal{D}(A^{n-1})$  due to the inclusions  $\tilde{\tilde{\varphi}} \in \mathcal{D}(A^n)$  and  $B \in C^{(n-1)}(A)$ . Therefore  $A\tilde{\varphi} \in D^{n-1}\mathcal{D}(A^{n-1})$  and  $\psi \in \mathcal{D}(A^n) \subset \mathcal{D}(A^{n-1})$ , and we infer from the induction hypothesis that there exists  $c_{A\tilde{\varphi}, \psi} \geq 0$  such that

$$|\langle A\tilde{\varphi}, U_j \psi \rangle_{\mathcal{H}}| \leq \frac{1}{\ell_j^{n-1}} c_{A\tilde{\varphi}, \psi}. \quad (2.3)$$

Finally, combining (2.1), (2.2) and (2.3), we obtain that

$$|\langle \varphi, U_j \psi \rangle_{\mathcal{H}}| \leq \frac{1}{\ell_j} \frac{1}{\ell_j^{n-1}} c_{A\tilde{\varphi}, \psi} + \frac{1}{\ell_j} \frac{1}{\ell_j^{n-1}} c_{\tilde{\varphi}, A\psi} = \frac{1}{\ell_j} c_{\varphi, \psi}$$

with  $c_{\varphi, \psi} := c_{A\tilde{\varphi}, \psi} + c_{\tilde{\varphi}, A\psi}$ .  $\square$

*Remark 2.2.* (a) If the operators  $U_j$  are given by a unitary representation, then the property of Theorem 2.1(a)  $\lim_j \langle \xi, U_j \zeta \rangle_{\mathcal{H}} = 0$  for  $\xi \in \ker(D)^\perp$  and  $\zeta \in \mathcal{H}$  amounts to a strong mixing property of the unitary representation in  $\ker(D)^\perp$ . See [27] for more information on this point.

(b) The set  $D^n\mathcal{D}(A^n)$  in Theorem 2.1(c) is always dense in  $\ker(D)^\perp$ , independently of the value of  $n \in \mathbb{N}^*$ . Indeed, since  $\mathcal{D}(A^n)$  is dense in  $\mathcal{H}$ , we have that  $D^n\mathcal{D}(A^n)$  is dense in  $\overline{D^n\mathcal{H}} = \ker(D^n)^\perp$ . But  $D$  is self-adjoint. So  $\ker(D^n) = \ker(D)$ , and thus  $D^n\mathcal{D}(A^n)$  is dense in  $\ker(D)^\perp$ .

(c) Sometimes the unitary operators  $U_j$  are given by the evolution group of a self-adjoint operator  $H$ , namely,  $(U_j)_{j \in J} = (e^{-itH})_{t > 0}$ . In this situation, a convenient operator  $A$  in Theorem 2.1 is often of the form  $\tilde{A} = (H+i)^{-1}A(H-i)^{-1}$  with  $A$  some self-adjoint operator such that  $(H-i)^{-1} \in C^1(A)$ . In that case, the estimates of Theorem 2.1 hold for vectors  $\varphi \in D^n\mathcal{D}((\tilde{A})^n)$  and  $\psi \in \mathcal{D}((\tilde{A})^n)$ . However, since  $A$  is simpler than  $\tilde{A}$  and since  $\mathcal{D}(A^n) \subset \mathcal{D}((\tilde{A})^n)$ , in concrete examples we will only present the estimates for vectors  $\varphi \in D^n\mathcal{D}(A^n)$  and  $\psi \in \mathcal{D}(A^n)$  for the sake of simplicity (see Sects. 3.2, 3.4, 3.7, 3.8 and 3.9).

In the sequel, we assume that the unitary operators  $U_j$  are given by a unitary representation  $\mathcal{U} : X \rightarrow \mathbf{U}(\mathcal{H})$  of a topological group  $X$ . We also assume that the scalars  $\ell_j$  are given by a proper length function on  $X$ , that is, a function  $\ell : X \rightarrow [0, \infty)$  satisfying the following properties (with  $e$  the identity of  $X$ ):

$$(L1) \quad \ell(e) = 0,$$

- (L2)  $\ell(x^{-1}) = \ell(x)$  for all  $x \in X$ ,
- (L3)  $\ell(xy) \leq \ell(x) + \ell(y)$  for all  $x, y \in X$ ,
- (L4) if  $K \subset [0, \infty)$  is compact, then  $\ell^{-1}(K) \subset X$  is relatively compact.

Finally, we recall that a net  $(x_j)_{j \in J}$  in a topological space  $X$  diverges to infinity, with notation  $x_j \rightarrow \infty$ , if  $(x_j)_{j \in J}$  has no limit point in  $X$ . This implies that for each compact set  $K \subset X$ , there exists  $j_K \in J$  such that  $x_j \notin K$  for  $j \geq j_K$ . In particular,  $X$  is not compact. In this setting, we also note that  $\ell(x_j) \rightarrow \infty$  as  $x_j \rightarrow \infty$ , due to assumption (L4).

In this situation, the existence of the strong limit  $D$  leads to additional properties of the unitary operators given by  $\mathcal{U}$ . Namely,  $\mathcal{U}$  has no nontrivial finite-dimensional unitary subrepresentation in  $\ker(D)^\perp$ , and the operator  $D$  commutes with  $\mathcal{U}$ :

**Proposition 2.3.** *Let  $X$  be a topological group equipped with a proper length function  $\ell$ , let  $\mathcal{U} : X \rightarrow \text{U}(\mathcal{H})$  be a unitary representation of  $X$ , let  $(x_j)_{j \in J}$  be a net in  $X$  with  $x_j \rightarrow \infty$ , assume there exists a self-adjoint operator  $A$  in  $\mathcal{H}$  such that  $\mathcal{U}(x_j) \in C^1(A)$  for each  $j \in J$ , and suppose that the strong limit*

$$D := \text{s-lim}_j D_j \quad \text{with} \quad D_j := \frac{1}{\ell(x_j)} [A, \mathcal{U}(x_j)] \mathcal{U}(x_j)^{-1}$$

*exists. Then*

- (a)  $\mathcal{U}$  has no nontrivial finite-dimensional unitary subrepresentation in  $\ker(D)^\perp$ .
- (b) Let  $x \in X$ , assume that  $\mathcal{U}(x) \in C^1(A)$ , and suppose that the strong limit

$$\tilde{D} := \text{s-lim}_j \tilde{D}_j \quad \text{with} \quad \tilde{D}_j := \frac{1}{\ell(x^{-1}x_j)} [A, \mathcal{U}(x^{-1}x_j)] \mathcal{U}(x^{-1}x_j)^{-1}$$

*exists and satisfies  $D = \tilde{D}$ . Then  $[D, \mathcal{U}(x)] = 0$ .*

*Proof.* (a) The claim follows from Theorem 2.1(a) and the fact that matrix coefficients of finite-dimensional unitary representations of a group do not vanish at infinity (see for instance [3, Rem. I.2.15(iii)]).

(b) First, note that the commutator  $[A, \mathcal{U}(x^{-1}x_j)]$  in the expression for  $\tilde{D}$  is well-defined for each  $j \in J$  because  $\mathcal{U}(x^{-1}x_j) = \mathcal{U}(x)^{-1} \mathcal{U}(x_j)$  with  $\mathcal{U}(x) \in C^1(A)$  and  $\mathcal{U}(x_j) \in C^1(A)$ . Next, we have

$$\begin{aligned} D\mathcal{U}(x) &= \text{s-lim}_j \frac{1}{\ell(x_j)} [A, \mathcal{U}(x) \mathcal{U}(x^{-1}x_j)] \mathcal{U}(x^{-1}x_j)^{-1} \\ &= \text{s-lim}_j \frac{1}{\ell(x_j)} [A, \mathcal{U}(x)] \mathcal{U}(x^{-1}x_j) \mathcal{U}(x^{-1}x_j)^{-1} \\ &\quad + \mathcal{U}(x) \cdot \text{s-lim}_j \frac{1}{\ell(x_j)} [A, \mathcal{U}(x^{-1}x_j)] \mathcal{U}(x^{-1}x_j)^{-1} \\ &= \lim_j \frac{1}{\ell(x_j)} \cdot [A, \mathcal{U}(x)] + \mathcal{U}(x) \cdot \text{s-lim}_j \frac{1}{\ell(x_j)} [A, \mathcal{U}(x^{-1}x_j)] \mathcal{U}(x^{-1}x_j)^{-1} \end{aligned}$$

with the first term vanishing because  $\lim_j \frac{1}{\ell(x_j)} = 0$  and with the second term satisfying

$$\begin{aligned} & \mathcal{U}(x) \cdot \text{s-lim}_j \frac{1}{\ell(x_j)} [A, \mathcal{U}(x^{-1}x_j)] \mathcal{U}(x^{-1}x_j)^{-1} \\ &= \mathcal{U}(x) \tilde{D} + \mathcal{U}(x) \cdot \text{s-lim}_j \left( \frac{1}{\ell(x_j)} - \frac{1}{\ell(x^{-1}x_j)} \right) [A, \mathcal{U}(x^{-1}x_j)] \mathcal{U}(x^{-1}x_j)^{-1}. \end{aligned}$$

Thus, to conclude the proof, it is sufficient to show that

$$\text{s-lim}_j \left( \frac{1}{\ell(x_j)} - \frac{1}{\ell(x^{-1}x_j)} \right) [A, \mathcal{U}(x^{-1}x_j)] \mathcal{U}(x^{-1}x_j)^{-1} = 0. \quad (2.4)$$

Let  $\varphi \in \mathcal{H}$ . Then we have

$$\begin{aligned} & \lim_j \left\| \left( \frac{1}{\ell(x_j)} - \frac{1}{\ell(x^{-1}x_j)} \right) [A, \mathcal{U}(x^{-1}x_j)] \mathcal{U}(x^{-1}x_j)^{-1} \varphi \right\|_{\mathcal{H}} \\ &= \lim_j \left| \frac{\ell(x^{-1}x_j) - \ell(x_j)}{\ell(x_j)} \right| \cdot \left\| \frac{1}{\ell(x^{-1}x_j)} [A, \mathcal{U}(x^{-1}x_j)] \mathcal{U}(x^{-1}x_j)^{-1} \varphi \right\|_{\mathcal{H}} \\ &= \|\tilde{D}\varphi\|_{\mathcal{H}} \cdot \lim_j \left| \frac{\ell(x^{-1}x_j) - \ell(x_j)}{\ell(x_j)} \right|. \end{aligned}$$

Since  $|\ell(x^{-1}x_j) - \ell(x_j)| \leq \ell(x)$  due to the triangle inequality for  $\ell$ , we infer that

$$\begin{aligned} & \lim_j \left\| \left( \frac{1}{\ell(x_j)} - \frac{1}{\ell(x^{-1}x_j)} \right) [A, \mathcal{U}(x^{-1}x_j)] \mathcal{U}(x^{-1}x_j)^{-1} \varphi \right\|_{\mathcal{H}} \\ & \leq \|\tilde{D}\varphi\|_{\mathcal{H}} \cdot \ell(x) \lim_j \frac{1}{\ell(x_j)} = 0, \end{aligned}$$

which proves (2.4).  $\square$

## 2.2. Unitary Representations with Self-adjoint Generator

In this section, we consider the important case where the representation is a strongly continuous unitary representation  $\mathcal{U} : \mathbb{R} \rightarrow \text{U}(\mathcal{H})$  of the additive group  $\mathbb{R}$ . In such a case, Stone's theorem implies the existence of a self-adjoint operator  $H$  in  $\mathcal{H}$  such that  $\mathcal{U}(t) = e^{-itH}$  for each  $t \in \mathbb{R}$ . One could also consider the higher-dimensional case of a strongly continuous unitary representation of the additive group  $\mathbb{R}^d$  for  $d \geq 1$ . But we refrained from doing it for the sake of simplicity.

We use the notation  $P_p(H)$  (resp.  $P_c(H)$ ,  $P_{ac}(H)$ ) for the projection onto the pure point (resp. continuous, absolutely continuous) subspace  $\mathcal{H}_p(H)$  (resp.  $\mathcal{H}_c(H)$ ,  $\mathcal{H}_{ac}(H)$ ) of  $H$ ,  $E^H(\cdot)$  for the spectral projections of  $H$ , and  $\chi_{\mathcal{B}}$  for the characteristic function of a Borel set  $\mathcal{B} \subset \mathbb{R}$ .

**Lemma 2.4** (Properties of  $D$ ). *Let  $H$  and  $A$  be self-adjoint operators in a Hilbert space  $\mathcal{H}$  with  $(H - i)^{-1} \in C^1(A)$ , and let*

$$D_t := \frac{1}{t} \int_0^t d\tau e^{-i\tau H} (H + i)^{-1} [iH, A] (H - i)^{-1} e^{i\tau H}, \quad t > 0.$$

Then

$$(a) \text{ s-lim}_{t \rightarrow \infty} D_t P_p(H) = 0.$$



(b) If there exists  $B \in \mathcal{B}(\mathcal{H})$  such that  $(H + i)^{-1}([iH, A] - B)(H - i)^{-1} \in \mathcal{K}(\mathcal{H})$  and

$$s\text{-}\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t d\tau e^{-i\tau H} (H + i)^{-1} B (H - i)^{-1} e^{i\tau H} P_c(H) \text{ exists,}$$

then

$$s\text{-}\lim_{n \rightarrow \infty} D_t = s\text{-}\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t d\tau e^{-i\tau H} (H + i)^{-1} B (H - i)^{-1} e^{i\tau H} P_c(H).$$

Furthermore, if  $D := s\text{-}\lim_{t \rightarrow \infty} D_t$  exists, then

- (c)  $[D, e^{isH}] = 0$  for all  $s \in \mathbb{R}$ . In particular,  $D$  is decomposable in the spectral representation of  $H$ .
- (d)  $D = DP_c(H)$ . In particular,  $H|_{\ker(D)^\perp}$  has purely continuous spectrum.
- (e) If  $D\mathcal{D}(A) \subset \mathcal{D}(A)$  and  $\int_1^\infty dt \|(D - D_t)\varphi\|_{\mathcal{H}}^2 < \infty$  for all  $\varphi \in \mathcal{D}(A)$ , then  $H|_{\ker(D)^\perp}$  has purely a.c. spectrum.

Point (c) implies that  $\ker(D)^\perp$  is a reducing subspace for  $H$ . Therefore the operator  $H|_{\ker(D)^\perp}$  in points (d)–(e) is a well-defined self-adjoint operator (see [41, Thm. 7.28]).

*Proof.* (a) Let  $\varphi \in \mathcal{H}$ . Then  $P_p(H)\varphi = \sum_{j \geq 1} \alpha_j \varphi_j$  with  $(\varphi_j)_{j \geq 1}$  an orthonormal basis of  $\mathcal{H}_p(H)$ ,  $\alpha_j \in \mathbb{C}$ , and  $H\varphi_j = \lambda_j \varphi_j$  for some  $\lambda_j \in \mathbb{R}$ . Thus we obtain

$$\begin{aligned} & s\text{-}\lim_{t \rightarrow \infty} D_t P_p(H)\varphi \tag{2.5} \\ &= s\text{-}\lim_{t \rightarrow \infty} \sum_{j \geq 1} \alpha_j \left( \frac{1}{t} \int_0^t d\tau e^{-i\tau(H-\lambda_j)} \right) (H + i)^{-1} [iH, A] (H - i)^{-1} \varphi_j. \end{aligned}$$

Now  $\|\frac{1}{t} \int_0^t d\tau e^{-i\tau(H-\lambda_j)}\|_{\mathcal{B}(\mathcal{H})} \leq 1$  for all  $t > 0$ , and

$$s\text{-}\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t d\tau e^{-i\tau(H-\lambda_j)} = E^H(\{\lambda_j\})$$

due to von Neumann’s mean ergodic theorem. Therefore we can exchange the limit and the sum in (2.5) to get

$$\begin{aligned} s\text{-}\lim_{t \rightarrow \infty} D_t P_p(H)\varphi &= \sum_{j \geq 1} \alpha_j E^H(\{\lambda_j\}) (H + i)^{-1} [iH, A] (H - i)^{-1} \varphi_j \\ &= \sum_{j \geq 1} \alpha_j \langle \lambda_j \rangle^{-2} E^H(\{\lambda_j\}) [iH, A] E^H(\{\lambda_j\}) \varphi_j. \end{aligned}$$

But  $E^H(\{\lambda_j\}) [iH, A] E^H(\{\lambda_j\}) = 0$  for each  $\lambda_j$  due to the virial theorem for self-adjoint operators [1, Prop. 7.2.10]. Thus we obtain that  $s\text{-}\lim_{t \rightarrow \infty} D_t P_p(H)\varphi = 0$ , which proves the claim.

(b) Let  $K := (H + i)^{-1}([iH, A] - B)(H - i)^{-1} \in \mathcal{K}(\mathcal{H})$ . Then it follows from point (a) that

$$\begin{aligned} \text{s-lim}_{t \rightarrow \infty} D_t &= \text{s-lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t d\tau e^{-i\tau H} (H + i)^{-1} B (H - i)^{-1} e^{i\tau H} P_c(H) \\ &\quad + \text{s-lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t d\tau e^{-i\tau H} K e^{i\tau H} P_c(H). \end{aligned}$$

with  $\text{s-lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t d\tau e^{-i\tau H} K e^{i\tau H} P_c(H) = 0$  due to [31, Thm. 5.9] (or Theorem 5.6 in the first edition of the monograph).

(c) The claim follows from Proposition 2.3(b) in the case of the additive group  $X = \mathbb{R}$  and the auxiliary operator

$$\tilde{A}\varphi := (H + i)^{-1}A(H - i)^{-1}\varphi, \quad \varphi \in \mathcal{D}(A).$$

Indeed, we know from [27, Cor. 2.7 & Rem. 2.8] that  $\tilde{A}$  is essentially self-adjoint (with closure denoted by the same symbol) and that  $e^{-itH} \in C^1(\tilde{A})$  with  $[\tilde{A}, e^{-itH}] = tD_t e^{-itH}$  for any  $t > 0$ . Therefore, if we take the proper length function  $\ell : \mathbb{R} \rightarrow [0, \infty)$  given by  $\ell(t) := |t|$ , the unitary representation  $\mathcal{U} : \mathbb{R} \rightarrow \text{U}(\mathcal{H})$  given by  $\mathcal{U}(t) := e^{-itH}$ , and the net  $(x_j)_{j \in J} = (t)_{t > 0}$ , then we get  $\mathcal{U}(s) = e^{-isH} \in C^1(\tilde{A})$  for all  $s \in \mathbb{R}$  and

$$\tilde{D} = \text{s-lim}_{t \rightarrow \infty} \frac{1}{t-s} [\tilde{A}, e^{-i(t-s)H}] e^{i(t-s)H} = \text{s-lim}_{t \rightarrow \infty} \frac{1}{t} [\tilde{A}, e^{-itH}] e^{itH} = \text{s-lim}_{t \rightarrow \infty} D_t = D.$$

So all the assumptions of Proposition 2.3(b) are verified, and thus  $[D, e^{isH}] = 0$  for all  $s \in \mathbb{R}$ .

Finally, since  $[D, e^{isH}] = 0$  for all  $s \in \mathbb{R}$ , we have  $D\chi_{\mathcal{B}}(H) = \chi_{\mathcal{B}}(H)D$  for each Borel set  $\mathcal{B} \subset \mathbb{R}$ , and thus  $D$  is decomposable in the spectral representation of  $H$  [4, Thm. 7.2.3(b)].

(d) The equality  $D = DP_c(H)$  follows from point (a). As a consequence, we get that  $\ker(D)^\perp \subset \mathcal{H}_c(H)$ , and thus the operator  $H|_{\ker(D)^\perp}$  has purely continuous spectrum.

(e) Take  $\varphi \in \mathcal{D}(A)$ . Then  $\psi = D\varphi \in DD(\tilde{A}) \cap \mathcal{D}(\tilde{A})$ , and it follows from Theorem 2.1(a) that there exists  $c_\psi \geq 0$  such that

$$|\langle \psi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \|(D - D_t)\varphi\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \frac{1}{t} c_\psi, \quad t > 0.$$

So, we infer from the assumption and from Cauchy–Schwarz inequality that  $\int_1^\infty dt |\langle \psi, e^{-itH} \psi \rangle_{\mathcal{H}}|^2 < \infty$ , and thus that  $t \mapsto \langle \psi, e^{-itH} \psi \rangle_{\mathcal{H}}$  belongs to  $L^2(\mathbb{R})$ . Therefore, Plancherel’s theorem for the group  $X = \mathbb{R}$  [9, Thm. 4.26] implies that  $\psi \in DD(A)$  belongs to the a.c. subspace  $\mathcal{H}_{\text{ac}}(H)$  of  $H$ . Thus  $H|_{\ker(D)^\perp}$  has purely a.c. spectrum, since  $DD(A)$  is dense in  $\overline{D\mathcal{H}} = \ker(D)^\perp$  and  $\mathcal{H}_{\text{ac}}(H)$  is closed in  $\mathcal{H}$ .  $\square$

In the next proposition, we consider the particular case where  $[iH, A] = f(H)$  for some Borel function  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Our results in this case generalise the results of [37, Cor. 4.3–4.4].

**Proposition 2.5** (The case  $[iH, A] = f(H)$ ). *Let  $H$  and  $A$  be self-adjoint operators in a Hilbert space  $\mathcal{H}$ , assume that  $(H - i)^{-1} \in C^1(A)$  with  $[iH, A] = f(H)$  for some Borel function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , and set  $g := f \langle \cdot \rangle^{-2}$ . Then*

(a) *For each  $\varphi \in g(H)\mathcal{D}(A)$  and  $\psi \in \mathcal{D}(A)$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that*

$$|\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \frac{1}{t} c_{\varphi, \psi}, \quad t > 0.$$

(b) *If  $g(H)\mathcal{D}(A) \subset \mathcal{D}(A)$ , then  $H|_{\ker(f(H))^\perp}$  has purely a.c. spectrum.*

(c) *Suppose that  $f \in C^n(\mathbb{R})$  ( $n \in \mathbb{N}^*$ ) with  $g^{(k)} \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$  for all  $k = 0, \dots, n$  and  $g^{(n)}$  uniformly continuous. Then for each  $\varphi \in g(H)^n \mathcal{D}(A^n)$  and  $\psi \in \mathcal{D}(A^n)$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that*

$$|\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \frac{1}{t^n} c_{\varphi, \psi}, \quad t > 0.$$

Before the proof, note that the function  $g$  of Proposition 2.5 has to be bounded due to the relation (A.2) for operators of class  $C^1(A)$ .

*Proof.* (a) We know from the proof of Lemma 2.4(c) that the auxiliary operator

$$\tilde{A}\varphi = (H + i)^{-1}A(H - i)^{-1}\varphi, \quad \varphi \in \mathcal{D}(A),$$

is essentially self-adjoint (with closure denoted by the same symbol) and that  $e^{-itH} \in C^1(\tilde{A})$  with  $[\tilde{A}, e^{-itH}] = tD_t e^{-itH}$  for any  $t > 0$ . So the strong limit  $D$  exists and satisfies the equalities

$$\begin{aligned} D_t &= \frac{1}{t} [\tilde{A}, e^{-itH}] e^{itH} = \frac{1}{t} \int_0^t d\tau e^{-i\tau H} (H + i)^{-1} [iH, A] (H - i)^{-1} e^{i\tau H} \\ &= g(H) = D. \end{aligned} \tag{2.6}$$

Therefore the assumptions of Theorem 2.1(b) are satisfied for the net  $(U_j)_{j \in J} = (e^{-itH})_{t > 0}$ , the set  $(\ell_j)_{j \in J} = (t)_{t > 0}$ , the operator  $\tilde{A}$ , and  $D = g(H)$ . Thus for each  $\varphi \in g(H)\mathcal{D}(A) \subset g(H)\mathcal{D}(\tilde{A})$  and  $\psi \in \mathcal{D}(A) \subset \mathcal{D}(\tilde{A})$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that

$$|\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \frac{1}{t} c_{\varphi, \psi}, \quad t > 0.$$

(b) The claim follows from Lemma 2.4(e) because  $D = g(H) = D_t$  for all  $t > 0$  and

$$\ker(g(H)) = \chi_{g^{-1}(\{0\})}(H)\mathcal{H} = \chi_{f^{-1}(\{0\})}(H)\mathcal{H} = \ker(f(H)).$$

(c) Since  $g$  is uniformly continuous and belongs to  $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ , we have

$$\lim_{\varepsilon \searrow 0} \|g(H) - g^\varepsilon(H)\|_{\mathcal{B}(\mathcal{H})} = 0 \tag{2.7}$$

with

$$g^\varepsilon(H) := \int_{\mathbb{R}} dt (\mathcal{F}g)(t) e^{-(\varepsilon t)^2} e^{2\pi i t H} \quad (\text{strong or Bochner integral})$$

and  $\mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  the Fourier transform (see [8, Thm. 8.35(b)]). Using successively the fact that  $D = g(H)$ , Eq. (2.7), the inclusion  $e^{2\pi itH} \mathcal{D}(A) \subset \mathcal{D}(\tilde{A})$ , Eq. (2.6), the relation  $2\pi it(\mathcal{F}g)(t) = (\mathcal{F}g')(t)$ , and (2.7) with  $g$  replaced by  $g'$ , we get for  $\varphi \in \mathcal{D}(A)$  the equalities

$$\begin{aligned}
& \langle \tilde{A}\varphi, D\varphi \rangle_{\mathcal{H}} - \langle \varphi, D\tilde{A}\varphi \rangle_{\mathcal{H}} \\
&= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} dt \overline{(\mathcal{F}g)(t)} e^{-(\varepsilon t)^2} (\langle \tilde{A}\varphi, e^{2\pi itH} \varphi \rangle_{\mathcal{H}} - \langle \varphi, e^{2\pi itH} \tilde{A}\varphi \rangle_{\mathcal{H}}) \\
&= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} dt \overline{(\mathcal{F}g)(t)} e^{-(\varepsilon t)^2} \langle \varphi, [\tilde{A}, e^{2\pi itH}] \varphi \rangle_{\mathcal{H}} \\
&= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} dt \overline{(\mathcal{F}g)(t)} e^{-(\varepsilon t)^2} \langle \varphi, (-2\pi t) D e^{2\pi itH} \varphi \rangle_{\mathcal{H}} \\
&= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} dt i \overline{(\mathcal{F}g')(t)} e^{-(\varepsilon t)^2} \langle \varphi, D e^{2\pi itH} \varphi \rangle_{\mathcal{H}} \\
&= \langle \varphi, i D g'(H) \varphi \rangle_{\mathcal{H}}.
\end{aligned}$$

Since  $g'(H) \in \mathcal{B}(\mathcal{H})$  due to the inclusion  $g' \in L^\infty(\mathbb{R})$ , we infer that  $D \in C^1(\tilde{A})$  with  $[\tilde{A}, D] = i D g'(H)$ . Thus, in the present case, the operator  $B \in \mathcal{B}(\mathcal{H})$  appearing in the statement of Theorem 2.1(c) is  $B = i g'(H)$ . So we trivially get that  $[D, B] = 0$ , and by reproducing  $(n-1)$  times the previous argument we obtain that  $B \in C^{n-1}(\tilde{A})$ .

Summing up, the assumptions of Theorem 2.1(c) are satisfied for the net  $(U_j)_{j \in J} = (e^{-itH})_{t > 0}$ , the set  $(\ell_j)_{j \in J} = (t)_{t > 0}$ , the operator  $\tilde{A}$ , and  $D = g(H)$ . Thus for each  $\varphi \in g(H)^n \mathcal{D}(A^n) \subset g(H)^n \mathcal{D}(\tilde{A}^n)$  and  $\psi \in \mathcal{D}(A^n) \subset \mathcal{D}(\tilde{A}^n)$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that

$$|\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \frac{1}{t^n} c_{\varphi, \psi}, \quad t > 0.$$

□

*Remark 2.6.* If the operators  $H$  and  $A$  satisfy the commutation relation  $[iH, A] = f(H)$  for some Borel function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , then they satisfy the relation  $[iH, A] = \tilde{f}(H)$  for any Borel function  $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\tilde{f}|_{\sigma(H)} = f|_{\sigma(H)}$  due to functional calculus. This basic observation will be useful in some applications in which the initial function  $f$  fails to satisfy the assumptions of Proposition 2.5(b) or 2.5(c) (see for instance Sect. 3.3).

### 2.3. Unitary Representations with Unitary Generator

In this section, we consider the important case where the representation is a unitary representation  $\mathcal{U} : \mathbb{Z} \rightarrow \mathbf{U}(\mathcal{H})$  of the additive group  $\mathbb{Z}$ . In such a case, the fact that  $\mathbb{Z}$  has generator 1 implies the existence of a unitary operator  $U$  in  $\mathcal{H}$  such that  $\mathcal{U}(m) = U^m$  for each  $m \in \mathbb{Z}$ . One could also consider the higher-dimensional case of a unitary representation of the additive group  $\mathbb{Z}^d$  for  $d \geq 1$ . But we refrained from doing it for the sake of simplicity.

We use the notation  $P_p(U)$  (resp.  $P_c(U)$ ,  $P_{ac}(U)$ ) for the projection onto the pure point (resp. continuous, absolutely continuous) subspace  $\mathcal{H}_p(U)$  (resp.

$\mathcal{H}_c(U)$ ,  $\mathcal{H}_{ac}(U)$  of  $U$ ,  $E^U(\cdot)$  for the spectral projections of  $U$ , and  $\chi_\Theta$  for the characteristic function of a Borel set  $\Theta \subset \mathbb{S}^1$ .

**Lemma 2.7** (Properties of  $D$ ). *Let  $U$  and  $A$  be a unitary and a self-adjoint operator in a Hilbert space  $\mathcal{H}$  with  $U \in C^1(A)$ , and let*

$$D_n := \frac{1}{n}[A, U^n]U^{-n}, \quad n \in \mathbb{N}^*.$$

Then

- (a)  $\text{s-lim}_{n \rightarrow \infty} D_n P_p(U) = 0$ .
- (b) *If there exists  $B \in \mathcal{B}(\mathcal{H})$  such that  $[A, U]U^{-1} - B \in \mathcal{K}(\mathcal{H})$  and if  $\text{s-lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} U^m B U^{-m} P_c(U)$  exists, then*

$$\text{s-lim}_{n \rightarrow \infty} D_n = \text{s-lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} U^m B U^{-m} P_c(U).$$

Furthermore, if  $D := \text{s-lim}_{n \rightarrow \infty} D_n$  exists, then

- (c)  $[D, U^m] = 0$  for all  $m \in \mathbb{Z}$ . In particular,  $D$  is decomposable in the spectral representation of  $U$ .
- (d)  $D = D P_c(U)$ . In particular,  $U|_{\ker(D)^\perp}$  has purely continuous spectrum.
- (e) *If  $D\mathcal{D}(A) \subset \mathcal{D}(A)$  and  $\sum_{n \geq 1} \|(D - D_n)\varphi\|_{\mathcal{H}}^2 < \infty$  for all  $\varphi \in \mathcal{D}(A)$ , then  $U|_{\ker(D)^\perp}$  has purely a.c. spectrum.*

Point (c) implies that  $\ker(D)^\perp$  is a reducing subspace for  $U$ . Therefore the operator  $U|_{\ker(D)^\perp}$  in points (d)–(e) is a well-defined unitary operator (see [41, Example 5.39(b)]).

*Proof.* (a) Let  $\varphi \in \mathcal{H}$ . Then  $P_p(U)\varphi = \sum_{j \geq 1} \alpha_j \varphi_j$  with  $(\varphi_j)_{j \geq 1}$  an orthonormal basis of  $\mathcal{H}_p(U)$ ,  $\alpha_j \in \mathbb{C}$ , and  $U\varphi_j = \theta_j \varphi_j$  for some  $\theta_j \in \mathbb{S}^1$ . Furthermore, we have for any  $n \in \mathbb{N}^*$  the equalities

$$D_n = \frac{1}{n}[A, U^n]U^{-n} = \frac{1}{n} \sum_{m=0}^{n-1} U^m ([A, U]U^{-1})U^{-m}.$$

Therefore, we obtain

$$\text{s-lim}_{n \rightarrow \infty} D_n P_p(U)\varphi = \text{s-lim}_{n \rightarrow \infty} \sum_{j \geq 1} \alpha_j \left( \frac{1}{n} \sum_{m=0}^{n-1} (U\theta_j^{-1})^m \right) [A, U]U^{-1}\varphi_j. \quad (2.8)$$

Now  $\|\frac{1}{n} \sum_{m=0}^{n-1} (U\theta_j^{-1})^m\|_{\mathcal{B}(\mathcal{H})} \leq 1$  for all  $n \in \mathbb{N}^*$ , and

$$\text{s-lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} (U\theta_j^{-1})^m = E^U(\{\theta_j\})$$

due to von Neumann's mean ergodic theorem. Therefore we can exchange the limit and the sum in (2.8) to get

$$\begin{aligned} \text{s-lim}_{n \rightarrow \infty} D_n P_p(U) \varphi &= \sum_{j \geq 1} \alpha_j E^U(\{\theta_j\}) [A, U] U^{-1} \varphi_j \\ &= \sum_{j \geq 1} \alpha_j E^U(\{\theta_j\}) [A, U] U^{-1} E^U(\{\theta_j\}) \varphi_j. \end{aligned}$$

But  $E^U(\{\theta_j\}) [A, U] U^{-1} E^U(\{\theta_j\}) = 0$  for each  $\theta_j$  due to the virial theorem for unitary operators [7, Prop. 2.3]. Thus we obtain that  $\text{s-lim}_{n \rightarrow \infty} D_n P_p(U) \varphi = 0$ , which proves the claim.

(b) Let  $K := [A, U] U^{-1} - B \in \mathcal{K}(\mathcal{H})$ . Then it follows from point (a) that

$$\begin{aligned} \text{s-lim}_{n \rightarrow \infty} D_n &= \text{s-lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} U^m ([A, U] U^{-1}) U^{-m} P_c(U) \\ &= \text{s-lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} U^m B U^{-m} P_c(U) + \text{s-lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} U^m K U^{-m} P_c(U). \end{aligned}$$

with  $\text{s-lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} U^m K U^{-m} P_c(U) = 0$  due to Theorem B.2.

(c) The claim follows from Proposition 2.3(b) in the case of the additive group  $X = \mathbb{Z}$  and the operator  $A$ . Indeed, if we take the proper length function  $\ell : \mathbb{Z} \rightarrow [0, \infty)$  given by  $\ell(n) := |n|$ , the unitary representation  $\mathcal{U} : \mathbb{Z} \rightarrow \mathcal{U}(\mathcal{H})$  given by  $\mathcal{U}(n) := U^n$ , and the net  $(x_j)_{j \in J} = (n)_{n \in \mathbb{N}^*}$ , then we get  $\mathcal{U}(m) = U^m \in C^1(A)$  for all  $m \in \mathbb{Z}$  and

$$\tilde{D} = \text{s-lim}_{n \rightarrow \infty} \frac{1}{n-m} [A, U^{n-m}] U^{-(n-m)} = \text{s-lim}_{n \rightarrow \infty} \frac{1}{n} [A, U^n] U^{-n} = \text{s-lim}_{n \rightarrow \infty} D_n = D.$$

So all the assumptions of Proposition 2.3(b) are verified, and thus  $[D, U^m] = 0$  for all  $m \in \mathbb{Z}$ .

Finally, since  $[D, U^m] = 0$  for all  $m \in \mathbb{Z}$ , we have  $D \chi_\Theta(U) = \chi_\Theta(U) D$  for each Borel set  $\Theta \subset \mathbb{S}^1$ , and thus  $D$  is decomposable in the spectral representation of  $U$  [4, Thm. 7.2.3(b)].

(d) The equality  $D = D P_c(U)$  follows from point (a). As a consequence, we get that  $\ker(D)^\perp \subset \mathcal{H}_c(U)$ , and thus the operator  $U|_{\ker(D)^\perp}$  has purely continuous spectrum.

(e) Let  $\varphi \in \mathcal{D}(A)$ . Then  $\psi = D\varphi \in D\mathcal{D}(A) \cap \mathcal{D}(A)$ , and it follows from Theorem 2.1(a) that there exists  $c_\psi \geq 0$  such that

$$|\langle \psi, U^n \psi \rangle_{\mathcal{H}}| \leq \|(D - D_n)\varphi\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \frac{1}{n} c_\psi, \quad n \in \mathbb{N}^*.$$

So, we infer from the assumption and Cauchy-Schwarz inequality that  $\sum_{n \geq 1} |\langle \psi, U^n \psi \rangle_{\mathcal{H}}|^2 < \infty$ , and thus that  $n \mapsto \langle \psi, U^n \psi \rangle_{\mathcal{H}}$  belongs to  $\ell^2(\mathbb{Z})$ . Therefore, Plancherel's theorem for the group  $X = \mathbb{Z}$  [9, Thm. 4.26] implies that  $\psi \in D\mathcal{D}(A)$  belongs to the a.c. subspace  $\mathcal{H}_{\text{ac}}(U)$  of  $U$ . Thus  $U|_{\ker(D)^\perp}$  has purely a.c. spectrum, since  $D\mathcal{D}(A)$  is dense in  $\overline{D\mathcal{H}} = \ker(D)^\perp$  and  $\mathcal{H}_{\text{ac}}(U)$  is closed in  $\mathcal{H}$ .  $\square$

In the next proposition, we consider the particular case where  $[A, U] = \gamma(U)$  for some bounded Borel function  $\gamma : \mathbb{S}^1 \rightarrow \mathbb{C}$ . Our results in this case generalise the results of [37, Cor. 3.3–3.4].

**Proposition 2.8** (The case  $[A, U] = \gamma(U)$ ). *Let  $U$  and  $A$  be a unitary and a self-adjoint operator in a Hilbert space  $\mathcal{H}$ , assume that  $U \in C^1(A)$  with  $[A, U] = \gamma(U)$  for some bounded Borel function  $\gamma : \mathbb{S}^1 \rightarrow \mathbb{C}$ , and set  $\eta(U) := \gamma(U)U^{-1}$ . Then*

- (a) *For each  $\varphi \in \eta(U)\mathcal{D}(A)$  and  $\psi \in \mathcal{D}(A)$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that*

$$|\langle \varphi, U^n \psi \rangle_{\mathcal{H}}| \leq \frac{1}{n} c_{\varphi, \psi}, \quad n \in \mathbb{N}^*.$$

- (b) *If  $\eta(U)\mathcal{D}(A) \subset \mathcal{D}(A)$ , then  $U|_{\ker(\gamma(U))^\perp}$  has purely a.c. spectrum.*
- (c) *Suppose that  $\gamma \in C^k(\mathbb{S}^1)$  ( $k \in \mathbb{N}^*$ ). Then for each  $\varphi \in \eta(U)^k \mathcal{D}(A^k)$  and  $\psi \in \mathcal{D}(A^k)$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that*

$$|\langle \varphi, U^n \psi \rangle_{\mathcal{H}}| \leq \frac{1}{n^k} c_{\varphi, \psi}, \quad n \in \mathbb{N}^*.$$

*Proof.* (a) The strong limit  $D$  exists and satisfies for any  $n \in \mathbb{N}^*$  the equalities

$$D_n = \frac{1}{n} \sum_{m=0}^{n-1} U^m ([A, U]U^{-1})U^{-m} = \eta(U) = D. \tag{2.9}$$

Therefore the assumptions of Theorem 2.1(b) are satisfied for the net  $(U_j)_{j \in J} = (U^n)_{n \in \mathbb{N}^*}$ , the set  $(\ell_j)_{j \in J} = (n)_{n \in \mathbb{N}^*}$ , the operator  $A$ , and  $D = \eta(U)$ . Thus for each  $\varphi \in \eta(U)\mathcal{D}(A)$  and  $\psi \in \mathcal{D}(A)$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that

$$|\langle \varphi, U^n \psi \rangle_{\mathcal{H}}| \leq \frac{1}{n} c_{\varphi, \psi}, \quad n \in \mathbb{N}^*.$$

- (b) The claim follows from Lemma 2.7(e) because  $D = \eta(U) = D_n$  for all  $n \in \mathbb{N}^*$  and

$$\ker(\eta(U)) = \chi_{\eta^{-1}(\{0\})}(U)\mathcal{H} = \chi_{\gamma^{-1}(\{0\})}(U)\mathcal{H} = \ker(\gamma(U)).$$

- (c) Since  $\eta \in C(\mathbb{S}^1)$ , we have that

$$\lim_{\varepsilon \searrow 0} \|\eta(U) - \eta^\varepsilon(U)\|_{\mathcal{B}(\mathcal{H})} = 0 \tag{2.10}$$

with

$$\eta^\varepsilon(U) := \sum_{m \in \mathbb{Z}} (\mathcal{F}\eta)(m) e^{-\varepsilon m^2} U^m \quad (\text{strong or operator norm sum})$$

and  $\mathcal{F} : L^2(\mathbb{S}^1) \rightarrow L^2(\mathbb{Z})$  the Fourier transform (see [8, Thm. 8.36(a)]). Using successively the fact that  $D = \eta(U)$ , Eq. (2.10), the inclusion  $U^m \mathcal{D}(A) \subset \mathcal{D}(A)$ , Eq. (2.9), the relation  $m(\mathcal{F}\eta)(m) = (\mathcal{F}(\text{id}_{\mathbb{S}^1} \cdot \eta'))(m)$  with  $\text{id}_{\mathbb{S}^1}$  the identity function on  $\mathbb{S}^1$ , and (2.10) with  $\eta$  replaced by  $\text{id}_{\mathbb{S}^1} \cdot \eta'$ , we get for

$\varphi \in \mathcal{D}(A)$  the equalities

$$\begin{aligned}
& \langle A\varphi, D\varphi \rangle_{\mathcal{H}} - \langle \varphi, DA\varphi \rangle_{\mathcal{H}} \\
&= \lim_{\varepsilon \searrow 0} \sum_{m \in \mathbb{Z}} \overline{(\mathcal{F}\eta)(m)} e^{-(\varepsilon m)^2} (\langle A\varphi, U^m \varphi \rangle_{\mathcal{H}} - \langle \varphi, U^m A\varphi \rangle_{\mathcal{H}}) \\
&= \lim_{\varepsilon \searrow 0} \sum_{m \in \mathbb{Z}} \overline{(\mathcal{F}\eta)(m)} e^{-(\varepsilon m)^2} \langle \varphi, [A, U^m] \varphi \rangle_{\mathcal{H}} \\
&= \lim_{\varepsilon \searrow 0} \sum_{m \in \mathbb{Z}} \overline{(\mathcal{F}\eta)(m)} e^{-(\varepsilon m)^2} \langle \varphi, mDU^m \varphi \rangle_{\mathcal{H}} \\
&= \lim_{\varepsilon \searrow 0} \sum_{m \in \mathbb{Z}} \overline{(\mathcal{F}(\text{id}_{\mathbb{S}^1} \cdot \eta'))(m)} e^{-(\varepsilon m)^2} \langle \varphi, DU^m \varphi \rangle_{\mathcal{H}} \\
&= \langle \varphi, D(\text{id}_{\mathbb{S}^1} \cdot \eta')(U)\varphi \rangle_{\mathcal{H}} \\
&= \langle \varphi, DU\eta'(U)\varphi \rangle_{\mathcal{H}}.
\end{aligned}$$

Since  $\eta'(U) \in \mathcal{B}(\mathcal{H})$  due to the inclusion  $\eta' \in C^{k-1}(\mathbb{S}^1)$ , we infer that  $D \in C^1(A)$  with  $[A, D] = DU\eta'(U)$ . Thus, in the present case, the operator  $B \in \mathcal{B}(\mathcal{H})$  appearing in the statement of Theorem 2.1(c) is  $B = U\eta'(U)$ . So we trivially get that  $[D, B] = 0$ , and by reproducing  $(k-1)$  times the previous argument we obtain that  $B \in C^{k-1}(A)$ .

Summing up, the assumptions of Theorem 2.1(c) are satisfied for the net  $(U_j)_{j \in J} = (U^n)_{n \in \mathbb{N}^*}$ , the set  $(\ell_j)_{j \in J} = (n)_{n \in \mathbb{N}^*}$ , the operator  $A$ , and  $D = \eta(U)$ . Thus for each  $\varphi \in \eta(U)^k \mathcal{D}(A^k)$  and  $\psi \in \mathcal{D}(A^k)$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that

$$|\langle \varphi, U^n \psi \rangle_{\mathcal{H}}| \leq \frac{1}{n^k} c_{\varphi, \psi}, \quad n \in \mathbb{N}^*.$$

□

### 3. Applications

In this section, we apply our results to various models in quantum mechanics and dynamical systems. First, we consider an example of unitary representation with no self-adjoint or unitary generator. Then we consider examples of unitary representations having a self-adjoint generator. And finally we consider examples of unitary representations having a unitary generator.

#### 3.1. Left Regular Representation

This example is motivated by [27, Ex. 2.9]. Let  $X$  be a  $\sigma$ -compact locally compact Hausdorff group with left Haar measure  $\mu$  and proper length function  $\ell$  (see properties (L1)–(L4) in Sect. 2.1). Let  $\mathcal{H} := L^2(X, \mu)$ , let  $\mathcal{D} \subset \mathcal{H}$  be the set of functions  $X \rightarrow \mathbb{C}$  with compact support, let  $\mathcal{U} : X \rightarrow \mathcal{U}(\mathcal{H})$  be the left regular representation of  $X$  on  $\mathcal{H}$

$$\mathcal{U}(x)\varphi := \varphi(x^{-1}\cdot), \quad x \in X, \varphi \in \mathcal{H},$$

and let  $A$  be the maximal multiplication operator by  $\ell$  in  $\mathcal{H}$

$$A\varphi := \ell\varphi, \quad \varphi \in \mathcal{D}(A) := \{\varphi \in \mathcal{H} \mid \|\ell\varphi\|_{\mathcal{H}} < \infty\}.$$



For any  $x \in X$ , one has on  $\mathcal{D}$  the equality

$$A\mathcal{U}(x) - \mathcal{U}(x)A = (\ell(\cdot) - \ell(x^{-1}\cdot))\mathcal{U}(x),$$

with  $|(\ell(\cdot) - \ell(x^{-1}\cdot))| \leq \ell(x)$  due to properties (L2)–(L3). Since  $\mathcal{D}$  is dense in  $\mathcal{D}(A)$ , it follows that  $\mathcal{U}(x) \in C^1(A)$  with

$$[A, \mathcal{U}(x)]\mathcal{U}(x)^{-1} = \ell(\cdot) - \ell(x^{-1}\cdot). \tag{3.1}$$

In addition, we know from [27, Ex. 2.9] that for any net  $(x_j)_{j \in J}$  in  $X$  with  $x_j \rightarrow \infty$  we have

$$D := \text{s-lim}_j D_j = \text{s-lim}_j \frac{1}{\ell(x_j)} [A, \mathcal{U}(x_j)]\mathcal{U}(x_j)^{-1} = -1.$$

Therefore, Theorem 2.1(a) applies for the net  $(U_j)_{j \in J} = (\mathcal{U}(x_j))_{j \in J}$ , the set  $(\ell_j)_{j \in J} = (\ell(x_j))_{j \in J}$ , the operator  $A$ , and  $D = -1$ . Thus, for each  $\varphi, \psi \in \mathcal{D}(A)$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that

$$|\langle \varphi, \mathcal{U}(x_j)\psi \rangle_{\mathcal{H}}| \leq \|(D - D_j)\varphi\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \frac{1}{\ell(x_j)} c_{\varphi, \psi}, \quad \ell(x_j) > 0.$$

But (3.1) and the bound  $|\ell(\cdot) - \ell(x_j^{-1}\cdot) + \ell(x_j)| \leq 2\ell(\cdot)$  (which follows from properties (L2)–(L3)) imply that

$$\|(D - D_j)\varphi\|_{\mathcal{H}} = \left\| \frac{\ell(\cdot) - \ell(x_j^{-1}\cdot) + \ell(x_j)}{\ell(x_j)} \varphi \right\|_{\mathcal{H}} \leq \frac{2}{\ell(x_j)} \|\ell\varphi\|_{\mathcal{H}}.$$

Therefore, by setting  $\tilde{c}_{\varphi, \psi} := 2\|\ell\varphi\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + c_{\varphi, \psi}$ , we obtain that

$$|\langle \varphi, \mathcal{U}(x_j)\psi \rangle_{\mathcal{H}}| \leq \frac{1}{\ell(x_j)} \tilde{c}_{\varphi, \psi}, \quad \ell(x_j) > 0.$$

This estimate is similar to the estimates of Propositions 2.5(a) and 2.8(a), but with the interesting difference that in general the representation  $\mathcal{U}$  doesn't have neither a self-adjoint generator nor a unitary generator. In addition, we note that one can even obtain higher order decay estimates by carrying on the above calculations. But we refrained from presenting them here for the sake of simplicity.

### 3.2. Schrödinger Operator in $\mathbb{R}^n$

In the Hilbert space  $\mathcal{H} := L^2(\mathbb{R}^n)$ , consider the Schrödinger operator and the generator of dilations given by

$$H\varphi := (-\Delta + V)\varphi \quad \text{and} \quad A\varphi := \frac{1}{2}(Q \cdot P + P \cdot Q), \quad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

with  $V \in L^\infty(\mathbb{R}^n, \mathbb{R})$ ,  $Q := (Q_1, \dots, Q_n)$  the position operator,  $P := -i\nabla$  the momentum operator and  $\mathcal{S}(\mathbb{R}^n)$  the Schwartz space on  $\mathbb{R}^n$ . Both operators are essentially self-adjoint with closures denoted by the same symbols. If we assume that

$$x \mapsto x \cdot (\nabla V)(x) \in L^\infty(\mathbb{R}^n, \mathbb{R}) \quad \text{and} \quad \lim_{|x| \rightarrow \infty} |2V(x) + x \cdot (\nabla V)(x)| = 0,$$

then we have  $(H - i)^{-1} \in C^1(A)$  with  $[iH, A] = 2H - 2V - Q \cdot \nabla V$ , we have

$$(H + i)^{-1}(2V + Q \cdot \nabla V)(H - i)^{-1} \in \mathcal{K}(\mathcal{H})$$

due to the standard theorem [1, Prop. 4.1.3], and

$$\mathfrak{s}\text{-}\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t d\tau e^{-i\tau H} (H+i)^{-1} 2H(H-i)^{-1} e^{i\tau H} P_c(H) = 2H\langle H \rangle^{-2} P_c(H).$$

Therefore, it follows from Lemma 2.4 that

$$D = \mathfrak{s}\text{-}\lim_{t \rightarrow \infty} D_t = 2H\langle H \rangle^{-2} P_c(H)$$

and that  $H|_{\ker(D)^\perp}$  has purely continuous spectrum. Moreover, Theorem 2.1(a) applies for the net  $(U_j)_{j \in J} = (e^{-itH})_{t>0}$ , the set  $(\ell_j)_{j \in J} = (t)_{t>0}$ , the operator  $\tilde{A} = (H+i)^{-1}A(H-i)^{-1}$ , and  $D$  as above. Therefore, for each  $\varphi = D\tilde{\varphi} \in \mathcal{D}\mathcal{D}(A)$  and  $\psi \in \mathcal{D}(A)$  there exists a constant  $c_{\varphi,\psi} \geq 0$  such that

$$|\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \|(D - D_t)\tilde{\varphi}\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \frac{1}{t} c_{\varphi,\psi}, \quad t > 0. \quad (3.2)$$

In this example, one cannot easily improve the decay estimate (3.2) when  $V \neq 0$ . Indeed, in order to establish the convergence  $D_t \xrightarrow{s} D$ , we used Lemma 2.4(b). But the proof of Lemma 2.4(b) relies on the RAGE theorem for self-adjoint operators, whose proof relies in turn on Wiener's theorem. And as sad as it is, in general one cannot infer an explicit rate of convergence from Wiener's theorem.

On the other hand, if  $V = 0$ , then we have  $[iH, A] = 2H$ , and thus the stronger decay estimates of Proposition 2.5(c) are satisfied.

### 3.3. Dirac Operator in $\mathbb{R}^3$

This example is motivated by [15, Ex. 7.7] and [26, Sec. 7.3]. Consider in the Hilbert space  $\mathcal{H} := L^2(\mathbb{R}^3, \mathbb{C}^4)$  the Dirac operator for a spin- $\frac{1}{2}$  particle of mass  $m > 0$

$$H\varphi := (\alpha \cdot P + \beta m)\varphi, \quad \varphi \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4),$$

with  $\alpha := (\alpha_1, \alpha_2, \alpha_3)$  and  $\beta$  the usual  $4 \times 4$  Dirac matrices. Then  $H$  is essentially self-adjoint (with closure denoted by the same symbol), and we have the following result:<sup>1</sup>

**Lemma 3.1.** *Let  $X_{\text{NW}} := (X_{\text{NW},1}, X_{\text{NW},2}, X_{\text{NW},3})$  be the Newton–Wigner position operator. Then the operator*

$$A\varphi := \frac{1}{2}(X_{\text{NW}} \cdot PH^{-1} + PH^{-1} \cdot X_{\text{NW}})\varphi, \quad \varphi \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4),$$

*is essentially self-adjoint (with closure denoted by the same symbol) and one has  $(H-i)^{-1} \in C^1(A)$  with  $[iH, A] = (H^2 - m^2)H^{-2}$ .*

<sup>1</sup>In [15, Ex. 7.7], the authors say that it is shown in [26] that  $(H-i)^{-1} \in C^1(X_{\text{NW}})$  with

$$[iH, X_{\text{NW}}] = \sqrt{H^2 - m^2} H^{-1}. \quad (3.3)$$

This is not what is written in [26, Sec. 7.3], and it cannot be since  $X_{\text{NW}}$  on the l.h.s. of (3.3) is a vector operator with three components whereas  $\sqrt{H^2 - m^2} H^{-1}$  on the r.h.s. of (3.3) is a scalar operator with one component.

*Proof.* First, we recall that  $X_{NW} = \mathcal{F}_{FW}^{-1}Q\mathcal{F}_{FW}$ , with  $\mathcal{F}_{FW}$  the (unitary) Foldy–Wouthuysen transform, and that  $\mathcal{F}_{FW}$  leaves  $\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$  invariant [32, Sec. 1.4.3]. Therefore, one gets on  $\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$  the equalities

$$\begin{aligned} A &= \frac{1}{2}(X_{NW} \cdot PH^{-1} + PH^{-1} \cdot X_{NW}) \\ &= \frac{1}{2}\mathcal{F}_{FW}^{-1}(Q \cdot \mathcal{F}_{FW}PH^{-1}\mathcal{F}_{FW}^{-1} + \mathcal{F}_{FW}PH^{-1}\mathcal{F}_{FW}^{-1} \cdot Q)\mathcal{F}_{FW} \\ &= \frac{1}{2}\mathcal{F}_{FW}^{-1}(Q \cdot P\beta|H|^{-1} + P\beta|H|^{-1} \cdot Q)\mathcal{F}_{FW} \end{aligned}$$

with the operator within the parenthesis equal to a direct sum of operators of the form

$$\pm(Q \cdot P(P^2 + m^2)^{-1/2} + P(P^2 + m^2)^{-1/2} \cdot Q).$$

Since each of these operators is essentially self-adjoint on  $\mathcal{S}(\mathbb{R}^3)$  due to a commutator criterion for essential self-adjointness [23, Thm. X.37], one infers that  $A$  is essentially self-adjoint on  $\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$ .

For the second claim, we recall from [26, Sec. 7.3] that  $(H - i)^{-1} \in C^1(X_{NW,j})$  with  $[iH, X_{NW,j}] = P_jH^{-1}$ . Therefore, a calculation in the form sense on  $\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$  gives

$$\begin{aligned} &[(H - i)^{-1}, A] \\ &= \frac{1}{2} \sum_{j=1}^3 ([ (H - i)^{-1}, X_{NW,j} ] P_j H^{-1} + P_j H^{-1} [ (H - i)^{-1}, X_{NW,j} ]) \\ &= \frac{i}{2} (H - i)^{-1} \sum_{j=1}^3 ([iH, X_{NW,j}] P_j H^{-1} + P_j H^{-1} [iH, X_{NW,j}]) (H - i)^{-1} \\ &= i(H - i)^{-1} (H^2 - m^2) H^{-2} (H - i)^{-1}. \end{aligned}$$

Since  $\mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$  is a core for  $A$ , this implies that  $(H - i)^{-1} \in C^1(A)$  with  $[iH, A] = (H^2 - m^2)H^{-2}$ .  $\square$

Now, take any function  $f \in C^\infty(\mathbb{R})$  such that  $f(x) = (x^2 - m^2)x^{-2}$  if  $|x| \geq m$ . Then Remark 2.6 and Lemma 3.1 imply that  $(H - i)^{-1} \in C^1(A)$  with  $[iH, A] = f(H)$ , and  $g(H) = f(H)\langle H \rangle^{-2} \in C^1(A)$ . Thus  $g(H)\mathcal{D}(A) \subset \mathcal{D}(A)$  and Proposition 2.5 applies. Since an application of the Foldy–Wouthuysen transform shows that  $H$  has spectrum equal to  $(-\infty, -m] \cup [m, \infty)$  and  $\ker(f(H)) = \{0\}$ , it follows that  $H$  has purely a.c. spectrum equal to  $(-\infty, -m] \cup [m, \infty)$  (which is standard knowledge) and for each  $\varphi \in g(H)^n \mathcal{D}(A^n)$  and  $\psi \in \mathcal{D}(A^n)$  ( $n \in \mathbb{N}^*$ ) there exists a constant  $c_{\varphi,\psi} \geq 0$  such that

$$|\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \frac{1}{t^n} c_{\varphi,\psi}, \quad t > 0.$$

### 3.4. Quantum Waveguides in $\mathbb{R}^n$

This example is motivated by [33]. Let  $\Sigma$  be a bounded open connected set in  $\mathbb{R}^{n-1}$  ( $n \geq 2$ ), let  $\Omega := \Sigma \times \mathbb{R}$  be the corresponding waveguide with coordinates  $x \equiv (\omega, x_n)$ , let  $-\Delta_\Sigma^2$  be the Dirichlet Laplacian in  $L^2(\Sigma)$ , let

$$H_0 := 1 \otimes P_n^2 + (-\Delta_\Sigma^2) \otimes 1 \quad \text{in} \quad \mathcal{H} := L^2(\Omega) \simeq L^2(\Sigma) \otimes L^2(\mathbb{R}),$$

and let  $H := H_0 + V$  with  $V \in L^\infty(\Omega, \mathbb{R})$  satisfying

$$(1 \otimes \langle Q_n \rangle^{1+\varepsilon})V \in \mathcal{B}(\mathcal{H}) \quad \text{and} \quad (1 \otimes \langle Q_n \rangle^{1+\varepsilon})(\partial_n V) \in \mathcal{B}(\mathcal{H}) \quad \text{for some } \varepsilon > 0.$$

Then  $V \in \mathcal{K}(\mathcal{D}(H_0), \mathcal{H})$ . Indeed, if we use the strongly commuting self-adjoint operators  $X_1 := 1 \otimes \langle P_n \rangle^2$  and  $X_2 := (-\Delta_D^\Sigma + 1) \otimes 1$ , we get

$$\begin{aligned} & V(H_0 + 2)^{-1} \\ &= (1 \otimes \langle Q_n \rangle)V \cdot ((-\Delta_D^\Sigma + 1)^{-1/2} \otimes \langle Q_n \rangle^{-1} \langle P_n \rangle^{-1}) \cdot X_1^{1/2} X_2^{1/2} (X_1 + X_2)^{-1}. \end{aligned}$$

But  $(1 \otimes \langle Q_n \rangle)V \in \mathcal{B}(\mathcal{H})$  by assumption,  $X_1^{1/2} X_2^{1/2} (X_1 + X_2)^{-1} \in \mathcal{B}(\mathcal{H})$  due to the bound

$$\|X_1^{1/2} X_2^{1/2} (X_1 + X_2)^{-1}\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{x_1, x_2 \geq 1} (x_1 x_2)^{1/2} (x_1 + x_2)^{-1} < \infty,$$

and  $(-\Delta_D^\Sigma + 1)^{-1/2} \otimes \langle Q_n \rangle^{-1} \langle P_n \rangle^{-1} \in \mathcal{K}(\mathcal{H})$  because  $-\Delta_D^\Sigma$  has compact resolvent and  $\langle Q_n \rangle^{-1} \langle P_n \rangle^{-1}$  is compact. Hence  $V(H_0 + 2)^{-1} \in \mathcal{K}(\mathcal{H})$ , and thus  $V \in \mathcal{K}(\mathcal{D}(H_0), \mathcal{H})$ . Therefore, the assumptions of [33] are satisfied and the following holds true: The operator  $H$  has no singular continuous spectrum, the eigenvalues of  $H$  (if any) are of finite multiplicity and can accumulate only at the set of eigenvalues of  $-\Delta_D^\Sigma$ , and the wave operators  $W_\pm := \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0}$  exist and are complete.

Now, set  $A := 1 \otimes A_n$  with  $A_n$  the generator of dilations in  $L^2(\mathbb{R})$ . Then  $(H - i)^{-1} \in C^1(A)$  with

$$[iH, A] = 2(1 \otimes P_n^2) - (1 \otimes Q_n)(\partial_n V),$$

and an argument as above shows that the following three operators belong to  $\mathcal{K}(\mathcal{H})$ :

$$\begin{aligned} & (H + i)^{-1}(1 \otimes Q_n)(\partial_n V)(H - i)^{-1}, \\ & ((H + i)^{-1} - (H_0 + i)^{-1})(1 \otimes P_n^2)(H - i)^{-1}, \\ & (H_0 + i)^{-1}(1 \otimes P_n^2)((H - i)^{-1} - (H_0 - i)^{-1}). \end{aligned}$$

Furthermore, since  $P_c(H) = P_{ac}(H)$ , one has

$$\begin{aligned} & \text{s-lim}_{\tau \rightarrow \infty} e^{-i\tau H} (H_0 + i)^{-1} 2(1 \otimes P_n^2)(H_0 - i)^{-1} e^{i\tau H} P_c(H) \\ &= 2 \text{s-lim}_{\tau \rightarrow \infty} e^{-i\tau H} e^{i\tau H_0} (1 \otimes P_n^2) \langle H_0 \rangle^{-2} e^{-i\tau H_0} e^{i\tau H} P_{ac}(H) \\ &= 2W_-(1 \otimes P_n^2) \langle H_0 \rangle^{-2} W_-^*. \end{aligned}$$

And since strong convergence implies strong Cesaro convergence, this implies that

$$\begin{aligned} & \text{s-lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t d\tau e^{-i\tau H} (H_0 + i)^{-1} 2(1 \otimes P_n^2)(H_0 - i)^{-1} e^{i\tau H} P_c(H) \\ &= 2W_-(1 \otimes P_n^2) \langle H_0 \rangle^{-2} W_-^*. \end{aligned}$$

Thus it follows from Lemma 2.4(b) that

$$D = \text{s-lim}_{t \rightarrow \infty} D_t = 2W_-(1 \otimes P_n^2) \langle H_0 \rangle^{-2} W_-^*.$$

So, Theorem 2.1(a) applies for the net  $(U_j)_{j \in J} = (e^{-itH})_{t>0}$ , the set  $(\ell_j)_{j \in J} = (t)_{t>0}$ , the operator  $\tilde{A} = (H + i)^{-1}A(H - i)^{-1}$ , and  $D$  as above. Therefore, for each  $\varphi = D\tilde{\varphi} \in DD(A)$  and  $\psi \in \mathcal{D}(A)$  there exists a constant  $c_{\varphi,\psi} \geq 0$  such that

$$|\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \|(D - D_t)\tilde{\varphi}\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \frac{1}{t} c_{\varphi,\psi}, \quad t > 0.$$

Of course, one obtains stronger decay estimates when  $V = 0$ . Indeed, one gets in such a case  $[iH, A] = 2(1 \otimes P_n^2)$  and  $D = D_t = 2(1 \otimes P_n^2)\langle H \rangle^{-2}$  for all  $t > 0$ , from which one can infer the estimates of Theorem 2.1(c).

**3.5. Stark Hamiltonian in  $\mathbb{R}^n$**

This example is motivated by [15, Sec. 7.3] and [26, Sec. 7.1]. Let  $\mathcal{H} := L^2(\mathbb{R}^n)$  and take a unit vector  $v \in \mathbb{R}^n$ . Then the Stark Hamiltonian  $H$  with electric field along  $v$  and the operator  $A$  given by

$$\begin{aligned} (H\varphi)(x) &:= -(\Delta\varphi)(x) + (v \cdot x)\varphi(x) \quad \text{and} \\ A\varphi &:= i(v \cdot \nabla)\varphi, \quad \varphi \in \mathcal{S}(\mathbb{R}^n), \quad x \in \mathbb{R}^n. \end{aligned}$$

are essentially self-adjoint (with closures denoted by the same symbols), and  $(H - i)^{-1} \in C^\infty(A)$  with  $[iH, A] = 1$ . Thus Proposition 2.5 applies with  $f(H) = 1$ ,  $g(H) = \langle H \rangle^{-2} \in C^\infty(A)$  and  $\ker(f(H)) = \{0\}$ . It follows that  $H$  has purely a.c. spectrum (which is standard knowledge) and for each  $\varphi \in \langle H \rangle^{-2n}\mathcal{D}(A^n)$  and  $\psi \in \mathcal{D}(A^n)$  ( $n \in \mathbb{N}^*$ ) there exists a constant  $c_{\varphi,\psi} \geq 0$  such that

$$|\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \frac{1}{t^n} c_{\varphi,\psi}, \quad t > 0.$$

This estimate is similar to the result of [15, Sec. 7.3] (see [15, Thm. 6.1]).

**3.6. Fractional Laplacian in  $\mathbb{R}^n$**

This example is motivated by [15, Sec. 7.5]. For  $s \in (0, 2)$ , let  $H := (-\Delta)^{s/2}$  be the fractional Laplacian in the Hilbert space  $\mathcal{H} := L^2(\mathbb{R}^n)$  and let  $A$  be the generator of dilations in  $\mathcal{H}$ . Then  $(H - i)^{-1} \in C^\infty(A)$  with  $[iH, A] = sH$ . Thus Proposition 2.5 applies with  $f(H) = sH$ ,  $g(H) = sH\langle H \rangle^{-2} \in C^\infty(A)$  and  $\ker(f(H)) = \ker(H)$ . Since an application of the Fourier transform shows that  $H$  has spectrum equal to  $[0, \infty)$  and  $\ker(H) = \{0\}$ , it follows that  $H$  has purely a.c. spectrum equal to  $[0, \infty)$  (which is standard knowledge) and for each  $\varphi \in g(H)^n\mathcal{D}(A^n)$  and  $\psi \in \mathcal{D}(A^n)$  ( $n \in \mathbb{N}^*$ ) there exists a constant  $c_{\varphi,\psi} \geq 0$  such that

$$|\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \frac{1}{t^n} c_{\varphi,\psi}, \quad t > 0.$$

This estimate improves the result of [15, Sec. 7.5] (see [15, Thm. 6.3]).

**3.7. Horocycle Flow**

This example is motivated by [34,37], but see also [35,38]. Let  $\Sigma$  be a finite volume Riemann surface of genus  $\geq 2$  and let  $M := T^1\Sigma$  be the unit tangent bundle of  $\Sigma$ . The 3-manifold  $M$  carries a probability measure  $\mu_\Omega$ , induced by a canonical volume form  $\Omega$ , which is preserved by two distinguished one-parameter groups of diffeomorphisms: the horocycle flow  $F_1 := (F_{1,t})_{t \in \mathbb{R}}$  and

the geodesic flow  $F_2 := (F_{2,t})_{t \in \mathbb{R}}$ . Each flow admits a self-adjoint generator  $H_j$  in  $\mathcal{H} := L^2(M, \mu_\Omega)$  essentially self-adjoint on  $C_c^\infty(M)$  and given by

$$H_j \varphi := -i \mathcal{L}_{X_j} \varphi, \quad \varphi \in C_c^\infty(M),$$

with  $X_j$  the divergence-free vector field associated to  $F_j$  and  $\mathcal{L}_{X_j}$  the corresponding Lie derivative. Moreover, one has  $(H_1 - i)^{-1} \in C^\infty(H_2)$  with  $[iH_1, H_2] = H_1$ , see [34, Sec. 3]. Thus Proposition 2.5 applies with  $f(H_1) = H_1$ ,  $g(H_1) = H_1 \langle H_1 \rangle^{-2} \in C^\infty(H_2)$  and  $\ker(f(H_1)) = \ker(H_1)$ . It follows that  $H_1|_{\ker(H_1)^\perp}$  has purely a.c. spectrum (which is standard knowledge) and for each  $\varphi \in g(H_1)^n \mathcal{D}(H_2^n)$  and  $\psi \in \mathcal{D}(H_2^n)$  ( $n \in \mathbb{N}^*$ ) there exists a constant  $c_{\varphi, \psi} \geq 0$  such that

$$|\langle \varphi, \psi \circ F_{1,t} \rangle_{\mathcal{H}}| = |\langle \varphi, e^{-itH_1} \psi \rangle_{\mathcal{H}}| \leq \frac{1}{t^n} c_{\varphi, \psi}, \quad t > 0.$$

Roughly, this estimate shows that if the vectors  $\varphi, \psi$  are  $n$  times differentiable along the geodesic flow, then their correlation along the horocycle flow decays as  $\frac{1}{t^n}$ . It thus provides a new version of the polynomial decay of correlations for the horocycle flow on the unit tangent bundle of a finite volume surface of constant negative curvature. And its (rather short) proof did not use the identification of  $M$  with a homogeneous space  $\Gamma \backslash \mathrm{PSL}(2, \mathbb{R})$  and the representation theory associated to it as is customary (see for instance [22]).

Now, assume that  $\Sigma$  is compact and consider a  $C^1$  vector field proportional to  $X_1$ , that is,  $fX_1$  with  $f \in C^1(M, (0, \infty))$ . The vector field  $fX_1$  admits a complete flow  $\tilde{F}_1 := (\tilde{F}_{1,t})_{t \in \mathbb{R}}$  uniquely ergodic with respect to the measure  $\tilde{\mu}_\Omega := \frac{f^{-1} \mu_\Omega}{\int_M f^{-1} d\mu_\Omega}$  and with generator  $H := fH_1$  essentially self-adjoint on  $C^1(M) \subset \mathcal{H} := L^2(M, f^{-1} \mu_\Omega)$ . Furthermore, the following holds true [37, Ex. 4.8]: The operator  $A := f^{1/2} H_2 f^{-1/2}$  is essentially self-adjoint on  $C^1(M) \subset \mathcal{H}$ , one has  $(H - i)^{-1} \in C^1(A)$  with

$$[iH, A] = H\xi + \xi H, \quad \xi := \frac{1}{2} - \frac{1}{2} f^{-1} \mathcal{L}_{X_2}(f),$$

and  $D = \mathrm{s}\text{-}\lim_{t \rightarrow \infty} D_t = H \langle H \rangle^{-2}$ . Thus Theorem 2.1(a) applies for the net  $(U_j)_{j \in J} = (e^{-itH})_{t > 0}$ , the set  $(\ell_j)_{j \in J} = (t)_{t > 0}$ , the operator  $\tilde{A} = (H + i)^{-1} A (H - i)^{-1}$ , and  $D = H \langle H \rangle^{-2}$ . Therefore, for each  $\varphi = D\tilde{\varphi} \in D\mathcal{D}(A)$  and  $\psi \in \mathcal{D}(A)$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that

$$|\langle \varphi, \psi \circ \tilde{F}_{1,t} \rangle_{\mathcal{H}}| = |\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \|(D - D_t)\tilde{\varphi}\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \frac{1}{t} c_{\varphi, \psi}, \quad t > 0. \quad (3.4)$$

In this case, one cannot easily improve the decay estimate (3.4) with the tools of this paper. Indeed, in order to establish the convergence  $D_t \xrightarrow{s} D$ , one uses the ergodic theorem for uniquely ergodic flows (see [37]) which does not come with an explicit rate of convergence. We refer to [10, Thm. 19] for a more quantitative decay estimate obtained using the representation theory of  $\mathrm{SL}(2, \mathbb{R})$ .

### 3.8. Adjacency Matrices

This example is motivated by [21] and [37, Ex. 4.7]. Let  $(X, \sim)$  be a graph of finite degree, with symmetric relation  $\sim$ , and with no multiple edges or loops.

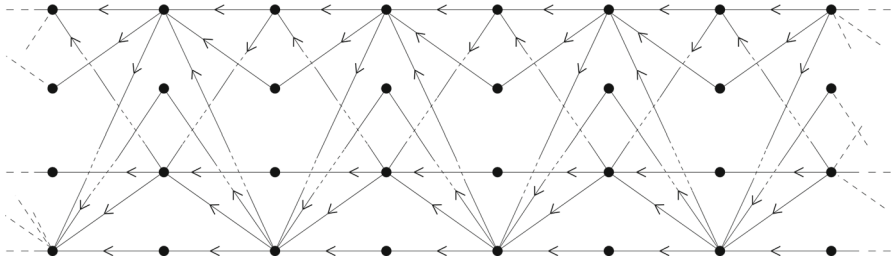


FIGURE 1. Example of admissible graph

Then the adjacency matrix of  $(X, \sim)$  is the bounded self-adjoint operator in the Hilbert space  $\mathcal{H} := \ell^2(X)$  given by

$$(H\varphi)(x) := \sum_{y \sim x} \varphi(y), \quad \varphi \in \mathcal{H}, x \in X.$$

A directed graph  $(X, <)$  subjacent to  $(X, \sim)$  is the graph  $(X, \sim)$  together with a relation  $<$  on  $X$  such that, for each  $x, y \in X$ ,  $x \sim y$  is equivalent to  $x < y$  or  $y < x$  (and one cannot have both  $x < y$  and  $y < x$ ). When using drawings, we draw an arrow  $x \leftarrow y$  if  $x < y$ . For each  $x \in X$ , we define the sets  $N^-(x) := \{y \in X \mid x < y\}$  and  $N^+(x) := \{y \in X \mid y < x\}$ . Then a directed graph  $(X, <)$  is called admissible if [21, Def. 5.1]:

- (i) each closed path in  $X$  has the same number of positively and negatively oriented edges,
- (ii) for each  $x, y \in X$ , one has  $\#\{N^-(x) \cap N^-(y)\} = \#\{N^+(x) \cap N^+(y)\}$ .

In the case of admissible graphs, see for example Fig. 1, it is shown in [21, Secs. 3–5] that there exist a self-adjoint operator  $A$  and a bounded self-adjoint operator  $K$  in  $\mathcal{H}$  such that  $H \in C^1(A)$  with  $[iH, A] = K^2$ ,  $K^2 \in C^1(A)$  with  $[iK^2, A] = -2HK^2$ , and  $[K, H] = 0$ . Thus Lemma 2.4 applies with  $D = D_t = K^2 \langle H \rangle^{-2} \in C^1(A)$  for all  $t > 0$  and  $\ker(D) = \ker(K)$ . So  $H|_{\ker(K)^\perp}$  has purely a.c. spectrum, as was first proved in [21, Thm. 1.1] with a more complicated method.

Now, since  $H, K^2 \in C^1(A)$ , we have  $(H \pm i)^{-1} \mathcal{D}(A) \subset \mathcal{D}(A)$  and  $K^2 \mathcal{D}(A) \subset \mathcal{D}(A)$ . Thus a calculation using the above commutation relations and the operator  $\tilde{A} := (H + i)^{-1} A (H - i)^{-1}$  gives on  $\mathcal{D}(A)$ :

$$\begin{aligned} [\tilde{A}, D] &= [(H + i)^{-1} A (H - i)^{-1}, K^2 \langle H \rangle^{-2}] \\ &= K^2 (H + i)^{-1} [A, \langle H \rangle^{-2}] (H - i)^{-1} \\ &\quad + (H + i)^{-1} [A, K^2] (H - i)^{-1} \langle H \rangle^{-2} \\ &= K^2 (H + i)^{-1} (-2iK^2 H \langle H \rangle^{-4}) (H - i)^{-1} \\ &\quad + (H + i)^{-1} (-2iHK^2) (H - i)^{-1} \langle H \rangle^{-2} \\ &= -2iK^2 H \langle H \rangle^{-4} (K^2 \langle H \rangle^{-2} + 1) \\ &= -2iDH \langle H \rangle^{-2} (D + 1). \end{aligned}$$

Since  $DH\langle H \rangle^{-2}(D+1) \in \mathcal{B}(\mathcal{H})$  and  $\mathcal{D}(A)$  is a core for  $\tilde{A}$ , we infer that  $D \in C^1(\tilde{A})$  with  $[\tilde{A}, D] = -2iDH\langle H \rangle^{-2}(D+1)$ . So, in the present case, the operator  $B \in \mathcal{B}(\mathcal{H})$  appearing in the statement of Theorem 2.1(c) is  $B = -2iH\langle H \rangle^{-2}(D+1)$  and it satisfies  $[D, B] = 0$ . Furthermore, a repeated use of the information gathered so far shows that each factor appearing in the expression for  $B$  belongs to  $C^{(n-1)}(\tilde{A})$  for any  $n \in \mathbb{N}^*$ . Thus  $B \in C^{(n-1)}(\tilde{A})$  for any  $n \in \mathbb{N}^*$ . It follows that the assumptions of Theorem 2.1(c) are satisfied for the net  $(U_j)_{j \in J} = (e^{-itH})_{t>0}$ , the set  $(\ell_j)_{j \in J} = (t)_{t>0}$ , the operator  $\tilde{A}$ , and  $D = K^2\langle H \rangle^{-2}$ . Thus for each  $\varphi \in D^n\mathcal{D}(A^n)$  and  $\psi \in \mathcal{D}(A^n)$  ( $n \in \mathbb{N}^*$ ) there exists a constant  $c_{\varphi, \psi} \geq 0$  such that

$$|\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \frac{1}{t^n} c_{\varphi, \psi}, \quad t > 0.$$

### 3.9. Jacobi Matrices

This example is motivated by [28]. Let  $\mathcal{H} := \ell^2(\mathbb{N}^*)$ , let  $\ell_0^2 \subset \mathcal{H}$  be the set of functions  $\mathbb{N}^* \rightarrow \mathbb{C}$  with compact support, and let  $H$  be the Jacobi matrix

$$(H\varphi)(n) := a_{n-1}\varphi(n-1) + b_n\varphi(n) + a_n\varphi(n+1), \quad \varphi \in \ell_0^2, \quad n \in \mathbb{N}^*, \quad \varphi(0) := 0,$$

with coefficients

$$a_n := n^\alpha + \eta_n, \quad b_n := \lambda(n^\alpha + (n-1)^\alpha) + \beta_n, \quad \alpha > 0, \quad \lambda, \eta_n, \beta_n \in \mathbb{R}.$$

For any sequence  $(r_n)_{n \in \mathbb{N}^*}$  and  $k \in \mathbb{N}^*$ , define by induction  $(\partial^{k+1}r)_n := (\partial\partial^k r)_n$  with  $(\partial r)_n := r_{n+1} - r_n$ , set

$$\beta'_n := \beta_n - \lambda(\eta_n + \eta_{n-1}), \quad n \in \mathbb{N}^*, \quad \eta_0 := 0,$$

and assume the following:

**Assumption 3.2.** Suppose that  $|\lambda| < 1$ ,  $\alpha \in (0, 1)$ , and assume that the sequences with elements

$$\frac{\eta_n}{n^\alpha}, \quad (\partial\eta)_n, \quad n^{1-2\alpha}(\partial\eta)_n, \quad n^{1-\alpha}(\partial^2\eta)_n, \quad \beta'_n, \quad n^{1-\alpha}(\partial\beta')_n, \quad (n \in \mathbb{N}^*)$$

vanish as  $n \rightarrow \infty$ .

Then it is shown in [28, Thms 1.1–1.4] that  $H$  is essentially self-adjoint (with closure denoted by the same symbol), that  $\sigma_{\text{ess}}(H) = \mathbb{R}$ , and that the eigenvalues of  $H$  (if any) are of finite multiplicity and can accumulate only at  $\pm\infty$ . Furthermore, it is shown in the proof of [28, Cor. 8.1] that there exists a self-adjoint operator  $A$  in  $\mathcal{H}$  such that  $(H-i)^{-1} \in C^1(A)$  with

$$[iH, A] = 4(1-\lambda^2)(1-\alpha) + K, \quad K \in \mathcal{K}(\mathcal{D}(H), \mathcal{H}).$$

Since  $(H+i)^{-1}K(H-i)^{-1} \in \mathcal{K}(\mathcal{H})$  and

$$\begin{aligned} & \text{s-lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t d\tau e^{-i\tau H} (H+i)^{-1} 4(1-\lambda^2)(1-\alpha) (H-i)^{-1} e^{i\tau H} P_c(H) \\ &= 4(1-\lambda^2)(1-\alpha)\langle H \rangle^{-2} P_c(H), \end{aligned}$$

it follows from Lemma 2.4 that

$$D = \text{s-lim}_{t \rightarrow \infty} D_t = 4(1-\lambda^2)(1-\alpha)\langle H \rangle^{-2} P_c(H)$$



and that  $H|_{\ker(D)^\perp} = H|_{P_c(H)\mathcal{H}}$  has (trivially) purely continuous spectrum. Moreover, Theorem 2.1(a) applies for the net  $(U_j)_{j \in J} = (e^{-itH})_{t>0}$ , the set  $(\ell_j)_{j \in J} = (t)_{t>0}$ , the operator  $\tilde{A} = (H + i)^{-1}A(H - i)^{-1}$ , and  $D$  as above. Therefore, for each  $\varphi = D\tilde{\varphi} \in \mathcal{D}\mathcal{D}(A)$  and  $\psi \in \mathcal{D}(A)$  there exists a constant  $c_{\varphi,\psi} \geq 0$  such that

$$|\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \|(D - D_t)\tilde{\varphi}\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \frac{1}{t} c_{\varphi,\psi}, \quad t > 0.$$

One can obtain stronger decay estimates in particular cases. For instance, it is shown in [28, Appx. A] that for certain choices of  $a_n$  and  $b_n$  one gets the relation  $[iH, A] = aH + b$  with  $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ , from which one can infer the estimates of Theorem 2.1(c).

### 3.10. Schrödinger Operators on Fock Spaces

This example is motivated by [14]. Let  $U$  be an isometry in a Hilbert space  $\mathcal{H}$  and let  $N$  be a number operator for  $U$ , that is, a self-adjoint operator in  $\mathcal{H}$  such that  $U^*\mathcal{D}(N) \subset \mathcal{D}(N)$  and  $UNU^* = N - 1$  on  $\mathcal{D}(N)$ . Number operators do not always exist, but if  $U$  is completely non-unitary, namely if  $\text{s-lim}_{n \rightarrow \infty} (U^*)^n = 0$ , then  $N$  exists and is unique. Next, consider the bounded self-adjoint operators  $H := \text{Re}(U)$  and  $S := \text{Im}(U)$ , and set

$$A\varphi := \frac{1}{2}(SN + NS)\varphi, \quad \varphi \in \mathcal{D}(N).$$

Then it is shown in [14, Secs. 2–3] that  $A$  is essentially self-adjoint (with closure denoted by the same symbol) and that  $H \in C^\infty(A)$  with  $[iH, A] = 1 - H^2$ . Thus Proposition 2.5 applies with  $f(H) = 1 - H^2$ ,  $g(H) = (1 - H^2)\langle H \rangle^{-2} \in C^\infty(A)$  and  $\ker(f(H)) = E^H(\{-1, 1\})\mathcal{H}$ . It follows that  $H$  has purely a.c. spectrum in  $(-1, 1)$  and for each  $\varphi \in g(H)^n \mathcal{D}(A^n)$  and  $\psi \in \mathcal{D}(A^n)$  ( $n \in \mathbb{N}^*$ ) there exists a constant  $c_{\varphi,\psi} \geq 0$  such that

$$|\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \frac{1}{t^n} c_{\varphi,\psi}, \quad t > 0.$$

Operators like  $H$  (and additive perturbations of it) appear naturally in the framework of Fock spaces and their applications to Schrödinger operators on trees. Indeed, let  $\mathfrak{h}$  be a complex Hilbert space and  $\mathcal{H} := \bigoplus_{n=0}^\infty \mathfrak{h}^{\otimes n}$  the complete Fock space associated to it (with  $\mathfrak{h}^{\otimes 0} := \mathbb{C}$  and  $\mathfrak{h}^{\otimes n} := \{0\}$  if  $n < 0$ ). For any  $u \in \mathfrak{h}$  with  $\|u\|_{\mathfrak{h}} = 1$ , let  $U \in \mathcal{B}(\mathcal{H})$  act on the sector  $\mathfrak{h}^{\otimes n}$  as

$$U(h_1 \otimes \cdots \otimes h_n) := h_1 \otimes \cdots \otimes h_n \otimes u, \quad h_1 \otimes \cdots \otimes h_n \in \mathfrak{h}^{\otimes n}.$$

Then  $U$  is completely non-unitary, the operator  $H = \text{Re}(U)$  acts on the sector  $\mathfrak{h}^{\otimes n}$  as

$$\begin{cases} H(h_1 \otimes \cdots \otimes h_n) \\ = \frac{1}{2}(h_1 \otimes \cdots \otimes h_n \otimes u + h_1 \otimes \cdots \otimes h_{n-1} \langle h_n, u \rangle_{\mathfrak{h}}) & \text{if } n \geq 1, \\ Hh = \frac{1}{2}hu & \text{if } h \in \mathfrak{h}^{\otimes 0} = \mathbb{C}, \end{cases}$$

and the number operator  $N$  for  $U$  can be explicitly described (see [14, Sec. 4] for more details).

**3.11. Multiplication by  $\lambda$  in  $L^2(\mathbb{R}_+, d\mu)$**

This example is motivated by [15, Sec. 7.6]. Let  $H$  be the maximal multiplication operator by the variable  $\lambda \in \mathbb{R}_+$  in the Hilbert space  $\mathcal{H} := L^2(\mathbb{R}_+, d\mu)$ , with  $d\mu := h d\lambda$  and  $h \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ , and assume that the function

$$q : \mathbb{R}_+ \rightarrow \mathbb{R}, \lambda \mapsto \frac{\lambda h'(\lambda)}{h(\lambda)} + 1,$$

is bounded. Then the operator

$$A\varphi := -\frac{i}{4}(2H\varphi' + q\varphi), \quad \varphi \in C_c^\infty(\mathbb{R}_+),$$

is essentially self-adjoint (with closure denoted by the same symbol) and  $(H - i)^{-1} \in C^\infty(A)$  with  $[iH, A] = -\frac{1}{2}H$ .<sup>2</sup> Thus Proposition 2.5 applies with  $f(H) = -\frac{1}{2}H$ ,  $g(H) = -\frac{1}{2}H\langle H \rangle^{-2} \in C^\infty(A)$  and  $\ker(f(H)) = \ker(H)$ . Furthermore, since

$$\mu(\{\lambda \in [0, \infty) \mid \lambda = 0\}) = 0,$$

we have that  $\ker(H) = \{0\}$ . It follows that  $H$  has purely a.c. spectrum equal to  $[0, \infty)$  and for each  $\varphi \in g(H)^n \mathcal{D}(A^n)$  and  $\psi \in \mathcal{D}(A^n)$  ( $n \in \mathbb{N}^*$ ) there exists a constant  $c_{\varphi, \psi} \geq 0$  such that

$$|\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \frac{1}{t^n} c_{\varphi, \psi}, \quad t > 0.$$

This estimate improves the result of [15, Sec. 7.6] (see [15, Thm. 6.3]).

**3.12.  $H = -\partial_{xx} + \partial_{yy}$  in  $\mathbb{R}^2$**

This example is motivated by [15, Sec. 7.2]. Consider in the Hilbert space  $\mathcal{H} := L^2(\mathbb{R}^2)$  the partial differential operator

$$H\varphi := (-\partial_{xx} + \partial_{yy})\varphi, \quad \varphi \in \mathcal{S}(\mathbb{R}^2),$$

which is essentially self-adjoint (with closure denoted by the same symbol), and  $A$  the generator of dilations in  $\mathcal{H}$ . Then we have  $(H - i)^{-1} \in C^\infty(A)$  with  $[iH, A] = 2H$ . Thus Proposition 2.5 applies with  $f(H) = 2H$ ,  $g(H) = 2H\langle H \rangle^{-2} \in C^\infty(A)$  and  $\ker(f(H)) = \ker(H)$ . Since an application of the Fourier transform shows that  $H$  has spectrum equal to  $\mathbb{R}$  and  $\ker(H) = \{0\}$ , it follows that  $H$  has purely a.c. spectrum equal to  $\mathbb{R}$ , and for each  $\varphi \in g(H)^n \mathcal{D}(A^n)$  and  $\psi \in \mathcal{D}(A^n)$  ( $n \in \mathbb{N}^*$ ) there exists a constant  $c_{\varphi, \psi} \geq 0$  such that

$$|\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \frac{1}{t^n} c_{\varphi, \psi}, \quad t > 0.$$

This estimate improves the result of [15, Sec. 7.2] (see [15, Thm. 6.3]).

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<sup>2</sup>In [15, Sec. 7.6], there is a small mistake in the calculation of the commutator  $[iH, A]$ . This is why our operator  $A$  and commutator  $[iH, A]$  slightly differ from the ones appearing in [15, Sec. 7.6].

**3.13.  $H = -X^{2-s}\Delta - \Delta X^{2-s}$  in  $\mathbb{R}_+$**

This example is motivated by [15, Sec. 7.4]. Let  $X$  be the maximal multiplication operator by the variable  $x \in \mathbb{R}_+$  in the Hilbert space  $\mathcal{H} := L^2(\mathbb{R}_+)$  and let  $A$  be the generator of dilations in  $\mathcal{H}$ . Then the operator

$$H\varphi := (-X^{2-s}\Delta - \Delta X^{2-s})\varphi, \quad \varphi \in C_c^\infty(\mathbb{R}_+), \quad s \in (0, 2),$$

is essentially self-adjoint (with closure denoted by the same symbol) and  $(H - i)^{-1} \in C^\infty(A)$  with  $[iH, A] = sH$ . In particular, one obtains that  $e^{-itA} H = e^{st} H e^{-itA}$  for all  $t \in \mathbb{R}$  which implies that  $\ker(H) = \{0\}$ , since otherwise  $\ker(H)$  would be a nontrivial invariant subspace of the irreducible representation  $\mathbb{R} \ni t \mapsto e^{-itA} \in U(\mathcal{H})$ . Thus Proposition 2.5 applies with  $f(H) = sH, g(H) = sH\langle H \rangle^{-2} \in C^\infty(A)$  and  $\ker(f(H)) = \ker(H) = \{0\}$ . It follows that  $H$  has purely a.c. spectrum, and for each  $\varphi \in g(H)^n \mathcal{D}(A^n)$  and  $\psi \in \mathcal{D}(A^n)$  ( $n \in \mathbb{N}^*$ ) there exists a constant  $c_{\varphi, \psi} \geq 0$  such that

$$|\langle \varphi, e^{-itH} \psi \rangle_{\mathcal{H}}| \leq \frac{1}{t^n} c_{\varphi, \psi}, \quad t > 0.$$

This estimate improves the result of [15, Sec. 7.4] (see [15, Thm. 6.3]).

**3.14. Quantum Walks on  $\mathbb{Z}$**

This example is motivated by [24, 25]. Consider a quantum walk on  $\mathbb{Z}$  with evolution operator  $U := SC$  in the Hilbert space  $\mathcal{H} := \ell^2(\mathbb{Z}, \mathbb{C}^2)$ , where  $S$  is the shift operator defined as

$$(S\varphi)(x) := \begin{pmatrix} \varphi^{(0)}(x+1) \\ \varphi^{(1)}(x-1) \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi^{(0)} \\ \varphi^{(1)} \end{pmatrix} \in \mathcal{H}, \quad x \in \mathbb{Z},$$

and  $C$  the coin operator defined as

$$(C\varphi)(x) := C(x)\varphi(x), \quad \varphi \in \mathcal{H}, \quad x \in \mathbb{Z}, \quad C(x) \in U(2).$$

Assume that  $C$  is anisotropic, namely, converging with short-range rate to an asymptotic coin on the left and to an asymptotic coin on the right:

**Assumption 3.3.** There exist  $C_\ell, C_r \in U(2)$ ,  $\kappa_\ell, \kappa_r > 0$ , and  $\varepsilon_\ell, \varepsilon_r > 0$  such that

$$\begin{aligned} \|C(x) - C_\ell\|_{\mathcal{B}(\mathbb{C}^2)} &\leq \kappa_\ell |x|^{-1-\varepsilon_\ell}, \quad x < 0, \\ \|C(x) - C_r\|_{\mathcal{B}(\mathbb{C}^2)} &\leq \kappa_r |x|^{-1-\varepsilon_r}, \quad x > 0, \end{aligned}$$

where the indexes  $\ell$  and  $r$  stand for “left” and “right”.

Under this assumption, it is shown in [24, Sec. 4] that  $U$  has no singular continuous spectrum and that the eigenvalues of  $U$  (if any) are of finite multiplicity and can accumulate only at a finite set of threshold values. Furthermore, there exist a self-adjoint operator  $A$  in  $\mathcal{H}$ , a unitary operator  $U_0$  in an auxiliary Hilbert space  $\mathcal{H}_0$ , a self-adjoint operator  $A_0$  in  $\mathcal{H}_0$  and  $J \in \mathcal{B}(\mathcal{H}_0, \mathcal{H})$  such that  $U \in C^1(A), U_0 \in C^1(A_0)$  and

$$[A_0, U_0]U_0^{-1} = V_0^2 \quad \text{and} \quad [A, U]U^{-1} - J[A_0, U_0]U_0^{-1}J^* \in \mathcal{K}(\mathcal{H})$$

with  $V_0 \in \mathcal{B}(\mathcal{H}_0)$  an asymptotic velocity operator for  $U_0$  satisfying  $[V_0, U_0] = 0$ . Since  $P_c(U) = P_{ac}(U)$ , one then infers from [25, Sec. 3] that

$$\begin{aligned} & \text{s-lim}_{n \rightarrow \infty} U^n J V_0^2 J^* U^{-n} P_c(U) \\ &= \left( \text{s-lim}_{n \rightarrow \infty} U^n J U_0^{-n} P_{ac}(U_0) \right) V_0^2 \left( \text{s-lim}_{n \rightarrow \infty} U_0^n J^* U^{-n} P_{ac}(U) \right) \\ &= W_- V_0^2 W_-^* \end{aligned}$$

with

$$W_{\pm} := \text{s-lim}_{n \rightarrow \pm\infty} U^{-n} J U_0^n P_{ac}(U_0) \quad (3.5)$$

the wave operators for the triple  $(U, U_0, J)$ . Since strong convergence implies strong Cesaro convergence, we obtain that

$$\text{s-lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} U^m J V_0^2 J^* U^{-m} P_c(U) = W_- V_0^2 W_-^*,$$

and thus it follows from Lemma 2.7(b) that

$$D = \text{s-lim}_{n \rightarrow \infty} D_n = W_- V_0^2 W_-^*.$$

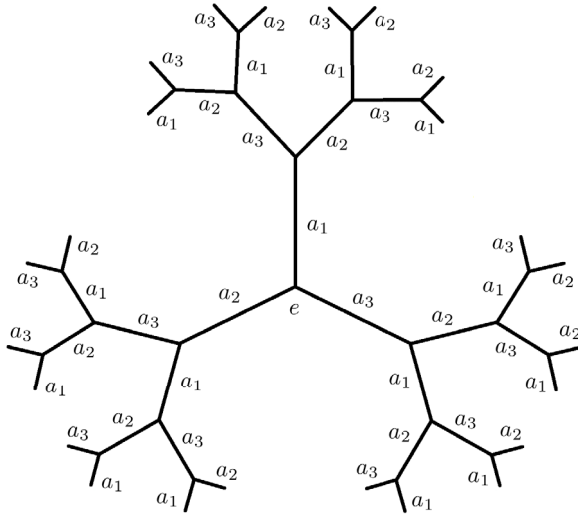
So, Theorem 2.1(a) applies for the net  $(U_j)_{j \in J} = (U^n)_{n \in \mathbb{N}^*}$ , the set  $(\ell_j)_{j \in J} = (n)_{n \in \mathbb{N}^*}$ , the operator  $A$ , and  $D = W_- V_0^2 W_-^*$ . Therefore, for each  $\varphi = D\tilde{\varphi} \in D\mathcal{D}(A)$  and  $\psi \in \mathcal{D}(A)$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that

$$|\langle \varphi, U^n \psi \rangle_{\mathcal{H}}| \leq \|(D - D_n)\tilde{\varphi}\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \frac{1}{n} c_{\varphi, \psi}, \quad n \in \mathbb{N}^*. \quad (3.6)$$

In this example, one cannot easily improve the decay estimate (3.6). Indeed, in order to establish the convergence  $D_n \xrightarrow{s} D$ , we used Lemma 2.7(b). But the proof of Lemma 2.7(b) relies on the RAGE theorem B.1, whose proof relies in turn on a discrete version of Wiener's theorem. And as in the continuous case, in general one cannot infer an explicit rate of convergence from Wiener's theorem. In addition, we used the strong limits (3.5) which do not come with an explicit rate of convergence either.

### 3.15. Quantum Walks on Trees

This example is motivated by [40]. Let  $\mathcal{T}$  be a homogeneous tree of odd degree  $d \geq 3$ , that is, a finitely generated group  $\mathcal{T}$  with generators  $a_1, \dots, a_d$ , identity  $e$ , and presentation  $\mathcal{T} := \langle a_1, \dots, a_d \mid a_1^2 = \dots = a_d^2 = e \rangle$ .



Homogeneous tree  $\mathcal{T}$  of degree  $d = 3$

Using the word length  $|\cdot|$  on  $\mathcal{T}$ , we define sets of even/odd elements of  $\mathcal{T}$

$$\mathcal{T}_e := \{x \in \mathcal{T} \mid |x| \in 2\mathbb{N}\} \quad \text{and} \quad \mathcal{T}_o := \{x \in \mathcal{T} \mid |x| \in 2\mathbb{N} + 1\}$$

with corresponding characteristic functions  $\chi_e$  and  $\chi_o$ . Then we consider a quantum walk on  $\mathcal{T}$  with evolution operator  $U := SC$  in the Hilbert space  $\mathcal{H} := \ell^2(\mathcal{T}, \mathbb{C}^d)$ , where  $S$  is the shift operator defined as

$$S := \begin{pmatrix} S_{1+1,1+2} & & & 0 \\ & S_{2+1,2+2} & & \\ 0 & & \ddots & \\ & & & S_{d+1,d+2} \end{pmatrix}, \quad S_{d,d+1} := S_{d,1}, \quad S_{d+1,d+2} := S_{1,2},$$

$$S_{i,j}f := \chi_e f(\cdot a_i) + \chi_o f(\cdot a_j), \quad i, j \in \{1, \dots, d\}, \quad f \in \ell^2(\mathcal{T}),$$

and  $C$  the coin operator defined as

$$(C\varphi)(x) := C(x)\varphi(x), \quad \varphi \in \mathcal{H}, \quad x \in \mathcal{T}, \quad C(x) \in U(d).$$

Assume that  $C$  is anisotropic, namely, converging with short-range rate to a diagonal asymptotic coin on each main branch of  $\mathcal{T}$ :

**Assumption 3.4.** For  $i = 1, \dots, d$ , there exist a diagonal matrix  $C_i \in U(d)$  and  $\varepsilon_i > 0$  such that

$$\|C(x) - C_i\|_{\mathcal{B}(\mathbb{C}^d)} \leq \text{Const.} \langle x \rangle^{-(1+\varepsilon_i)}, \quad x \in \mathcal{T}_i,$$

where  $\mathcal{T}_i := \{x \in \mathcal{T} \mid |a_i x| = |x| - 1\}$ .

Under this assumption, it is shown in [40, Sec. 5] that the spectrum of  $U$  covers the whole unit circle and is purely absolutely continuous, outside possibly a finite set where  $U$  may have eigenvalues of finite multiplicity. Furthermore, there exist a self-adjoint operator  $A$  in  $\mathcal{H}$ , a unitary operator  $U_0$  in an

auxiliary Hilbert space  $\mathcal{H}_0$ , a self-adjoint operator  $A_0$  in  $\mathcal{H}_0$  and  $J \in \mathcal{B}(\mathcal{H}_0, \mathcal{H})$  such that  $U \in C^1(A)$ ,  $U_0 \in C^\infty(A_0)$  and

$$JJ^* = 1_{\mathcal{H}}, \quad [A_0, U_0]U_0^{-1} = 2, \quad [A, U]U^{-1} - J[A_0, U_0]U_0^{-1}J^* \in \mathcal{K}(\mathcal{H}).$$

It then follows from Lemma 2.7(b) that

$$D = \text{s-lim}_{n \rightarrow \infty} D_n = \text{s-lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} U^m (J[A_0, U_0]U_0^{-1}J^*)U^{-m} P_c(U) = 2P_c(U).$$

Thus, Theorem 2.1(a) applies for the net  $(U_j)_{j \in J} = (U^n)_{n \in \mathbb{N}^*}$ , the set  $(\ell_j)_{j \in J} = (n)_{n \in \mathbb{N}^*}$ , the operator  $A$ , and  $D = 2P_c(U)$ . So for each  $\varphi = D\tilde{\varphi} \in D\mathcal{D}(A)$  and  $\psi \in \mathcal{D}(A)$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that

$$|\langle \varphi, U^n \psi \rangle_{\mathcal{H}}| \leq \|(D - D_n)\tilde{\varphi}\|_{\mathcal{H}} \|\psi\|_{\mathcal{H}} + \frac{1}{n} c_{\varphi, \psi}, \quad n \in \mathbb{N}^*. \quad (3.7)$$

As in the example of the previous section, one cannot easily improve the decay estimate (3.7).

### 3.16. Skew Products

This example is motivated by [39], but see also [5, 35–37]. Let  $X$  be a smooth compact second countable Hausdorff manifold with Borel probability measure  $\mu_X$ , and let  $(F_t)_{t \in \mathbb{R}}$  be a  $C^1$  measure-preserving flow on  $X$ . The operators  $V_t : L^2(X, \mu_X) \rightarrow L^2(X, \mu_X)$  given by  $V_t \varphi := \varphi \circ F_t$  define a strongly continuous one-parameter unitary group with self-adjoint generator  $H$  in  $L^2(X, \mu_X)$  essentially self-adjoint on  $C^1(X)$  and given by

$$H\varphi := i\mathcal{L}_Y \varphi, \quad \varphi \in C^1(X),$$

with  $Y$  the  $C^0$  vector field associated to  $F$  and  $\mathcal{L}_Y$  the corresponding Lie derivative.

Let  $G$  be a compact Lie group with identity  $e_G$ , normalised Haar measure  $\mu_G$ , and Lie algebra  $\mathfrak{g}$ . Then any measurable function  $\phi : X \rightarrow G$  induces a measurable cocycle  $X \times \mathbb{Z} \ni (x, n) \mapsto \phi^{(n)}(x) \in G$  over  $F_1$  given by

$$\phi^{(n)}(x) := \begin{cases} \phi(x)(\phi \circ F_1)(x) \cdots (\phi \circ F_{n-1})(x) & \text{if } n \geq 1 \\ e_G & \text{if } n = 0 \\ (\phi^{(-n)} \circ F_n)(x)^{-1} & \text{if } n \leq -1. \end{cases}$$

The skew product  $T_\phi$  defined by

$$T_\phi : X \times G \rightarrow X \times G, \quad (x, g) \mapsto (F_1(x), g\phi(x)),$$

is an automorphism of  $(X \times G, \mu_X \otimes \mu_G)$ , and the corresponding Koopman operator

$$U_\phi \psi := \psi \circ T_\phi, \quad \psi \in \mathcal{H} := L^2(X \times G, \mu_X \otimes \mu_G),$$

is a unitary operator in  $\mathcal{H}$ .

Let  $\widehat{G}$  be the set of (equivalence classes of) finite-dimensional irreducible unitary representations of  $G$ . Then each  $\pi \in \widehat{G}$  is a  $C^\infty$  group homomorphism from  $G$  to the unitary group  $U(d_\pi)$  of degree  $d_\pi := \dim(\pi)$ , and Peter–Weyl’s theorem implies that the set of all matrix elements  $\{\pi_{jk}\}_{j,k=1}^{d_\pi}$  of all  $\pi \in \widehat{G}$

forms an orthogonal basis of  $L^2(G, \mu_G)$ . Accordingly, one has the orthogonal decomposition

$$\mathcal{H} = \bigoplus_{\pi \in \widehat{G}} \bigoplus_{j=1}^{d_\pi} \mathcal{H}_j^{(\pi)}, \quad \mathcal{H}_j^{(\pi)} := \bigoplus_{k=1}^{d_\pi} L^2(X, \mu_X) \otimes \{\pi_{jk}\}, \tag{3.8}$$

and  $U_\phi$  is reduced by the decomposition (3.8), with restriction  $U_{\phi, \pi, j} := U_\phi|_{\mathcal{H}_j^{(\pi)}}$  given by

$$U_{\phi, \pi, j} \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} = \sum_{k, \ell=1}^{d_\pi} (\varphi_k \circ F_1)(\pi_{\ell k} \circ \phi) \otimes \pi_{j\ell}, \quad \varphi_k \in L^2(X, \mu_X).$$

Furthermore, the following holds true [39, Sec. 3]: The operator  $A$  given by

$$A \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} := \sum_{k=1}^{d_\pi} H \varphi_k \otimes \pi_{jk}, \quad \varphi_k \in C^1(X),$$

is essentially self-adjoint in  $\mathcal{H}_j^{(\pi)}$  (with closure denoted by the same symbol). If  $\mathcal{L}_Y(\pi \circ \phi) \in L^\infty(X, \mathcal{B}(\mathbb{C}^{d_\pi}))$ , then  $U_{\phi, \pi, j} \in C^1(A)$  with

$$[A, U_{\phi, \pi, j}] = iM_{\pi \circ \phi} U_{\phi, \pi, j}$$

and  $M_{\pi \circ \phi}$  the bounded matrix-valued multiplication operator in  $\mathcal{H}_j^{(\pi)}$  given by

$$M_{\pi \circ \phi} \sum_{k=1}^{d_\pi} \varphi_k \otimes \pi_{jk} := \sum_{k, \ell=1}^{d_\pi} (\mathcal{L}_Y(\pi \circ \phi) \cdot (\pi \circ \phi)^{-1})_{k\ell} \varphi_\ell \otimes \pi_{jk}, \quad \varphi_k \in L^2(X, \mu_X).$$

Finally, if  $\mathcal{L}_Y \phi$  exists  $\mu_X$ -almost everywhere and  $M_\phi \in L^2(X, \mathfrak{g})$ , then

$$D = \text{s-lim}_{n \rightarrow \infty} D_n = i(d\pi)_{e_G}((P_\phi M_\phi)(\cdot))$$

where  $(d\pi)_{e_G}((P_\phi M_\phi)(\cdot))$  is a bounded operator in  $\mathcal{H}_j^{(\pi)}$  that can be interpreted as a matrix-valued degree of the cocycle  $\pi \circ \phi : X \rightarrow \pi(G)$  (see [39, Rem. 3.12] for more details). Thus, Theorem 2.1(a) applies for the net  $(U_j)_{j \in J} = (U_{\phi, \pi, j}^n)_{n \in \mathbb{N}^*}$ , the set  $(\ell_j)_{j \in J} = (n)_{n \in \mathbb{N}^*}$ , the operator  $A$ , and  $D = i(d\pi)_{e_G}((P_\phi M_\phi)(\cdot))$ . So for each  $\varphi = D\tilde{\varphi} \in DD(A)$  and  $\psi \in \mathcal{D}(A)$  there exists a constant  $c_{\varphi, \psi} \geq 0$  such that

$$|\langle \varphi, U_{\phi, \pi, j}^n \psi \rangle_{\mathcal{H}_j^{(\pi)}}| \leq \|(D - D_n)\tilde{\varphi}\|_{\mathcal{H}_j^{(\pi)}} \|\psi\|_{\mathcal{H}_j^{(\pi)}} + \frac{1}{n} c_{\varphi, \psi}, \quad n \in \mathbb{N}^*. \tag{3.9}$$

In this example, one cannot easily improve the decay estimate (3.9). Indeed, the convergence  $D_n \xrightarrow{s} D$  follows from [39, Lemma 3.3], whose proof relies on Birkhoff’s pointwise ergodic theorem for Banach-valued functions. And without additional information, one cannot infer an explicit rate of convergence from that theorem. That being said, one can exhibit various examples

where (3.9) is satisfied, and one can even prove that  $U_\phi$  has purely a.c. spectrum in appropriate subspaces of  $\mathcal{H}$ . We refer to [39, Sec. 4] for more details.

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## A. Commutators and Regularity Classes

In this appendix, we recall the definitions of commutators of operators and regularity classes associated with them that we use in this work. We refer to Chapters 5–6 of the monograph [1] for more details.

Let  $A$  be a self-adjoint operator in a Hilbert space  $\mathcal{H}$  with domain  $\mathcal{D}(A)$ , and take a bounded operator  $S \in \mathcal{B}(\mathcal{H})$ . For any  $k \in \mathbb{N}$ , we say that  $S$  belongs to  $C^k(A)$ , with notation  $S \in C^k(A)$ , if the map

$$\mathbb{R} \ni t \mapsto e^{-itA} S e^{itA} \in \mathcal{B}(\mathcal{H}) \quad (\text{A.1})$$

is strongly of class  $C^k$ . The sets  $C^k(A) \subset \mathcal{B}(\mathcal{H})$  satisfy the inclusions

$$C^\infty(A) := \bigcap_{k \in \mathbb{N}} C^k(A) \subset \dots \subset C^2(A) \subset C^1(A) \subset C^0(A) = \mathcal{B}(\mathcal{H}),$$

and one has  $S\varphi \in \mathcal{D}(A^k)$  if  $\varphi \in \mathcal{D}(A^k)$  [1, Prop. 5.3.1]. In the case  $k = 1$ , one has  $S \in C^1(A)$  if and only if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle \varphi, iSA\varphi \rangle_{\mathcal{H}} - \langle A\varphi, iS\varphi \rangle_{\mathcal{H}} \in \mathbb{C}$$

is continuous for the topology induced by  $\mathcal{H}$  on  $\mathcal{D}(A)$ . We denote by  $[iS, A]$  the bounded operator associated with the continuous extension of this form, or equivalently the strong derivative of the map (A.1) at  $t = 0$ .

If  $H$  is a self-adjoint operator in  $\mathcal{H}$  with domain  $\mathcal{D}(H)$  and spectrum  $\sigma(H)$ , we say that  $H$  is of class  $C^k(A)$  if  $(H - z)^{-1} \in C^k(A)$  for some  $z \in \mathbb{C} \setminus \sigma(H)$ . In particular,  $H$  is of class  $C^1(A)$  if and only if the quadratic form

$$\mathcal{D}(A) \ni \varphi \mapsto \langle \varphi, (H - z)^{-1} A\varphi \rangle_{\mathcal{H}} - \langle A\varphi, (H - z)^{-1} \varphi \rangle_{\mathcal{H}} \in \mathbb{C}$$

extends continuously to a bounded form with corresponding bounded operator denoted by  $[(H - z)^{-1}, A] \in \mathcal{B}(\mathcal{H})$ . In such a case, the set  $\mathcal{D}(H) \cap \mathcal{D}(A)$  is a core for  $H$  and the quadratic form

$$\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H\varphi, A\varphi \rangle - \langle A\varphi, H\varphi \rangle \in \mathbb{C}$$

is continuous in the graph norm topology of  $\mathcal{D}(H)$  [1, Thm. 6.2.10(a)]. This form then extends uniquely to a continuous quadratic form on  $\mathcal{D}(H)$  which can be identified with a continuous operator  $[H, A]$  from  $\mathcal{D}(H)$  to the adjoint space  $\mathcal{D}(H)^*$ . In addition, the following relation holds in  $\mathcal{B}(\mathcal{H})$  [1, Thm. 6.2.10(b)]:

$$[(H - z)^{-1}, A] = -(H - z)^{-1}[H, A](H - z)^{-1}. \quad (\text{A.2})$$



## B. RAGE-Type Theorem for Unitary Operators

In this appendix, we give the proof of a RAGE-type theorem for unitary operators that we use in Sect. 2.3. The theorem is surely well-known, but since we did not find it in this form in the literature we present its proof for completeness.

We start by recalling the usual RAGE theorem for a unitary operator  $U$ . As in the previous sections, we use the notation  $P_p(U)$  for the projection onto the pure point subspace  $\mathcal{H}_p(U)$  of  $U$  and  $P_c(U)$  for the projection onto the continuous subspace  $\mathcal{H}_c(U)$  of  $U$ .

**Theorem B.1** (RAGE theorem, page 320 of [29]). *Let  $U$  be a unitary operator in a Hilbert space  $\mathcal{H}$  and  $K \in \mathcal{K}(\mathcal{H})$ . Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} \|KU^{-m}\varphi\|_{\mathcal{H}}^2 = \|KP_p(U)\varphi\|_{\mathcal{H}}^2 \quad \text{for all } \varphi \in \mathcal{H}.$$

**Theorem B.2** *Let  $U$  be a unitary operator in a Hilbert space  $\mathcal{H}$  and  $K \in \mathcal{K}(\mathcal{H})$ . Then*

$$\text{s-lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} U^m K U^{-m} = \sum_{\theta \in \{\text{eigenvalues of } U\}} E^U(\{\theta\}) K E^U(\{\theta\}).$$

*Proof.* We mimic the proof of the analogous theorem in the self-adjoint case [31, Thm. 5.9] (or Thm. 5.6 in the first edition of this reference). Any  $\varphi \in \mathcal{H}$  admits an orthogonal decomposition  $\varphi = \varphi_p + \varphi_c$  with  $\varphi_p \in \mathcal{H}_p(U)$  and  $\varphi_c \in \mathcal{H}_c(U)$ . For the component  $\varphi_c$ , we get from the Cauchy–Schwarz inequality and Theorem B.1 that

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{n} \sum_{m=0}^{n-1} U^m K U^{-m} \varphi_c \right\|_{\mathcal{H}} \leq \lim_{n \rightarrow \infty} \left( \sum_{m=0}^{n-1} \frac{1}{n} \right)^{1/2} \left( \frac{1}{n} \sum_{m=0}^{n-1} \|K U^{-m} \varphi_c\|_{\mathcal{H}}^2 \right)^{1/2} = 0.$$

For the component  $\varphi_p$ , we write  $\varphi_p = \sum_{j \geq 1} \alpha_j \varphi_j$  with  $(\varphi_j)_{j \geq 1}$  an orthonormal basis of  $\mathcal{H}_p(U)$ ,  $\alpha_j \in \mathbb{C}$ , and  $U \varphi_j = \theta_j \varphi_j$  for some  $\theta_j \in \mathbb{S}^1$ . Then we get

$$\text{s-lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} U^m K U^{-m} \varphi_p = \text{s-lim}_{n \rightarrow \infty} \sum_{j \geq 1} \alpha_j \left( \frac{1}{n} \sum_{m=0}^{n-1} (U \theta_j^{-1})^m \right) K \varphi_j. \quad (\text{B.1})$$

Now we have  $\| \frac{1}{n} \sum_{m=0}^{n-1} (U \theta_j^{-1})^m \|_{\mathcal{B}(\mathcal{H})} \leq 1$  for all  $n \in \mathbb{N}^*$ , and

$$\text{s-lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} (U \theta_j^{-1})^m = E^U(\{\theta_j\})$$

due to von Neumann’s mean ergodic theorem. Therefore we can exchange the limit and the sum in (B.1) to obtain

$$\begin{aligned} \text{s-lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{m=0}^{n-1} U^m K U^{-m} \varphi_p &= \sum_{j \geq 1} \alpha_j E^U(\{\theta_j\}) K \varphi_j \\ &= \sum_{\theta \in \{\text{eigenvalues of } U\}} E^U(\{\theta\}) K E^U(\{\theta\}) \varphi, \end{aligned}$$

as desired. □

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Communicated by Alain Joye.

Received: December 20, 2021.

Accepted: May 1, 2022.