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Scattering Operator and Wave Operators for 2D Schrödinger Operators with Threshold Obstructions

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Abstract
We determine the low-energy behaviour of the scattering operator of two-dimensional Schrödinger operators with any type of obstructions at 0-energy. We also derive explicit formulas for the wave operators in the absence of p-resonances, and outline in this case a topological version of Levinson's theorem.

Keywords Schrödinger operators · Wave operators · Resonances · Topological index theorem

Mathematics Subject Classification 81U05 · 35P25 · 35J10

1 Introduction and Main Results

1.1 The Analytic Study

Scattering theory for two-dimensional Schrödinger operators is a challenging subject that has been the focus of many studies. And since the list of papers dealing with it is

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very long, we will mention here only the ones relevant for our work. The story started with a *surprise* put into evidence in [7]: scattering properties of two-dimensional Schrödinger operators are very different from their three-dimensional analogues. The main difference is due to resonances at 0-energy, which can be divided into two categories: s-resonances (at most one) and p-resonances (at most two). It was shown in [7], and later confirmed in [6], that the presence of a s-resonance does not play an important role, while the presence of p-resonances has surprising consequences. For instance, they lead to a contribution of value 1 in the so-called Levinson's theorem, in a way similar to usual bound states. Unfortunately, the proofs of the results of [6] are based on double asymptotic expansions of the resolvent, which make them strenuous to follow.

A decade later, a renewed interest in the two-dimensional case has been triggered by the works [15,29] on the L^p-boundedness of the wave operators. However, these works were conducted under the assumption that 0-energy bound states and 0-energy resonances are absent (the so-called regular or generic case). The next breakthrough came with the derivation in [14] of a simplified resolvent expansion, no longer given as a two parameters expansion, but in terms of powers of a single parameter. Subsequently, numerous works took advantage of this simplified resolvent expansion, as for example [5,10,22,25] in which the assumption of absence 0-energy bound states and 0-energy resonances remains. In other works, it was assumed that 0-energy bound states and p-resonances are absent, as for example in [26], or only that the p-resonances are absent, as in [9]. Note however that the behaviour of the Schrödinger evolution group has been studied in the general case in [11]. More recently, two-dimensional Schrödinger operators with point interactions have been investigated: the boundedness of the wave operators in L^p-spaces in the regular case has been discussed in [8], while a full picture has been provided in [30]. Building on the latter, L^p-boundedness for more general Schrödinger operators with threshold obstructions has been studied in [31].

The present paper is a continuation of our work [22] on the wave operators for two-dimensional Schrödinger operators. In that paper, we considered the regular case. Here, we do not make this assumption anymore, and present several results either in the general case, or under the assumption of absence of p-resonances only. Our results are in line with the ones obtained in [7], but we present them in an updated (and presumably simpler) framework.

We consider the scattering system given by a pair of operators (H, H_0), where H_0 is the Laplacian in the Hilbert space $\mathsf{L}^2(\mathbb{R}^2)$ and $H := H_0 + V$ with V a real potential decaying rapidly at infinity. Under quite general conditions on V it is known that the wave operators

$$W_\pm := \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH}\,\mathrm{e}^{-itH_0}$$

exist and are complete. As a consequence, the scattering operator $S := W_+^* W_-$ is unitary in $\mathsf{L}^2(\mathbb{R}^2)$. Since S strongly commutes with H_0, the operator S decomposes in the spectral representation of H_0, meaning that S is unitarily equivalent to a family of unitary operators $\{S(\lambda)\}_{\lambda\in\mathbb{R}_+}$ in $\mathsf{L}^2(\mathbb{S})$. For historical reasons, the operator $S(\lambda)$ is called the scattering matrix at energy λ, even though it acts on an infinite-dimensional

Hilbert space. A function $f : \mathbb{R}^2 \to \mathbb{C}$ satisfying $Hf = 0$ in the distributional sense is called a 0-energy bound state if f belongs to the domain of H. If $f \in \mathsf{L}^\infty(\mathbb{R}^2)$, then f is called a s-resonance, while if $f \in \mathsf{L}^p(\mathbb{R}^2)$ for some $p \in (2, \infty)$, then f is called a p-resonance. These three distinct cases are related to orthogonal projections $S_1 \geq S_2 \geq S_3$ in $\mathsf{L}^2(\mathbb{R}^2)$ introduced in [14]: $S_3 \neq 0$ if there are 0-energy bound states, $T_3 := S_2 - S_3 \neq 0$ if there are p-resonances, and $T_2 := S_1 - S_2 \neq 0$ if there is a s-resonance. In this setup, the generic case corresponds to the assumption $S_1 = 0$.

Our first main result concerns the low-energy behaviour of the scattering matrix. It holds in the general case, without any assumption on the absence of 0-energy bound states or resonances. A more detailed version of the result is presented in Theorem 3.1.

Theorem 1.1 (Scattering matrix at 0-energy). *If V satisfies $|V(x)| \leq \mathrm{Const.}\,\langle x\rangle^{-\rho}$ for a.e. $x \in \mathbb{R}^2$ and $\rho > 11$, then* $\lim_{\lambda\searrow 0} S(\lambda) = 1$.

Let us stress that the behaviour $\lim_{\lambda\searrow 0} S(\lambda) = 1$ is completely different from its counterparts in one and three dimensions, where the value of the scattering matrix $S(0)$ depends on the presence or absence of resonances at 0-energy (see [2,13]). We also acknowledge that a similar result has been established before in [6], case by case for the different types of obstructions at 0. However, it was done under the much stronger assumption of exponential decay of the potential and the condition $\int_{\mathbb{R}^2} V(x)\,\mathrm{d}x \neq 0$. Clearly, the extension of this result to a power law decaying potential was expected by the community of experts, but up to the best of our knowledge, it has never been explicitly proved.

Remark 1.2 For the proof of Theorem 1.1, we rely on the resolvent expansion provided in the seminal paper [14], and the decay assumption on V is prescribed by this approach. By using more recent tools, as for example the ones developed in [31], it is certainly possible to weaken the condition $\rho > 11$. However, our aim in this paper was not only to provide the low energy behavior of the scattering matrix, but also to get explicit formulas for the wave operators. For that purpose, relying on a classical approach was an asset. As a comparison, in the one-dimensional case it took about 12 years between the derivation of the first closed formulas for the wave operators and their proof for potentials with optimal decay, see [12,18].

Our second main result is an explicit formula for the wave operator W_- (a similar formula for W_+ can be obtained by using the relation $W_+ = W_- S^*$). The formula is obtained in Theorem 4.11, but we present here the version to be found in Corollary 4.13. For its statement, we use the notation A for the generator of dilations in $\mathsf{L}^2(\mathbb{R}^2)$ and $\mathscr{K}(\mathsf{L}^2(\mathbb{S}))$ for the set of compact operators on $\mathsf{L}^2(\mathbb{S})$. Also, given a continuous function $\eta : \mathbb{R}_+ \to \mathscr{K}(\mathsf{L}^2(\mathbb{S}))$ and the operator $\mathscr{F}_0 : \mathsf{L}^2(\mathbb{R}^2) \to \mathsf{L}^2(\mathbb{R}_+; \mathsf{L}^2(\mathbb{S}))$ which diagonalises H_0 (see (2.1)), we write $\eta(H_0)$ for the bounded operator in $\mathsf{L}^2(\mathbb{R}^2)$ satisfying

$$\big(\mathscr{F}_0\,\eta(H_0)f\big)(\lambda) = \eta(\lambda)(\mathscr{F}_0 f)(\lambda), \quad f \in \mathsf{L}^2(\mathbb{R}^2),\ \text{a.e. } \lambda \in \mathbb{R}_+.$$

Theorem 1.3 *Let V satisfy $|V(x)| \leq \text{Const.}\, \langle x\rangle^{-\rho}$ for a.e. $x \in \mathbb{R}^2$ and $\rho > 11$, and $T_3 = 0$. Then, there exist two continuous functions $\eta, \widetilde{\eta} : \mathbb{R}_+ \to \mathscr{K}(\mathsf{L}^2(\mathbb{S}))$ vanishing at 0 and ∞, satisfying $\eta(H_0) + \widetilde{\eta}(H_0) = S - 1$, and such that*

$$\begin{aligned} W_- - 1 = \tfrac{1}{2}\big(1 + \tanh(\pi A/2)\big)\eta(H_0) + \tfrac{1}{2}\big(1 + \tanh(\pi A) \\ -i \cosh(\pi A)^{-1}\big)\widetilde{\eta}(H_0) + K \end{aligned} \tag{1.1}$$

with $K \in \mathscr{K}(\mathsf{L}^2(\mathbb{R}^2))$.

Note that we have not been able to obtain a similar formula in the general case with no assumption on the absence of p-resonances. Let us also mention that if there is no 0-energy bound state (i.e. $S_3 = 0$), then $\widetilde{\eta}(H_0) = 0$ and $\eta(H_0) = S - 1$.

1.2 The Topological Outcomes

Theorems 1.1 and 1.3 contain the main results of this paper, and their proofs are purely analytic. However, the motivation for getting these results comes from a topological index theorem which was out of reach at the time of [7]. In the second part of this introduction, we briefly outline some corollaries which follow from Theorem 1.3, and then compare these results with related results in the literature. Additional information and results about index theorems in scattering theory can be found in the review paper [20].

Formula (1.1) shows that the wave operator W_- coincides, up to a compact operator, with a combination of functions of A and H_0. This is even more apparent if one looks at the expression for $\mathscr{F}_0(W_- - 1)\mathscr{F}_0^*$ obtained in Corollary 4.12. In that representation, the operator $W_- - 1$ can be expressed as a combination of functions of the generator A_+ of dilations on $\mathbb{R}_+$ and functions of the position operator on $\mathbb{R}_+$ with values in $\mathscr{K}(\mathsf{L}^2(\mathbb{S}))$ (multiplication operators on $\mathbb{R}_+$ with values in $\mathscr{K}(\mathsf{L}^2(\mathbb{S}))$). And these functions are not arbitrary: the functions of A_+ have limits at $\pm\infty$ and the functions of the position operator vanish at 0 and ∞. A C^*-algebra generated by such functions has been extensively studied in [20, Sec. 4.4]; the multiplication operators were taking values in $\mathbb{C}$ instead of $\mathscr{K}(\mathsf{L}^2(\mathbb{S}))$, but a tensor product with the ideal $\mathscr{K}(\mathsf{L}^2(\mathbb{S}))$ leads to the functions appearing here. In particular the K-theory of this algebra has been determined, and a topological version of Levinson's theorem was illustrated with several examples. Therefore, our goal here is not to recall the C^*-algebraic machinery, but to present its consequences for our model.

The operator W_- is a Fredholm operator with trivial kernel and cokernel spanned by the eigenfunctions of H. Its Fedholm index is equal to minus the number of eigenvalues of H, multiplicity counted. Algebraically, this means that the K_0-element associated to W_- corresponds to (minus) the projection $E_\mathrm{p}(H)$ on the subspace spanned by the eigenfunctions of H. Being a Fredholm operator, the image of W_- in the Calkin algebra (the quotient of $\mathscr{B}(\mathsf{L}^2(\mathbb{R}^2))$ by the ideal of compact operators $\mathscr{K}(\mathsf{L}^2(\mathbb{R}^2))$) is a unitary operator. Of course, this algebra is too complicated to do anything with it, but the interest of formula (1.1) is precisely that it allows to compute explicitly the image. Indeed, by evaluating the functions $\frac{1}{2}(1 + \tanh(\cdot))$ and $\frac{1}{2}(1 + \tanh(\cdot) - i\cosh(\cdot)^{-1})$

at $+\infty$, and by using that $\eta(H_0)+\widetilde{\eta}(H_0)=S-1$ with $S(0)=S(\infty)=1$, one can identify the image of W_- with the family $\{S(\lambda)\}_{\lambda\in\mathbb{R}_+}$ (see [20, Sec. 4.4]). Collecting what precedes, one ends up with the K-theoretic equation

$$\operatorname{Ind}\big[\{S(\lambda)\}_{\lambda\in\mathbb{R}_+}\big]_1=-[E_{\mathrm{p}}(H)]_0. \tag{1.2}$$

This equality corresponds to a topological version of Levinson's theorem, an example of index theorem in scattering theory. However, it is not an equality between numbers yet. It is a relation between an equivalence class of unitary operators and an equivalence class of projections, with Ind the index map of K-theory. One could stop the presentation here, because a relation between equivalence classes of objects is stronger than an equality between numbers. However, Levinson's theorem is usually presented as an equality between numbers, so let us go one step further.

Extracting numbers from an equation like (1.2) requires the use of n-traces from cyclic cohomology. Fortunately, this is rather direct in our situation. For the right-hand side, the only way to get a number is to consider the usual trace Tr on $\mathscr{K}(\mathsf{L}^2(\mathbb{R}^2))$. Applying it gives (minus) the number of eigenvalues of H, multiplicity counted. For the left-hand side, the corresponding 1-trace is nothing but the winding number. However, since $S(\cdot)$ takes values not in $\mathbb{C}$ but in $\mathbb{C}+\mathscr{K}(\mathsf{L}^2(\mathbb{S}))$, this winding number has to be regularised. Explanations about the regularisation procedure are given in the Appendix of [19]. In our case, it amounts to check that an appropriate analytic formula can be applied to the representative $\{S(\lambda)\}_{\lambda\in\mathbb{R}_+}$. Doing so, one gets the numerical equality

$$\tfrac{1}{2\pi}\int_0^\infty \operatorname{tr}\big(i(1-S(\lambda))^n S(\lambda)^*S'(\lambda)\big)\,\mathrm{d}\lambda=\operatorname{Tr}\big(E_{\mathrm{p}}(H)\big), \tag{1.3}$$

where tr is the usual trace on $\mathscr{K}(\mathsf{L}^2(\mathbb{S}))$, n a sufficiently large integer, and $S'(\cdot)$ the derivative of $S(\cdot)$. We do not check here that $S(\cdot)$ is differentiable under our assumption on V, nor look for the smallest integer n for which (1.3) holds. But this can be done with the tools developed in the following sections.

Another version of Levinson's theorem was obtained in [6, Thm. 6.3] under the assumption of exponential decay of the potential and the condition $\int_{\mathbb{R}^2}V(x)\,\mathrm{d}x\neq 0$, but without assumption on the absence of p-resonances. In the framework of [6], Levinson's theorem is expressed as

$$\begin{aligned}&\int_0^\infty \operatorname{Im}\big((H-\lambda-i0)^{-1}-(H_0-\lambda-i0)^{-1}\big)\,\mathrm{d}\lambda\\&\quad=-N_-+\pi\,\Delta_{-1,-1}-\tfrac{1}{4}\int_{\mathbb{R}^2}V(x)\,\mathrm{d}x,\end{aligned} \tag{1.4}$$

where N_- is the number of strictly negative eigenvalues of H and $\Delta_{-1,-1}$ an integer related to the 0-energy eigenvalues and p-resonances. Clearly, the relations (1.3) and (1.4) cannot be directly compared, even if one takes into account the formal identity [6, Eq. (6.45)]:

$$\operatorname{Im}\operatorname{Tr}\big((H-\lambda-i0)^{-1}-(H_0-\lambda-i0)^{-1}\big)=-\tfrac{i}{2}\tfrac{\mathrm{d}}{\mathrm{d}\lambda}\operatorname{Tr}\big(\ln(S(\lambda))\big).$$

However, the common and important feature of these two relations is that the presence of a s-resonance does not contribute to Levinson's theorem. This point is surprising when compared with the one and three-dimensional cases. On the other hand, the relation (1.4) implies that each p-resonance leads to a contribution of value 1 to Levinson's theorem, like a 0-energy eigenvalue. A similar information cannot be inferred from (1.3) since p-resonances have been excluded in our analysis.

In the following final remark, we make some comparisons between our results and the content of the recent papers [8,30,31].

Remark 1.4 For two-dimensional Schrödinger operators with potential given by N point interactions, it is shown in [30, Lemmas 3.1 & 3.2] that

$$W_- - 1 = \sum_{j,k=1}^{N} \tau_{y_j} K \big(\Gamma(\sqrt{\Delta})^{-1}\big)_{jk} \tau_{y_k}^*, \tag{1.5}$$

with τ_y the shift operator by $y \in \mathbb{R}^2$, $y_j \in \mathbb{R}^2$ the positions of the point interactions, $(\Gamma(\sqrt{\Delta})^{-1})_{jk}$ the entries of the inverse of a matrix-valued Fourier multiplier, and K a singular integral operator in $\mathsf{L}^2(\mathbb{R}^2)$ with kernel

$$K(x, y) = \tfrac{2}{\pi^2 i} \tfrac{1}{x^2 - y^2 + i0}, \quad x, y \in \mathbb{R}^2.$$

A similar formula for W_+ is also given in [8, Sec. 4]. If one uses polar coordinates and writes P_0 for the orthogonal projection in $\mathsf{L}^2(\mathbb{S})$ on the constant functions, one can show as in [22, Thm. 2.5] that the operator K satisfies

$$K = -4\,\tfrac{1}{2}\big(1 + \tanh(\pi A/2)\big)(1 \otimes P_0). \tag{1.6}$$

Then, by inserting (1.6) into (1.5), by evaluating the function $\frac{1}{2}(1 + \tanh(\cdot))$ at $+\infty$, and by comparing the resulting operator with the expression for the scattering operator given in [3, Eq. (II.4.35)], one obtains that $W_- - 1$ is essentially a combination of the function $\frac{1}{2}(1 + \tanh(\pi A/2))$ and the scattering operator. This result, which was first obtained for 1 point interaction in [17], is similar to the content of Theorem 1.3.

As emphasized in [8] and [30], this picture is accurate in the regular case. But (1.5) ceases to be valid when threshold singularities are present: in such a case, the map $\lambda \mapsto (\Gamma(\lambda)^{-1})_{jk}$ exhibits singularities as $\lambda \searrow 0$, and the boundedness of each individual term is no more satisfied (the same issue leads to the introduction of the second term in (1.1) when 0-energy bound states are present). In [30, Lemma 4.4], it is shown how the summands in (1.5) have to be combined in order to obtain bounded operators. And the same procedure is applied to more general potentials in [31, Lemma 5.24]. However, the computations involve PDE technics, and so far it has not been possible to extract any closed formula from them. A closed formula in the presence of p-resonances would certainly contain one more term on the r.h.s. of (1.1), and lead to a new contribution in a topological version of Levinson's theorem, as displayed in (1.4). We refer to [23,24] for other examples of singular integral operators that have been shown to be equal to nice functions of simpler operators.

Notations $\mathbb{N} := \{0, 1, 2, \ldots\}$ is the set of natural numbers, $\mathscr{S}$ the Schwartz space on $\mathbb{R}^2$, $\mathbb{R}_+ := (0, \infty)$, and $\langle \cdot \rangle := \sqrt{1 + |\cdot|^2}$. The sets $\mathcal{H}^s_t$ are the weighted Sobolev spaces over $\mathbb{R}^2$ with index $s \in \mathbb{R}$ for derivatives and index $t \in \mathbb{R}$ for decay at infinity [4, Sec. 4.1], and with shorthand notations $\mathcal{H}^s := \mathcal{H}^s_0$, $\mathcal{H}_t := \mathcal{H}^0_t$, and $\mathcal{H} := \mathcal{H}^0_0 = \mathsf{L}^2(\mathbb{R}^2)$. For any $s, t \in \mathbb{R}$, the 2-dimensional Fourier transform $\mathscr{F}$ is a topological isomorphism of $\mathcal{H}^s_t$ onto $\mathcal{H}^t_s$, and the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ (antilinear in the first argument) extends continuously to a duality $\langle \cdot, \cdot \rangle_{\mathcal{H}^s_t, \mathcal{H}^{-s}_{-t}}$ between $\mathcal{H}^s_t$ and $\mathcal{H}^{-s}_{-t}$. Given two Banach spaces $\mathcal{G}_1$ and $\mathcal{G}_2$, $\mathscr{B}(\mathcal{G}_1, \mathcal{G}_2)$ (resp. $\mathscr{K}(\mathcal{G}_1, \mathcal{G}_2)$) denotes the set of bounded (resp. compact) operators from $\mathcal{G}_1$ to $\mathcal{G}_2$, with shorthand notation $\mathscr{B}(\mathcal{G}_1) := \mathscr{B}(\mathcal{G}_1, \mathcal{G}_1)$ (resp. $\mathscr{K}(\mathcal{G}_1) := \mathscr{K}(\mathcal{G}_1, \mathcal{G}_1)$). Finally, $\otimes$ stands for the closed tensor product of Hilbert spaces or of operators.

2 Preliminaries

2.1 Free Hamiltonian

Set $\mathfrak{h} := \mathsf{L}^2(\mathbb{S})$ and $\mathscr{H} := \mathsf{L}^2(\mathbb{R}_+; \mathfrak{h})$, and let H_0 be the (positive) self-adjoint operator in $\mathcal{H} = \mathsf{L}^2(\mathbb{R}^2)$ given by minus the Laplacian $-\Delta$ on $\mathbb{R}^2$. Then, the unitary operator $\mathscr{F}_0 : \mathcal{H} \to \mathscr{H}$ defined by

$$\big((\mathscr{F}_0 f)(\lambda)\big)(\omega) = 2^{-1/2} (\mathscr{F} f)(\sqrt{\lambda}\,\omega), \quad f \in \mathscr{S},\ \lambda \in \mathbb{R}_+,\ \omega \in \mathbb{S}, \tag{2.1}$$

is a spectral transformation for H_0 in the sense that

$$(\mathscr{F}_0 H_0 f)(\lambda) = \lambda\,(\mathscr{F}_0 f)(\lambda) = (L \mathscr{F}_0 f)(\lambda), \quad f \in \mathcal{H}^2,\ \text{a.e. } \lambda \in \mathbb{R}_+,$$

with L the maximal multiplication operator by the variable $\lambda \in \mathbb{R}_+$ in $\mathscr{H}$. Moreover, for each $\lambda \in \mathbb{R}_+$, the operator $\mathscr{F}_0(\lambda) : \mathscr{S} \to \mathfrak{h}$ given by $\mathscr{F}_0(\lambda) f := (\mathscr{F}_0 f)(\lambda)$ extends to an element of $\mathscr{B}(\mathcal{H}^s_t, \mathfrak{h})$ for any $s \in \mathbb{R}$ and $t > 1/2$, and the function $\mathbb{R}_+ \ni \lambda \mapsto \mathscr{F}_0(\lambda) \in \mathscr{B}(\mathcal{H}^s_t, \mathfrak{h})$ is continuous (see Lemma 4.8 for additional continuity properties of $\mathscr{F}_0(\cdot)$).

The asymptotic expansion of $\mathscr{F}_0(\lambda)$ as $\lambda \searrow 0$ will play an important role in our 2-dimensional case, in a similar way it does in the 3-dimensional case [13, Sec. 5]. By expanding the exponential $\mathrm{e}^{-i\sqrt{\lambda}\omega \cdot x}$ in Taylor series, one gets

$$\mathscr{F}_0(\lambda) = \gamma_0 + \sqrt{\lambda}\,\gamma_1 + \lambda \gamma_2 + o(\lambda), \quad \lambda \in \mathbb{R}_+, \tag{2.2}$$

with $\gamma_j : \mathscr{S} \to \mathfrak{h}$ $(j = 0, 1, 2)$ the operator given by

$$(\gamma_j f)(\omega) := \tfrac{(-i)^j}{2^{3/2}\pi\,(j!)} \int_{\mathbb{R}^2} \mathrm{d}x\, (\omega \cdot x)^j f(x), \quad f \in \mathscr{S},\ \omega \in \mathbb{S}.$$

One can check that γ_j extends to an element of $\mathscr{B}(\mathcal{H}^s_t, \mathfrak{h})$ for any $s \in \mathbb{R}$ and $t > j+1$, which implies that the expansion (2.2) holds in $\mathscr{B}(\mathcal{H}^s_t, \mathfrak{h})$ as $\lambda \searrow 0$ for any $s \in \mathbb{R}$ and

$t > 3$. We shall sometimes use the abbreviated notation $\gamma_2(\lambda)$, or $O(\lambda)$, for the sum $\lambda\gamma_2 + o(\lambda)$ in (2.2).

2.2 Perturbed Hamiltonian

Let us now consider a potential $V \in \mathrm{L}^\infty(\mathbb{R}^2;\mathbb{R})$ satisfying for some $\rho > 1$ the bound

$$|V(x)| \leq \text{Const.}\, \langle x\rangle^{-\rho}, \quad \text{a.e. } x \in \mathbb{R}^2. \tag{2.3}$$

Then, the perturbed Hamiltonian $H := H_0 + V$ is a short range perturbation of H_0, and it is known that the corresponding wave operators

$$W_\pm := \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{itH}\,\mathrm{e}^{-itH_0}$$

exist and are complete. As a consequence, the scattering operator $S := W_+^* W_-$ is unitary in $\mathcal{H}$. Now, define for $z \in \mathbb{C}\setminus\mathbb{R}$ the resolvents of H_0 and H

$$R_0(z) := (H_0 - z)^{-1} \quad \text{and} \quad R(z) := (H - z)^{-1}.$$

In order to recall properties of $R_0(z)$ and $R(z)$ as z approaches the real axis, it is convenient to decompose the potential V according to the following rule: for a.e. $x \in \mathbb{R}^2$ set

$$v(x) := |V(x)|^{1/2} \quad \text{and} \quad u(x) := \begin{cases} +1 & \text{if } V(x) \geq 0 \\ -1 & \text{if } V(x) < 0, \end{cases}$$

so that u is self-adjoint and unitary and $V = uv^2$. Then, using the fact that H has no positive eigenvalues [16, Sec. 1] and that a limiting absorption principle holds for H_0 and H [1, Thm. 4.2], we infer that the limits

$$vR_0(\lambda \pm i0)v := \lim_{\varepsilon\searrow 0} vR_0(\lambda \pm i\varepsilon)v \quad \text{and} \quad vR(\lambda \pm i0)v := \lim_{\varepsilon\searrow 0} vR(\lambda \pm i\varepsilon)v,$$

exist in $\mathscr{B}(\mathcal{H})$ and are continuous in the variable $\lambda \in \mathbb{R}_+$. This, together with the relation

$$u - uvR(\lambda \pm i\varepsilon)vu = \big(u + vR_0(\lambda \pm i\varepsilon)v\big)^{-1}, \quad \lambda \in \mathbb{R}_+,\ \varepsilon > 0,$$

implies the existence and the continuity of the function $\mathbb{R}_+ \ni \lambda \mapsto (u + vR_0(\lambda \pm i0)v)^{-1} \in \mathscr{B}(\mathcal{H})$. Furthermore, one has $\lim_{\lambda\to\infty}(u+vR_0(\lambda\pm i0)v)^{-1} = u$ in $\mathscr{B}(\mathcal{H})$, since $\lim_{\lambda\to\infty} vR_0(\lambda + i0)v = 0$ in $\mathscr{B}(\mathcal{H})$ [28, Prop. 7.1.2]. On the other hand, the existence in $\mathscr{B}(\mathcal{H})$ of the limits $\lim_{\lambda\searrow 0}(u+vR_0(\lambda\pm i0)v)^{-1}$ depends on the presence or absence of eigenvalues or resonances at 0-energy. This problem has been studied in detail in [14] in dimensions 1 and 2. We recall here the main result in dimension 2 [14, Thm. 6.2(ii)]: Take $\kappa \in \mathbb{C}^*$ with $\mathrm{Re}(\kappa) \geq 0$, let $\eta := 1/\ln(\kappa)$ (with ln the principal value of the complex logarithm), and set

$$\mathrm{M}(\kappa) := u + vR_0(-\kappa^2)v.$$

Then, if V satisfies (2.3) with $\rho > 11$ and if $0 < |\kappa| < \kappa_0$ with $\kappa_0 > 0$ small enough, the operator $\mathrm{M}(\kappa)^{-1}$ admits an expansion

$$\mathrm{M}(\kappa)^{-1} = I_1(\kappa) - g(\kappa)I_2(\kappa) - \tfrac{g(\kappa)\eta}{\kappa^2} I_3(\kappa), \tag{2.4}$$

with

$$I_1(\kappa) := (\mathrm{M}(\kappa) + S_1)^{-1}, \tag{2.5}$$

$$I_2(\kappa) := (\mathrm{M}(\kappa) + S_1)^{-1}S_1(M_1(\kappa) + S_2)^{-1}S_1(\mathrm{M}(\kappa) + S_1)^{-1}, \tag{2.6}$$

$$\begin{aligned} I_3(\kappa) := {} & (\mathrm{M}(\kappa) + S_1)^{-1}S_1(M_1(\kappa) + S_2)^{-1}S_2\big(T_3 m(\kappa)^{-1}T_3 \\ & - T_3 m(\kappa)^{-1}b(\kappa)d(\kappa)^{-1}S_3 - S_3 d(\kappa)^{-1}c(\kappa)m(\kappa)^{-1}T_3 \\ & + S_3 d(\kappa)^{-1}c(\kappa)m(\kappa)^{-1}b(\kappa)d(\kappa)^{-1}S_3 + S_3 d(\kappa)^{-1}S_3\big) \\ & \cdot S_2(M_1(\kappa) + S_2)^{-1}S_1(\mathrm{M}(\kappa) + S_1)^{-1}, \end{aligned} \tag{2.7}$$

and where $S_1 \geq S_2 \geq S_3$ are orthogonal projections in $\mathcal{H}$, $T_3 := S_2 - S_3$, $g : \mathbb{C} \to \mathbb{C}$ satisfies $g(\kappa) = O(\eta^{-1})$ for $0 < |\kappa| < \kappa_0$, $m : \mathbb{C} \to \mathscr{B}(\mathcal{H})$ satisfies $m(\kappa) = O(\eta^{-1})$ for $0 < |\kappa| < \kappa_0$, and all other factors are operator-valued functions having limits in $\mathscr{B}(\mathcal{H})$ as $\kappa \to 0$. Precise formulas for these factors are provided in [14, Sec. 6] and will be recalled in due time.

3 Scattering Operator

In this section we analyse the behaviour at low energy of the scattering matrix. Since the scattering operator S strongly commutes with H_0, it decomposes in the spectral representation of H_0. That is, there exist for a.e. $\lambda \in \mathbb{R}_+$ a unitary operator $S(\lambda) \in \mathscr{B}(\mathfrak{h})$ such that

$$\big(\mathscr{F}_0 S \mathscr{F}_0^* \varphi\big)(\lambda) = S(\lambda)\varphi(\lambda), \quad \varphi \in \mathscr{H}, \text{ a.e. } \lambda \in \mathbb{R}_+.$$

Furthermore, if V satisfies (2.3) for some $\rho > 1$, then the operators $S(\lambda)$ are given by the stationary formula [28, Thm. 1.8.1]

$$S(\lambda) = 1_{\mathfrak{h}} - 2\pi i \mathscr{F}_0(\lambda)v\big(u + vR_0(\lambda + i0)v\big)^{-1}v\mathscr{F}_0(\lambda)^*, \quad \text{a.e. } \lambda \in \mathbb{R}_+. \tag{3.1}$$

Our goal consists in determining the behaviour of this expression as $\lambda \searrow 0$. To this end, we will use both the expansion (2.2) for $\mathscr{F}_0(\lambda)$ and the asymptotic expansion (2.4) for $\mathrm{M}(\kappa)^{-1}$ in the case $\kappa = -i\sqrt{\lambda}$. This choice of κ corresponds to the value

$\lambda + i0 = -\kappa^2$ appearing in (3.1). For the sake of brevity, we will keep using the shorthand notations

$$\kappa := -i\sqrt{\lambda} \quad \text{and} \quad \eta := \tfrac{1}{\ln(\kappa)} = \tfrac{1}{\ln(\lambda)/2 - i\pi/2}. \tag{3.2}$$

From now on, we thus assume that V satisfies (2.3) with $\rho > 11$, so that both expansions are verified (the expansion for $\mathscr{F}_0(\lambda)$ is verified because the operator v in $\mathscr{F}_0(\lambda)v$ satisfies $v \in \mathscr{B}(\mathcal{H}, \mathcal{H}_t)$ with $t > 3$). As a consequence, the problem reduces to computating the limit $\lambda \searrow 0$ of the operator

$$\big(\gamma_0 + \sqrt{\lambda}\,\gamma_1 + \gamma_2(\lambda)\big) v \mathrm{M}(\kappa)^{-1} v \big(\gamma_0^* + \sqrt{\lambda}\,\gamma_1^* + \gamma_2(\lambda)^*\big) \tag{3.3}$$

with $\mathrm{M}(\kappa)^{-1}$ given by (2.4). Since the computation requires various preparatory lemmas, we start by stating the final result, and then proceed to its proof:

Theorem 3.1 (Scattering matrix at 0-energy) *If V satisfies (2.3) with $\rho > 11$, then $\lim_{\lambda \searrow 0} S(\lambda) = 1_{\mathfrak{h}}$ in $\mathscr{B}(\mathfrak{h})$.*

For our first lemma, we need to recall the definition of two orthogonal projections introduced in [14, Sec. 6]:

$$P := \tfrac{1}{\|v\|_{\mathcal{H}}^2} |v\rangle\langle v| \quad \text{and} \quad Q := 1 - P,$$

where $|v\rangle\langle v| f := \langle v, f\rangle_{\mathcal{H}}\, v$ for any $f \in \mathcal{H}$. We also need the vector notation $X = (X_1, X_2)$, with X_j the maximal multiplication operator in $\mathcal{H}$ by the j-th variable in $\mathbb{R}^2$.

Lemma 3.2

(a) One has $\gamma_0 v Q = 0 = Q v \gamma_0^*$.
(b) For $j = 1, 2, 3$, one has $\gamma_0 v S_j = 0 = S_j v \gamma_0^*$.
(c) One has $\gamma_1 v S_3 = 0 = S_3 v \gamma_1^*$.
(d) For $j = 1, 2, 3$, one has $P S_j = 0 = S_j P$.

Proof (a) For any $f \in \mathcal{H}$, we have

$$2^{3/2}\pi\,\gamma_0 v Q f = \int_{\mathbb{R}^2} \mathrm{d}x\, v(x)(Qf)(x) = \int_{\mathbb{R}^2} \mathrm{d}x\, v(x)\Big(f(x) - \tfrac{v(x)}{\|v\|_{\mathcal{H}}^2}\langle v, f\rangle_{\mathcal{H}}\Big) = 0,$$

which proves the first equality. The second equality is obtained by duality.

(b) The claim follows from point (a) and the fact that $Q \geq S_j$ for $j = 1, 2, 3$ (see [14, Thm. 6.2(i)]).

(c) For any $f \in \mathcal{H}$ and $\omega \in \mathbb{S}^1$, we have

$$2^{3/2}\pi i (\gamma_1 v S_3 f)(\omega) = \int_{\mathbb{R}^2} \mathrm{d}x\, (\omega \cdot x) v(x)(S_3 f)(x) = \sum_{j=1}^{2} \omega_j \big\langle v, X_j S_3 f\big\rangle_{\mathcal{H}} = 0,$$

where the last equality follows from [14, Eq. (6.100)]. This proves the first equality. The second equality is obtained by duality.

(d) The claim follows the facts that $P = 1 - Q$ and $Q \geq S_j$ for $j = 1, 2, 3$.

□

One can show that the operator $I_1(\kappa)$ appearing in (2.4) does not give any contribution. Indeed, we know from [14, Eq. (6.27)] that

$$(\mathrm{M}(\kappa) + S_1)^{-1} = g(\kappa)^{-1} I_0(\kappa) + Q D_0(\kappa) Q \tag{3.4}$$

with $I_0(\kappa)$ and $D_0(\kappa)$ operators having limits in $\mathscr{B}(\mathcal{H})$ as $\lambda \searrow 0$. Since $g(\kappa)^{-1} = O(\eta)$ as $\lambda \searrow 0$, it follows that

$$\lim_{\lambda \searrow 0} (\mathrm{M}(\kappa) + S_1)^{-1} = Q D_0(0) Q \quad \text{with} \quad D_0(0) := \lim_{\lambda \searrow 0} D_0(\kappa).$$

Therefore, (2.5) and Lemma 3.2(a) imply that

$$\begin{aligned} &\lim_{\lambda \searrow 0} \big(\gamma_0 + \sqrt{\lambda}\gamma_1 + \gamma_2(\lambda)\big) v I_1(\kappa) v \big(\gamma_0^* + \sqrt{\lambda}\gamma_1^* + \gamma_2(\lambda)^*\big) \\ &\quad = \lim_{\lambda \searrow 0} \gamma_0 v Q D_0(0) Q v \gamma_0^* = 0, \end{aligned} \tag{3.5}$$

showing that the first term in (2.4) does not lead to any contribution in (3.3).

We can now turn our attention to the second term in (2.4). For its analysis, we introduce the operator

$$M_0(\kappa) := \mathrm{M}(\kappa) + \tfrac{1}{2\pi\eta} \|v\|_{\mathcal{H}}^2 P,$$

and we note from [14, Eq. (6.28)] that $M_0(\kappa) = M_{0,0} + O(\lambda/\eta)$ as $\lambda \searrow 0$, with $M_{0,0} \in \mathscr{B}(\mathcal{H})$ self-adjoint.

Lemma 3.3 *One has* $[(\mathrm{M}(\kappa) + S_1)^{-1}, S_1] = O(\eta)$ *as* $\lambda \searrow 0$.

Proof Using (3.4), the fact that $D_0(\kappa) = \big(Q(M_0(\kappa) + S_1)Q\big)^{-1}$ [14, Eq. (6.29)], and the fact that $[Q, S_1] = 0$, we get as $\lambda \searrow 0$

$$\begin{aligned} [(\mathrm{M}(\kappa) + S_1)^{-1}, S_1] &= [Q D_0(\kappa) Q, S_1] + O(\eta) \\ &= Q\big[\big(Q(M_0(\kappa) + S_1)Q\big)^{-1}, S_1\big]Q + O(\eta) \\ &= Q D_0(\kappa)\big[S_1, Q(M_0(\kappa) + S_1)Q\big]D_0(\kappa) Q + O(\eta). \end{aligned}$$

Since $M_0(\kappa) = M_{0,0} + O(\lambda/\eta)$ and S_1 is the orthogonal projection on $\mathrm{Ker}(Q M_{0,0} Q)$ (see [14, Thm. 6.2(i)]), we infer that

$$[(\mathrm{M}(\kappa) + S_1)^{-1}, S_1] = Q D_0(\kappa)\, O(\lambda/\eta)\, D_0(\kappa) Q + O(\eta) = O(\eta) \quad \text{as } \lambda \searrow 0,$$

as desired. □

The equation (2.6) and Lemma 3.3 imply that the second term in (2.4) satisfies as $\lambda \searrow 0$

$$\begin{aligned} g(\kappa)I_2(\kappa) = g(\kappa)\big(S_1(\mathrm{M}(\kappa)+S_1)^{-1}+O(\eta)\big) \\ \cdot (M_1(\kappa)+S_2)^{-1}\big((\mathrm{M}(\kappa)+S_1)^{-1}S_1+O(\eta)\big). \end{aligned}$$

Thus, taking into account Lemma 3.2(b) and the fact that $g(\kappa)=O(\eta^{-1})$ as $\lambda \searrow 0$, one gets

$$\begin{aligned} &\lim_{\lambda\searrow 0}\big(\gamma_0+\sqrt{\lambda}\gamma_1+\lambda\gamma_2(\lambda)\big)vg(\kappa)I_2(\kappa)v\big(\gamma_0^*+\sqrt{\lambda}\gamma_1^*+\lambda\gamma_2(\lambda)^*\big) \\ &\quad=\lim_{\lambda\searrow 0} g(\kappa)\big(\gamma_0+\sqrt{\lambda}\gamma_1+\lambda\gamma_2(\lambda)\big)v\big(S_1(\mathrm{M}(\kappa)+S_1)^{-1}+O(\eta)\big) \\ &\qquad\cdot(M_1(\kappa)+S_2)^{-1}\big((\mathrm{M}(\kappa)+S_1)^{-1}S_1+O(\eta)\big)v\big(\gamma_0^*+\sqrt{\lambda}\gamma_1^*+\lambda\gamma_2(\lambda)^*\big) \\ &\quad=0, \end{aligned} \tag{3.6}$$

meaning that the second term in (2.4) does not lead to any contribution in (3.3).

Let us now consider the third term in (2.4). Since all factors in $I_3(\kappa)$ have limits as $\lambda \searrow 0$ and $g(\kappa)=O(\eta^{-1})$ as $\lambda \searrow 0$, that term behaves at worst like $O(\lambda^{-1})$ as $\lambda \searrow 0$. Therefore, the terms in

$$\tfrac{g(\kappa)\eta}{\lambda}\big(\gamma_0+\sqrt{\lambda}\gamma_1+\lambda\gamma_2(\lambda)\big)vI_3(\kappa)v\big(\gamma_0^*+\sqrt{\lambda}\gamma_1^*+\lambda\gamma_2(\lambda)^*\big)$$

that do not manifestly vanish in the limit $\lambda \searrow 0$ are:

(i) $\frac{g(\kappa)\eta}{\lambda}\gamma_0 vI_3(\kappa)v\gamma_0^*$,
(ii) $\frac{g(\kappa)\eta}{\sqrt{\lambda}}\big(\gamma_0 vI_3(\kappa)v\gamma_1^*+\gamma_1 vI_3(\kappa)v\gamma_0^*\big)$,
(iii) $g(\kappa)\eta\gamma_1 vI_3(\kappa)v\gamma_1^*$,
(iv) $g(\kappa)\eta\big(\gamma_0 vI_3(\kappa)v\gamma_2^*+\gamma_2 vI_3(\kappa)v\gamma_0^*\big)$.

In order to study these terms, some preparatory lemmas are necessary, starting with one on commutators:

Lemma 3.4 *For $j=2,3$, one has $[(M_1(\kappa)+S_2)^{-1},S_j]=O(\lambda/\eta^2)$ in $\mathscr{B}(S_1\mathcal{H})$ as $\lambda \searrow 0$.*

Proof We show the claim for $j=2$, since the case $j=3$ is similar. We know from [14, Eq. (6.31)] that the operator $M_1(\kappa)$ defined in $S_1\mathcal{H}$ satisfies as $\lambda \searrow 0$ the expansion $M_1(\kappa)=M_{1;0,0}+O(\lambda/\eta^2)$ with $M_{1;0,0}:=S_1M_{0,0}PM_{0,0}S_1$. It follows that

$$\begin{aligned} [(M_1(\kappa)+S_2)^{-1},S_2] &= (M_1(\kappa)+S_2)^{-1}[S_2,M_1(\kappa)+S_2](M_1(\kappa)+S_2)^{-1} \\ &= (M_1(\kappa)+S_2)^{-1}[S_2,M_{1;0,0}](M_1(\kappa)+S_2)^{-1}+O(\lambda/\eta^2). \end{aligned}$$

Since S_2 is the projection on the kernel of $M_{1;0,0}$ (see [14, Thm. 6.2(i)]), this implies the claim. ◻

In the next proposition, we deal with the simplest limits above, the ones of (iii) and (iv).

Proposition 3.5 *One has*

$$\lim_{\lambda\searrow 0} g(\kappa)\eta\,\gamma_1 v I_3(\kappa) v\gamma_1^* = 0 \tag{3.7}$$

and

$$\lim_{\lambda\searrow 0} g(\kappa)\eta\big(\gamma_0 v I_3(\kappa) v\gamma_2^* + \gamma_2 v I_3(\kappa) v\gamma_0^*\big) = 0. \tag{3.8}$$

Proof Since $\lim_{\lambda\searrow 0} g(\kappa)\eta$ exists, and since all the factors in $I_3(\kappa)$ have limits as $\lambda \searrow 0$, one can factorise the first limit as

$$\lim_{\lambda\searrow 0} g(\kappa)\eta\,\gamma_1 v I_3(\kappa) v\gamma_1^* = \lim_{\lambda\searrow 0} g(\kappa)\eta \cdot \lim_{\lambda\searrow 0} \gamma_1 v I_3(\kappa) v\gamma_1^*.$$

So, it is sufficient to show that $\lim_{\lambda\searrow 0} \gamma_1 v I_3(\kappa) v\gamma_1^* = 0$ to prove (3.7). Now, we have $S_1 \geq S_2 \geq S_3$, $m(\kappa)^{-1} = O(\eta)$ as $\lambda \searrow 0$, and also

$$\begin{aligned}
&\lim_{\lambda\searrow 0} (\mathrm{M}(\kappa) + S_1)^{-1} S_1 = S_1 = \lim_{\lambda\searrow 0} S_1 (\mathrm{M}(\kappa) + S_1)^{-1},\\
&\lim_{\lambda\searrow 0} (M_1(\kappa) + S_2)^{-1} S_2 = S_2 = \lim_{\lambda\searrow 0} S_2 (M_1(\kappa) + S_2)^{-1},
\end{aligned} \tag{3.9}$$

due to Lemmas 3.3-3.4 (and their proofs). Therefore, we get from (2.7)

$$\begin{aligned}
\lim_{\lambda\searrow 0} \gamma_1 v I_3(\kappa) v\gamma_1^* = \lim_{\lambda\searrow 0} \gamma_1 v\big(&- T_3 m(\kappa)^{-1} b(\kappa) d(\kappa)^{-1} S_3 - S_3 d(\kappa)^{-1} c(\kappa) m(\kappa)^{-1} T_3\\
&+ S_3 d(\kappa)^{-1} c(\kappa) m(\kappa)^{-1} b(\kappa) d(\kappa)^{-1} S_3 + S_3 d(\kappa)^{-1} S_3\big) v\gamma_1^*,
\end{aligned}$$

and thus obtain that $\lim_{\lambda\searrow 0} \gamma_1 v I_3(\kappa) v\gamma_1^* = 0$ thanks to Lemma 3.2(c).

Similarly, in order to prove (3.8) it is sufficient to show that $\lim_{\lambda\searrow 0} \gamma_0 v I_3(\kappa) v\gamma_2^* = 0$ (or that $\lim_{\lambda\searrow 0} \gamma_2 v I_3(\kappa) v\gamma_0^* = 0$, this is similar). In this case, we get

$$\begin{aligned}
\lim_{\lambda\searrow 0} \gamma_0 v I_3(\kappa) v\gamma_2^* = \lim_{\lambda\searrow 0} \gamma_0 v\big(&- T_3 m(\kappa)^{-1} b(\kappa) d(\kappa)^{-1} S_3 - S_3 d(\kappa)^{-1} c(\kappa) m(\kappa)^{-1} T_3\\
&+ S_3 d(\kappa)^{-1} c(\kappa) m(\kappa)^{-1} b(\kappa) d(\kappa)^{-1} S_3 + S_3 d(\kappa)^{-1} S_3\big) v\gamma_2^*,
\end{aligned}$$

and thus obtain that $\lim_{\lambda\searrow 0} \gamma_0 v I_3(\kappa) v\gamma_2^* = 0$ thanks to Lemma 3.2(b). □

For the remaining terms (i) and (ii), we need two more lemmas.

Lemma 3.6

(a) One has $P M_{0,0} S_2 = 0$.
(b) One has $\gamma_0 v M_{0,0} S_3 = 0 = S_3 M_{0,0} v\gamma_0^*$.

(c) *One has as* $\lambda \searrow 0$

$$PM_0(\kappa)QD_0(\kappa)S_2 = O(\lambda/\eta) \quad \textit{and} \quad S_2D_0(\kappa)QM_0(\kappa)P = O(\lambda/\eta).$$

Proof (a) Let $f \in \mathcal{H}$ and $g := S_2 f$. Then, we have by definition of S_2 the inclusion $g \in \mathrm{Ker}(M_{1;0,0}) = \mathrm{Ker}(S_1 M_{0,0} P M_{0,0} S_1)$. Therefore, we get the equalities

$$0 = \big\langle g, S_1 M_{0,0} P M_{0,0} S_1 g\big\rangle_{\mathcal{H}} = \|PM_{0,0}S_1 g\|^2_{\mathcal{H}} = \|PM_{0,0}S_2 f\|^2_{\mathcal{H}},$$

which imply the claim.

(b) We have for any $f \in \mathcal{H}$

$$2^{3/2}\pi\gamma_0 v M_{0,0}S_3 f = \int_{\mathbb{R}^2} \mathrm{d}x\, v(x)(M_{0,0}S_3 f)(x) = \big\langle v, M_{0,0}S_3 f\big\rangle_{\mathcal{H}} = 0,$$

with the last equality following from [14, Eq. (6.100)]. This shows the first equality. The second equality is then obtained by duality.

(c) A successive application of Equations (6.58), (6.69) and (6.56) of [14] gives as $\lambda \searrow 0$

$$\begin{aligned} PM_0(\kappa)QD_0(\kappa)S_2 &= PM_{0,0}QD_0(\kappa)S_2 + O(\lambda/\eta) \\ &= PM_{0,0}Q(QM_{0,0}Q + S_1)^{-1}S_2 + O(\lambda/\eta) \\ &= PM_{0,0}QS_2 + O(\lambda/\eta). \end{aligned}$$

Since $PM_{0,0}QS_2 = PM_{0,0}S_2 = 0$ by point (a), we obtain the first equality. The second equality is obtained similarly.

□

Lemma 3.7 *One has as* $\lambda \searrow 0$

$$\gamma_0 v(\mathrm{M}(\kappa) + S_1)^{-1}S_1(M_1(\kappa) + S_2)^{-1}S_2 = O(\lambda/\eta), \tag{3.10}$$

$$S_2(M_1(\kappa) + S_2)^{-1}S_1(\mathrm{M}(\kappa) + S_1)^{-1}v\gamma_0^* = O(\lambda/\eta). \tag{3.11}$$

Proof Using successively [14, Eq. (6.27)], Lemmas 3.2(a) & 3.2(d), 3.4, Lemma 3.6(c) and the estimate $g(\kappa)^{-1} = O(\eta)$, we obtain the equalities

$$\begin{aligned} &\gamma_0 v(\mathrm{M}(\kappa) + S_1)^{-1}S_1(M_1(\kappa) + S_2)^{-1}S_2 \\ &\quad = -g(\kappa)^{-1}\gamma_0 v PM_0(\kappa)QD_0(\kappa)S_1(M_1(\kappa) + OS_2)^{-1}S_2 \\ &\quad = -g(\kappa)^{-1}\gamma_0 v PM_0(\kappa)QD_0(\kappa)\big(S_2(M_1(\kappa) + S_2)^{-1} + O(\lambda/\eta^2)\big) \\ &\quad = -g(\kappa)^{-1}\gamma_0 v\big(O(\lambda/\eta) + O(\lambda/\eta^2)\big) \\ &\quad = O(\lambda/\eta). \end{aligned}$$

This proves (3.10). The equality (3.11) can be shown similarly. □

We are now ready to deal with the terms (i) and (ii):

Proposition 3.8 *One has*

$$\lim_{\lambda \searrow 0} \tfrac{g(\kappa)\eta}{\lambda}\, \gamma_0 v I_3(\kappa) v \gamma_0^* = 0$$

and

$$\lim_{\lambda \searrow 0} \tfrac{g(\kappa)\eta}{\sqrt{\lambda}} \big(\gamma_0 v I_3(\kappa) v \gamma_1^* + \gamma_1 v I_3(\kappa) v \gamma_0^*\big) = 0.$$

Proof The operator $I_3(\kappa)$ is of the form (see (2.7))

$$(\mathrm{M}(\kappa) + S_1)^{-1} S_1 (M_1(\kappa) + S_2)^{-1} S_2 (\ldots) S_2 (M_1(\kappa) + S_2)^{-1} S_1 (\mathrm{M}(\kappa) + S_1)^{-1},$$

with $(\ldots)$ having a limit as $\lambda \searrow 0$. Therefore, the two claims follow from an application of Lemma 3.7. □

We can finally give the proof of the main result of this section:

Proof *(Proof of Theorem 3.1)* The proof reduces to gathering the information contained in Eqs. (3.5), (3.6) and Propositions 3.5 & 3.8. As a result, we obtain that all the contributions appearing in the formula (3.1) for the S-matrix $S(\lambda)$ vanish in the limit $\lambda \searrow 0$, except the first term $1_{\mathfrak{h}}$. □

4 Wave Operators

If the potential V satisfies (2.3) with $\rho > 1$, then the stationary wave operators and the strong wave operators exist and coincide (see [27, Thm. 5.3.6]). So, starting from the formula for the stationary wave operators [27, Eq. 2.7.5] and taking into account the resolvent equation written in the symmetrised form [14, Eq. 4.3], one obtains for suitable $\varphi, \psi \in \mathscr{H}$:

$$\begin{aligned} &\big\langle \mathscr{F}_0 \big(W_\pm - 1\big) \mathscr{F}_0^* \varphi, \psi \big\rangle_{\mathscr{H}} \\ &\quad = -\int_{\mathbb{R}} \mathrm{d}\lambda \lim_{\varepsilon \searrow 0} \int_0^\infty \mathrm{d}\mu \, \big\langle \mathscr{F}_0(\mu) v \big(u + v R_0(\lambda \mp i\varepsilon) v\big)^{-1} v \mathscr{F}_0^* \\ &\quad\quad \delta_\varepsilon(L - \lambda)\varphi, (\mu - \lambda \mp i\varepsilon)^{-1} \psi(\mu) \big\rangle_{\mathfrak{h}} \end{aligned} \tag{4.1}$$

where

$$\delta_\varepsilon(L - \lambda) := \tfrac{\varepsilon}{\pi} (L - \lambda + i\varepsilon)^{-1} (L - \lambda - i\varepsilon)^{-1}.$$

In order to exchange the limit $\varepsilon \searrow 0$ and the integral over μ, one needs to collect some preparatory results. The first of them is a lemma on the operator $\mathscr{F}_0^* \delta_\varepsilon(L - \lambda)$ appearing in (4.1). We use the notation $C_c(\mathbb{R}_+; \mathcal{G})$ for the set of compactly supported continuous functions from $\mathbb{R}_+$ to some Hilbert space $\mathcal{G}$.

Lemma 4.1 (Lemma 2.3 of [21]). *For $s \geq 0$, $t > 1$, $\lambda \in \mathbb{R}_+$ and $\varphi \in C_c(\mathbb{R}_+; \mathfrak{h})$, one has* $\lim_{\varepsilon \searrow 0} \mathscr{F}_0^* \delta_\varepsilon(L - \lambda)\varphi = \mathscr{F}_0(\lambda)^* \varphi(\lambda)$ *in* $\mathcal{H}_{-t}^{-s}$.

The next task is to analyse the function $\lambda \mapsto (u + vR_0(\lambda \mp i0)v)^{-1} v\mathscr{F}_0(\lambda)^*$, which will appear in (4.1) once the limit $\varepsilon \searrow 0$ is taken. This is the content of the next section.

4.1 The Asymmetric Term

In this section, we determine the behaviour of the function $\lambda \mapsto (u + vR_0(\lambda \mp i0)v)^{-1} v\mathscr{F}_0(\lambda)^*$ as $\lambda \searrow 0$. The main difference with respect to the analysis conducted in Sect. 3 is the absence of a factor $\mathscr{F}_0(\lambda)v$ on the left of the operator $(u + vR_0(\lambda \mp i0)v)^{-1} v\mathscr{F}_0(\lambda)^*$. For this reason, we call this operator the *asymmetric term*. As in Sect. 3, we assume that V satisfies (2.3) with $\rho > 11$ so that both the expansions (2.2) for $\mathscr{F}_0(\lambda)$ and (2.4) for $(u + vR_0(-\kappa^2)v)^{-1}$ hold. The main result of this section is presented in Theorem 4.7, the lemmas and propositions coming before are preparation for it.

In our first lemma, we determine the behaviour of the simplest terms appearing in the expansion of $(u + vR_0(\lambda \mp i0)v)^{-1} v\mathscr{F}_0(\lambda)^*$. We keep using the shorthand notations $\kappa = -i\sqrt{\lambda}$ and $\eta = 1/\ln(\kappa)$ introduced in (3.2).

Lemma 4.2 *One has as* $\lambda \searrow 0$

(a) $I_1(\kappa)v\mathscr{F}_0(\lambda)^* = O(\eta)$,
(b) $g(\kappa)I_2(\kappa)v\mathscr{F}_0(\lambda)^* = O(1)$,
(c) $\frac{g(\kappa)\eta}{\lambda} I_3(\kappa)v\gamma_2(\lambda)^* = O(1)$.

Proof Using (2.2), (2.5), (3.4) and Lemma 3.2(a), we obtain the first claim:

$$I_1(\kappa)v\mathscr{F}_0(\lambda)^* = \big(QD_0(\kappa)Q + O(\eta)\big)v\big(\gamma_0^* + \sqrt{\lambda}\gamma_1^* + O(\lambda)\big) = O(\eta) \quad \text{as } \lambda \searrow 0.$$

For the second claim, we note from (2.2), (2.6) and Lemma 3.3 that as $\lambda \searrow 0$

$$\begin{aligned} g(\kappa)I_2(\kappa)v\mathscr{F}_0(\lambda)^* &= g(\kappa)(\mathrm{M}(\kappa) + S_1)^{-1} S_1 (M_1(\kappa) + S_2)^{-1} \\ &\quad \cdot \big((\mathrm{M}(\kappa) + S_1)^{-1} S_1 + O(\eta)\big)v\big(\gamma_0^* + \sqrt{\lambda}\gamma_1^* + O(\lambda)\big). \end{aligned}$$

Since $g(\kappa) = O(\eta^{-1})$ as $\lambda \searrow 0$ and $S_1 v\gamma_0^* = 0$ due to Lemma 3.2(b), we infer that

$$g(\kappa)I_2(\kappa)v\mathscr{F}_0(\lambda)^* = O(1) \quad \text{as } \lambda \searrow 0.$$

Finally, the third claim follows from the facts that $g(\kappa) = O(\eta^{-1})$ and $\gamma_2(\lambda) = O(\lambda)$ as $\lambda \searrow 0$. □

We can now focus on the remaining two terms of the expansion of $(u + vR_0(\lambda \mp i0)v)^{-1} v\mathscr{F}_0(\lambda)^*$:

$$\frac{g(\kappa)\eta}{\lambda} I_3(\kappa)v\gamma_0^* \quad \text{and} \quad \frac{g(\kappa)\eta}{\sqrt{\lambda}} I_3(\kappa)v\gamma_1^*. \tag{4.2}$$

For this, we first observe from (2.7) that $I_3(\kappa)$ can be rewritten as

$$I_3(\kappa) = \big(B_S(\kappa)S_3 + B_T(\kappa)T_3\big)S_2(M_1(\kappa) + S_2)^{-1}S_1(\mathrm{M}(\kappa) + S_1)^{-1} \qquad (4.3)$$

with

$$\begin{aligned} B_S(\kappa) &:= (\mathrm{M}(\kappa) + S_1)^{-1}S_1(M_1(\kappa) + S_2)^{-1}S_2\big(- T_3 m(\kappa)^{-1}b(\kappa)d(\kappa)^{-1} \\ &\quad + S_3 d(\kappa)^{-1}c(\kappa)m(\kappa)^{-1}b(\kappa)d(\kappa)^{-1} + S_3 d(\kappa)^{-1}\big) \end{aligned}$$

and

$$B_T(\kappa) := (\mathrm{M}(\kappa) + S_1)^{-1}S_1(M_1(\kappa) + S_2)^{-1}S_2\big(T_3 m(\kappa)^{-1} - S_3 d(\kappa)^{-1}c(\kappa)m(\kappa)^{-1}\big).$$

Then, we first consider a part of the second operator in (4.2):

Lemma 4.3 *One has*

$$B_S(\kappa)S_3(M_1(\kappa) + S_2)^{-1}S_1(\mathrm{M}(\kappa) + S_1)^{-1}v\gamma_1^* = O(\lambda/\eta^2) \quad \textit{as } \lambda \searrow 0.$$

Proof Using successively Lemma 3.4, [14, Eq. (6.27)], the fact that $g(\kappa)^{-1} = O(\eta)$ as $\lambda \searrow 0$, Lemma 3.6(c), the fact that $M_0(\kappa)$ has a limit as $\lambda \searrow 0$, and [14, Eq. (6.69)], we get as $\lambda \searrow 0$

$$\begin{aligned} &S_3(M_1(\kappa) + S_2)^{-1}S_1(\mathrm{M}(\kappa) + S_1)^{-1}v\gamma_1^* \\ &\quad = (M_1(\kappa) + S_2)^{-1}S_3(\mathrm{M}(\kappa) + S_1)^{-1}v\gamma_1^* + O(\lambda/\eta^2) \\ &\quad = (M_1(\kappa) + S_2)^{-1}S_3\big\{g(\kappa)^{-1}\big(- QD_0(\kappa)QM_0(\kappa)P \\ &\qquad + QD_0(\kappa)QM_0(\kappa)PM_0(\kappa)QD_0(\kappa)Q\big) + QD_0(\kappa)Q\big\}v\gamma_1^* + O(\lambda/\eta^2) \\ &\quad = (M_1(\kappa) + S_2)^{-1}S_3D_0(0)Qv\gamma_1^* + O(\lambda/\eta^2). \end{aligned}$$

Since $S_3D_0(0)Q = S_3$ due to [14, Eq. (6.56)], we infer from Lemma 3.2(c) that

$$S_3(M_1(\kappa) + S_2)^{-1}S_1(\mathrm{M}(\kappa) + S_1)^{-1}v\gamma_1^* = O(\lambda/\eta^2) \quad \text{as } \lambda \searrow 0.$$

To conclude, it only remains to observe that $B_S(\kappa)$ has a limit as $\lambda \searrow 0$. □

For the second term in (4.3), we cannot get a similar estimate since $T_3 = S_2 - S_3$, with the projection S_2 not leading to many simplifications. In this case, we only get:

Lemma 4.4 *One has*

$$\begin{aligned} &B_T(\kappa)T_3(M_1(\kappa) + S_2)^{-1}S_1(\mathrm{M}(\kappa) + S_1)^{-1}v\gamma_1^* \\ &\quad = \big(T_3 m(\kappa)^{-1} - S_3 d(\kappa)c(\kappa)m(\kappa)^{-1}\big)T_3 v\gamma_1^* + O(\lambda/\eta^2) \quad \textit{as } \lambda \searrow 0. \end{aligned}$$

Proof A calculation as in the proof of Lemma 4.3 gives as $\lambda \searrow 0$

$$\begin{aligned} &T_3(M_1(\kappa)+S_2)^{-1}S_1(\mathrm{M}(\kappa)+S_1)^{-1}v\gamma_1^* \\ &\quad = (M_1(\kappa)+S_2)^{-1}T_3(\mathrm{M}(\kappa)+S_1)^{-1}v\gamma_1^* + O(\lambda/\eta^2) \\ &\quad = (M_1(\kappa)+S_2)^{-1}T_3D_0(0)Qv\gamma_1^* + O(\lambda/\eta^2) \\ &\quad = (M_1(\kappa)+S_2)^{-1}T_3v\gamma_1^* + O(\lambda/\eta^2). \end{aligned}$$

Furthermore, we know from the proof of Lemma 3.4 that $M_1(\kappa) = M_{1;0,0} + O(\lambda/\eta^2)$ in $\mathscr{B}(S_1\mathcal{H})$ as $\lambda \searrow 0$ and that S_2 is the projection on the kernel of $M_{1;0,0}$. So,

$$(M_1(\kappa)+S_2)^{-1}S_2 = S_2 + O(\lambda/\eta^2) \quad \text{as } \lambda \searrow 0, \tag{4.4}$$

and we obtain

$$T_3(M_1(\kappa)+S_2)^{-1}S_1(\mathrm{M}(\kappa)+S_1)^{-1}v\gamma_1^* = T_3v\gamma_1^* + O(\lambda/\eta^2) \quad \text{as } \lambda \searrow 0. \tag{4.5}$$

Similarly, a calculation as in the proof of Lemma 4.3 gives as $\lambda \searrow 0$

$$\begin{aligned} B_T(\kappa) &= (\mathrm{M}(\kappa)+S_1)^{-1}S_1(M_1(\kappa)+S_2)^{-1}S_2\big(T_3m(\kappa)^{-1} - S_3d(\kappa)^{-1}c(\kappa)m(\kappa)^{-1}\big) \\ &= (\mathrm{M}(\kappa)+S_1)^{-1}\big(T_3m(\kappa)^{-1} - S_3d(\kappa)^{-1}c(\kappa)m(\kappa)^{-1}\big) + O(\lambda/\eta^2) \\ &= QD_0(0)\big(T_3m(\kappa)^{-1} - S_3d(\kappa)^{-1}c(\kappa)m(\kappa)^{-1}\big) + O(\lambda/\eta^2) \\ &= T_3m(\kappa)^{-1} - S_3d(\kappa)^{-1}c(\kappa)m(\kappa)^{-1} + O(\lambda/\eta^2). \end{aligned} \tag{4.6}$$

Thus, one infers the claim by combining (4.5) and (4.6). □

Using the previous two lemmas, we get the following estimate for the second term in (4.2):

Proposition 4.5 *One has as* $\lambda \searrow 0$

$$\tfrac{g(\kappa)\eta}{\sqrt{\lambda}}\, I_3(\kappa)v\gamma_1^* = \tfrac{g(\kappa)\eta}{\sqrt{\lambda}}\big(T_3 - S_3d(\kappa)^{-1}c(\kappa)\big)m(\kappa)^{-1}T_3v\gamma_1^* + O(\sqrt{\lambda}/\eta^2).$$

And for the first operator in (4.2), we get:

Lemma 4.6 *One has as* $\lambda \searrow 0$

$$\tfrac{g(\kappa)\eta}{\lambda}\, I_3(\kappa)v\gamma_0^* = \tfrac{1}{\eta}S_3\,O(1) + O(1).$$

Proof Using [14, Eq. (6.27)], Lemmas 3.2(a) & (d), Lemma 3.6(c), and the fact that $g(\kappa)^{-1} = O(\eta)$ as $\lambda \searrow 0$, we obtain as $\lambda \searrow 0$

$$\begin{aligned} &S_2(M_1(\kappa)+S_2)^{-1}S_1(\mathrm{M}(\kappa)+S_1)^{-1}v\gamma_0^* \\ &\quad = -g(\kappa)^{-1}S_2(M_1(\kappa)+S_2)^{-1}S_1D_0(\kappa)QM_0(\kappa)Pv\gamma_0^* \\ &\quad = -g(\kappa)^{-1}[S_2, (M_1(\kappa)+S_2)^{-1}]S_1D_0(\kappa)QM_0(\kappa)Pv\gamma_0^* + O(\lambda). \end{aligned}$$

Since $\frac{\eta^2}{\lambda}[S_2, (M_1(\kappa) + S_2)^{-1}]$ has a limit in $\mathscr{B}(S_1\mathcal{H})$ as $\lambda \searrow 0$ due to Lemma 3.4 and since $\frac{1}{\eta} m(\kappa)^{-1}$ has a limit in $\mathscr{B}(\mathcal{H})$ as $\lambda \searrow 0$, it follows from what precedes and (2.7) that

$$\begin{aligned}
&\tfrac{g(\kappa)\eta}{\lambda} I_3(\kappa) v\gamma_0^* \\
&\quad = -(\mathrm{M}(\kappa) + S_1)^{-1} S_1 (M_1(\kappa) + S_2)^{-1} S_2 \big(T_3 m(\kappa)^{-1} T_3 - T_3 m(\kappa)^{-1} b(\kappa) d(\kappa)^{-1} S_3 \\
&\qquad - S_3 d(\kappa)^{-1} c(\kappa) m(\kappa)^{-1} T_3 + S_3 d(\kappa)^{-1} c(\kappa) m(\kappa)^{-1} b(\kappa) d(\kappa)^{-1} S_3 + S_3 d(\kappa)^{-1} S_3\big) \\
&\qquad \cdot \tfrac{\eta}{\lambda}[S_2, (M_1(\kappa) + S_2)^{-1}] S_1 D_0(\kappa) Q M_0(\kappa) P v\gamma_0^* + O(1) \\
&\quad = -\tfrac{1}{\eta}(\mathrm{M}(\kappa) + S_1)^{-1} S_1 (M_1(\kappa) + S_2)^{-1} S_3 d(\kappa)^{-1} S_3 \tfrac{\eta^2}{\lambda}[S_2, (M_1(\kappa) + S_2)^{-1}] \\
&\qquad \cdot S_1 D_0(\kappa) Q M_0(\kappa) P v\gamma_0^* + O(1).
\end{aligned}$$

Now, (4.4) implies that $(M_1(\kappa) + S_2)^{-1} S_3 = S_3 + O(\lambda/\eta^2)$ as $\lambda \searrow 0$, and (3.4) & (3.9) imply that $(\mathrm{M}(\kappa) + S_1)^{-1} S_1 = S_1 + O(\eta)$ as $\lambda \searrow 0$. So, we have

$$(\mathrm{M}(\kappa) + S_1)^{-1} S_1 (M_1(\kappa) + S_2)^{-1} S_3 = S_3 + O(\eta) \quad \text{as } \lambda \searrow 0.$$

It follows that for $\lambda \searrow 0$

$$\begin{aligned}
&\tfrac{g(\kappa)\eta}{\lambda} I_3(\kappa) v\gamma_0^* \\
&\quad = -\tfrac{1}{\eta} S_3 d(\kappa)^{-1} S_3 \tfrac{\eta^2}{\lambda}[S_2, (M_1(\kappa) + S_2)^{-1}] S_1 D_0(\kappa) Q M_0(\kappa) P v\gamma_0^* + O(1).
\end{aligned}$$

Since the operators $d(\kappa)^{-1}$, $\frac{\eta^2}{\lambda}[S_2, (M_1(\kappa) + S_2)^{-1}]$, $D_0(\kappa)$, and $M_0(\kappa)$ have a limit as $\lambda \searrow 0$, we finally obtain that

$$\tfrac{g(\kappa)\eta}{\lambda} I_3(\kappa) v\gamma_0^* = \tfrac{1}{\eta} S_3\, O(1) + O(1) \quad \text{as } \lambda \searrow 0,$$

as desired. □

By collecting what precedes, we can finally get a description of the behaviour of the asymmetric term:

Theorem 4.7 *If V satisfies (2.3) with $\rho > 11$, then one has as $\lambda \searrow 0$*

$$\begin{aligned}
\big(u + vR_0(\lambda \mp i0)v\big)^{-1} v \mathscr{F}_0(\lambda)^* = \tfrac{g(\kappa)\eta}{\sqrt{\lambda}} \big(T_3 - S_3 d(\kappa)^{-1} c(\kappa)\big) m(\kappa)^{-1} T_3 v\gamma_1^* \\
+ \tfrac{1}{\eta} S_3\, O(1) + O(1).
\end{aligned}$$

Note that since $g(\kappa) = O(\eta^{-1})$ and $m(\kappa)^{-1} = O(\eta)$ as $\lambda \searrow 0$, the first term behaves as $O(\eta/\sqrt{\lambda})$ in the limit $\lambda \searrow 0$. This singular behaviour is due to p-resonances since the term vanishes when $T_3 = 0$ (see the discussion in Sect. 1). On the other hand, the singular term $\frac{1}{\eta} S_3\, O(1)$ is associated with 0-energy bound states, since it vanishes when $S_3 = 0$.

4.2 Explicit formula for the wave operators

Our final objective is the derivation of an explicit formula for the wave operators $W_\pm$. A formula of this type has already been obtained in [22], but only in the generic case (when the 0-energy is assumed to be neither an eigenvalue nor a resonance). We start by recalling continuity properties of the function $\lambda \mapsto \mathscr{F}_0(\lambda)$:

Lemma 4.8 (Continuity properties of $\mathscr{F}_0(\lambda)$)

(a) For any $s \geq 0$ and $t > 1/2$, the function $\mathbb{R}_+ \ni \lambda \mapsto \langle\lambda\rangle^{1/4}\mathscr{F}_0(\lambda) \in \mathscr{B}(\mathcal{H}^s_t, \mathfrak{h})$ is continuous and bounded.

(b) Let $s > -1/2$ and $t > 1$. Then, $\mathscr{F}_0(\lambda) \in \mathscr{K}(\mathcal{H}^s_t, \mathfrak{h})$ for each $\lambda \in \mathbb{R}_+$, and the function $\mathbb{R}_+ \ni \lambda \mapsto \mathscr{F}_0(\lambda) \in \mathscr{K}(\mathcal{H}^s_t, \mathfrak{h})$ is continuous, admits a limit as $\lambda \searrow 0$, and vanishes as $\lambda \to \infty$.

Proof Point (a) follows from the properties of $\mathscr{F}_0(\cdot)$ presented after (2.1) and the estimate [28, Thm. 1.1.4]. The proof of point (b) is analogous to that of [21, Lemma 2.2]. □

In the next lemma, we define and determine continuity properties of two operator-valued functions which play a key role in the sequel. As in Sect. 4.1, we assume that V satisfies (2.3) with $\rho > 11$, and we use the notation $S_3^\perp := 1 - S_3$.

Lemma 4.9 *(a) The function*

$$\mathbb{R}_+ \ni \lambda \mapsto N(\lambda) := \mathscr{F}_0(\lambda) v S_3^\perp \in \mathscr{K}(\mathcal{H}, \mathfrak{h})$$

is continuous, admits a limit as $\lambda \searrow 0$, and vanishes as $\lambda \to \infty$. The multiplication operator $N : C_c(\mathbb{R}_+; \mathcal{H}) \to \mathscr{H}$ given by $(N\xi)(\lambda) := N(\lambda)\xi(\lambda)$ for $\xi \in C_c(\mathbb{R}_+; \mathcal{H})$ and $\lambda \in \mathbb{R}_+$, extends continuously to an element of $\mathscr{B}\big(L^2(\mathbb{R}_+; \mathcal{H}), \mathscr{H}\big)$.

(b) The function

$$\mathbb{R}_+ \ni \lambda \mapsto \widetilde{N}(\lambda) := \mathscr{F}_0(\lambda) v \lambda^{-1/4} S_3 \in \mathscr{K}(\mathcal{H}, \mathfrak{h})$$

is continuous and vanishes as $\lambda \searrow 0$ and $\lambda \to \infty$. The multiplication operator $\widetilde{N} : C_c(\mathbb{R}_+; \mathcal{H}) \to \mathscr{H}$ given by $(\widetilde{N}\xi)(\lambda) := \widetilde{N}(\lambda)\xi(\lambda)$ for $\xi \in C_c(\mathbb{R}_+; \mathcal{H})$ and $\lambda \in \mathbb{R}_+$, extends continuously to an element of $\mathscr{B}\big(L^2(\mathbb{R}_+; \mathcal{H}), \mathscr{H}\big)$.

Proof The continuity of the functions $\lambda \mapsto N(\lambda)$ and $\lambda \mapsto \widetilde{N}(\lambda)$, the fact that $N(\lambda)$ and $\widetilde{N}(\lambda)$ vanish as $\lambda \to \infty$, and the fact that $N(\lambda)$ admits a limit as $\lambda \searrow 0$, follow from the inclusion $v \in \mathscr{B}(\mathcal{H}, \mathcal{H}_\rho)$ and Lemma 4.8(b). The fact that $\widetilde{N}(\lambda)$ vanishes as $\lambda \searrow 0$ follows from the expansion (2.2) for $\mathscr{F}_0(\lambda)$ and Lemma 3.2(b). The remaining claims are direct consequences of the continuity of the functions and the existence of the limits. □

In the next lemma, we assume for the first time in the paper the absence of p-resonances, that is, that $T_3 = 0$. Also, since the wave operators $W_\pm$ are related by the equation $W_+ = W_- S^*$, we present from now on only the calculations needed to establish the formula for W_-. This amounts to consider only the plus sign in the operators $(u + vR_0(\lambda \mp i0)v)^{-1}$ appearing below.

Lemma 4.10 *Assume that* $T_3 = 0$.

(a) The functions

$$\mathbb{R}_+ \ni \lambda \mapsto B(\lambda) := S_3^\perp \big(u + vR_0(\lambda + i0)v\big)^{-1} v\mathscr{F}_0(\lambda)^* \in \mathscr{K}(\mathfrak{h}, \mathcal{H}).$$

and

$$\mathbb{R}_+ \ni \lambda \mapsto \widetilde{B}(\lambda) := S_3 \lambda^{1/4} \big(u + vR_0(\lambda + i0)v\big)^{-1} v\mathscr{F}_0(\lambda)^* \in \mathscr{K}(\mathfrak{h}, \mathcal{H})$$

are continuous and bounded.

(b) The multiplication operator $B : C_c(\mathbb{R}_+; \mathfrak{h}) \to \mathsf{L}^2(\mathbb{R}_+; \mathcal{H})$ *given by* $(B\varphi)(\lambda) := B(\lambda)\varphi(\lambda)$ *for* $\varphi \in C_c(\mathbb{R}_+; \mathfrak{h})$ *and* $\lambda \in \mathbb{R}_+$*, extends continuously to an element of* $\mathscr{B}(\mathscr{H}, \mathsf{L}^2(\mathbb{R}_+; \mathcal{H}))$.

(c) The multiplication operator $\widetilde{B} : C_c(\mathbb{R}_+; \mathfrak{h}) \to \mathsf{L}^2(\mathbb{R}_+; \mathcal{H})$ *given by* $(\widetilde{B}\varphi)(\lambda) := \widetilde{B}(\lambda)\varphi(\lambda)$ *for* $\varphi \in C_c(\mathbb{R}_+; \mathfrak{h})$ *and* $\lambda \in \mathbb{R}_+$*, extends continuously to an element of* $\mathscr{B}(\mathscr{H}, \mathsf{L}^2(\mathbb{R}_+; \mathcal{H}))$.

Proof Points (b) and (c) are direct consequences of point (a). So, we only give the proof of (a).

The continuity of the functions $\lambda \mapsto B(\lambda)$ and $\lambda \mapsto \widetilde{B}(\lambda)$ follows from the inclusion $v \in \mathscr{B}(\mathcal{H}_{-\rho}, \mathcal{H})$, Lemma 4.8(b), and Section 2.2. The fact that $B(\lambda)$ and $\widetilde{B}(\lambda)$ stay bounded as $\lambda \to \infty$ follows from the inclusion $v \in \mathscr{B}(\mathcal{H}_{-\rho}, \mathcal{H})$, Lemma 4.8(a), and the fact that $\lim_{\lambda\to\infty}(u + vR_0(\lambda + i0)v)^{-1} = u$ in $\mathscr{B}(\mathcal{H})$. Finally, the fact that $B(\lambda)$ and $\widetilde{B}(\lambda)$ stay bounded as $\lambda \searrow 0$ follows from an application of Theorem 4.7 with $T_3 = 0$. □

We are now ready to state the main result of this section. For that purpose, we recall that the dilation group $\{U_t^+\}_{t\in\mathbb{R}}$ in $\mathsf{L}^2(\mathbb{R}_+)$, with self-adjoint generator A_+, is given by $(U_t^+\varphi)(\lambda) := \mathrm{e}^{t/2}\varphi(\mathrm{e}^t\lambda)$ for $\varphi \in C_c(\mathbb{R}_+)$, $\lambda \in \mathbb{R}_+$ and $t \in \mathbb{R}$.

Theorem 4.11 (Explicit formula for W_-). *If* V *satisfies* (2.3) *with* $\rho > 11$*, and* $T_3 = 0$*, then we have the equality*

$$\mathscr{F}_0(W_- - 1)\mathscr{F}_0^* = -2\pi i \left\{N\big(\vartheta(A_+) \otimes 1\big)B + \widetilde{N}\big(\widetilde{\vartheta}(A_+) \otimes 1\big)\widetilde{B}\right\}$$

where

$$\vartheta(s) := \tfrac{1}{2}\big(1 - \tanh(\pi s)\big) \text{ and } \widetilde{\vartheta}(s) := \tfrac{1}{2}\big(1 - \tanh(2\pi s) - i\cosh(2\pi s)^{-1}\big), \quad s \in \mathbb{R}.$$

Proof It can be shown as in the proof of [21, Thm. 2.6] that there exists a dense set $\mathscr{D} \subset \mathscr{H}$ such that (4.1) holds for $\varphi, \psi \in \mathscr{D}$. We can thus write

$$
\begin{aligned}
&\big\langle \mathscr{F}_0(W_- - 1)\mathscr{F}_0^* \varphi, \psi\big\rangle_{\mathscr{H}} \\
&= -\int_{\mathbb{R}} \mathrm{d}\lambda \lim_{\varepsilon \searrow 0} \int_0^\infty \mathrm{d}\mu \big\langle \mathscr{F}_0(\mu) v\big(u + vR_0(\lambda + i\varepsilon)v\big)^{-1} v\mathscr{F}_0^* \delta_\varepsilon(L-\lambda)\varphi, \\
&\quad (\mu - \lambda + i\varepsilon)^{-1}\psi(\mu)\big\rangle_{\mathfrak{h}} \\
&= -\int_{\mathbb{R}} \mathrm{d}\lambda \lim_{\varepsilon \searrow 0} \int_0^\infty \mathrm{d}\mu \Big\{\big\langle \mathscr{F}_0(\mu) v S_3^\perp \tfrac{1}{\mu-\lambda-i\varepsilon} S_3^\perp \big(u + vR_0(\lambda + i\varepsilon)v\big)^{-1} v\mathscr{F}_0^* \\
&\quad \delta_\varepsilon(L-\lambda)\varphi, \psi(\mu)\big\rangle_{\mathfrak{h}} \\
&\quad + \big\langle \mathscr{F}_0(\mu) v \mu^{-1/4} S_3 \tfrac{\mu^{1/4}\lambda^{-1/4}}{\mu-\lambda-i\varepsilon} S_3 \lambda^{1/4}\big(u + vR_0(\lambda + i\varepsilon)v\big)^{-1} v\mathscr{F}_0^* \delta_\varepsilon(L-\lambda)\varphi, \psi(\mu)\big\rangle_{\mathfrak{h}}\Big\}.
\end{aligned}
$$

Then, we can prove as in [22, Thm. 2.5] that the first term reduces to

$$\big\langle -2\pi i N\big(\vartheta(A_+) \otimes 1\big)B\varphi, \psi\big\rangle_{\mathscr{H}},$$

and we can prove as in [21, Thm. 2.6] that the second term reduces to

$$\big\langle -2\pi i \widetilde{N}\big(\widetilde{\vartheta}(A_+) \otimes 1\big)\widetilde{B}\varphi, \psi\big\rangle_{\mathscr{H}}.$$

So, we get the equality

$$
\begin{aligned}
&\big\langle \mathscr{F}_0(W_- - 1)\mathscr{F}_0^* \varphi, \psi\big\rangle_{\mathscr{H}} \\
&\quad = \big\langle -2\pi i \big\{N\big(\vartheta(A_+) \otimes 1\big)B + \widetilde{N}\big(\widetilde{\vartheta}(A_+) \otimes 1\big)\widetilde{B}\big\}\varphi, \psi\big\rangle_{\mathscr{H}},
\end{aligned}
$$

which implies the claim due to the density of $\mathscr{D}$ in $\mathcal{H}$. □

The formula of Theorem 4.11 can be recast into a different form by performing some commutations:

Corollary 4.12 *If V satisfies (2.3) with $\rho > 11$, and $T_3 = 0$, then we have the equality*

$$\mathscr{F}_0(W_- - 1)\mathscr{F}_0^* = -2\pi i \big\{\big(\vartheta(A_+) \otimes 1_{\mathfrak{h}}\big)NB + \big(\widetilde{\vartheta}(A_+) \otimes 1_{\mathfrak{h}}\big)\widetilde{N}\widetilde{B}\big\} + K \quad (4.7)$$

with $K \in \mathscr{K}(\mathscr{H})$. In addition, the functions

$$\mathbb{R}_+ \ni \lambda \mapsto N(\lambda)B(\lambda) \in \mathscr{K}(\mathfrak{h}) \quad \textit{and} \quad \mathbb{R}_+ \ni \lambda \mapsto \widetilde{N}(\lambda)\widetilde{B}(\lambda) \in \mathscr{K}(\mathfrak{h}) \quad (4.8)$$

are continuous and vanish as $\lambda \searrow 0$ and $\lambda \to \infty$.

Proof By taking into account that the functions $\lambda \mapsto N(\lambda)$ and $\lambda \mapsto \widetilde{N}(\lambda)$ have limits at 0 and ∞, and that the functions $s \mapsto \vartheta(s)$ and $s \mapsto \widetilde{\vartheta}(s)$ have a limit at $-\infty$ and ∞, one can show the inclusions

$$
\begin{aligned}
&N\big(\vartheta(A_+) \otimes 1\big) - \big(\vartheta(A_+) \otimes 1_{\mathfrak{h}}\big)N \in \mathscr{K}\big(\mathsf{L}^2(\mathbb{R}_+; \mathcal{H}), \mathscr{H}\big), \\
&\widetilde{N}\big(\widetilde{\vartheta}(A_+) \otimes 1\big) - \big(\widetilde{\vartheta}(A_+) \otimes 1_{\mathfrak{h}}\big)\widetilde{N} \in \mathscr{K}\big(\mathsf{L}^2(\mathbb{R}_+; \mathcal{H}), \mathscr{H}\big).
\end{aligned}
$$

This type of result has already been proved in [21, Lemma 2.7] and is based on an argument of Cordes, see for instance [4, Thm. 4.1.10]. These commutation relations and Theorem 4.11 imply (4.7). The continuity of the functions (4.8), as well as the equalities

$$\lim_{\lambda\searrow 0} \widetilde{N}(\lambda)\widetilde{B}(\lambda) = \lim_{\lambda\to\infty} \widetilde{N}(\lambda)\widetilde{B}(\lambda) = \lim_{\lambda\to\infty} N(\lambda)B(\lambda) = 0,$$

follow from Lemmas 4.9 and 4.10(a). Finally, the equality $\lim_{\lambda\searrow 0} N(\lambda)B(\lambda) = 0$ follows from Theorem 3.1, the equality $\lim_{\lambda\searrow 0} \widetilde{N}(\lambda)\widetilde{B}(\lambda) = 0$ and the identity (see (3.1))

$$N(\lambda)B(\lambda) + \widetilde{N}(\lambda)\widetilde{B}(\lambda) = -\tfrac{1}{2\pi i}\big(S(\lambda) - 1_{\mathfrak{h}}\big).$$

□

Let us mention that if $S_3 = 0$, then the formula (4.7) can be simplified further. Indeed, in such a case one has $\widetilde{N}\widetilde{B} = 0$ and $-2\pi i N(\lambda)B(\lambda) = S(\lambda) - 1_{\mathfrak{h}}$. Thus,

$$\mathscr{F}_0(W_- - 1)\mathscr{F}_0^* = \tfrac{1}{2}\big\{\big(1 - \tanh(\pi A_+)\big) \otimes 1_{\mathfrak{h}}\big\}\big(S(L) - 1_{\mathscr{H}}\big) + K$$

with L the multiplication operator introduced in Sect. 2.1. On another hand, the fact that $\mathscr{F}_0$ diagonalises H_0 implies $S = \mathscr{F}_0^* S(L)\mathscr{F}_0$, and a direct calculation shows that

$$\mathscr{F}_0^*\big\{\big(1 - \tanh(\pi A_+)\big) \otimes 1_{\mathfrak{h}}\big\}\mathscr{F}_0 = 1 + \tanh(\pi A/2)$$

with A the generator of dilations in $\mathcal{H}$. Therefore, we get that

$$W_- - 1 = \tfrac{1}{2}\big(1 + \tanh(\pi A/2)\big)\big(S - 1\big) + K'$$

with $K' \in \mathscr{K}(\mathcal{H})$, which is exactly the formula obtained in [22] in the generic case.

In the more general case, with no assumption on S_3, similar computations do not lead to such a neat formula. Indeed, the multiplication operators $-2\pi i NB$ and $-2\pi i \widetilde{N}\widetilde{B}$ are not individually related to the scattering operator; only their sum is related to $S(L)$ through the equality

$$-2\pi i(NB + \widetilde{N}\widetilde{B}) = S(L) - 1_{\mathscr{H}}.$$

The best we can get in this situation is therefore the following:

Corollary 4.13 *If V satisfies (2.3) with $\rho > 11$, and $T_3 = 0$, then we have the equality*

$$\begin{aligned} W_- - 1 &= \tfrac{1}{2}\big(1 + \tanh(\pi A/2)\big)\mathscr{F}_0^*(-2\pi i NB)\mathscr{F}_0 \\ &\quad + \tfrac{1}{2}\big(1 + \tanh(\pi A) - i\cosh(\pi A)^{-1}\big)\mathscr{F}_0^*(-2\pi i \widetilde{N}\widetilde{B})\mathscr{F}_0 + K' \end{aligned}$$

with $K' \in \mathscr{K}(\mathcal{H})$.

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