Stochastic Power Law Fluids: Construction of a Weak Solution

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0. Motivation

- **Object**: Dynamics of viscous, incompressible fluids with a random perturbation, as a model of **turbulence**.

- **The most studied model so far**: The **SNS** (stochastic Navier-Stokes eq.):

  \( u = (u_i(t,x))_{i=1}^d \): velocity of the fluid,

  \( \Pi = \Pi(t,x) \): pressure,

  \( W = (W_i(t,x))_{i=1}^d \): BM in \( L^2(dx)^d \) with the trace class cov., \( \text{div} \ W = 0 \).

1) \( \text{div} \ u = 0 \) (imcompressible);

1') \( \partial_t u + (u \cdot \nabla) u = -\nabla \Pi + \nu \Delta u + \partial_t W \).

where \( \nu > 0 \) and \( u \cdot \nabla = \sum_{j=1}^d u_j \partial_j \)

cf. [Flandoli 2008] and ref.’s therein.
More phys. behind (NS)
- A review of fluid mechanics.

- $u = (u_j)_{j=1}^d$: the velocity field of the fluid.

- The force exerted to the fluid per volume:

$$-\nabla \Pi + \text{div } \tau(u)_{\text{friction}}$$

$$= -\nabla \Pi + \left( \sum_{j=1}^d \partial_j \tau_{ij}(u) \right)_{i=1}^d \in \mathbb{R}^d,$$

where

$$\tau(u) \in \mathbb{R}^d \otimes \mathbb{R}^d \text{ (extra stress tensor)}$$

is a function of the (symmetrized) velocity gradient:

$$e(u) = \left( \frac{\partial_i u_j + \partial_j u_i}{2} \right) \in \mathbb{R}^d \otimes \mathbb{R}^d.$$

- **Stokes’ law:** $\tau(u) = 2\nu e(u)$.

  Valid for **Newtonian** fluids (e.g., air, water,...)

  Stokes’ law + div $u = 0$

  $\Rightarrow$ div $\tau(u) = \nu \Delta u$ (N.S.eq.).
More general relation of \( \tau(u) \) and \( e(u) \):

\[
\tau(u) = \frac{F(|e(u)|)}{\text{viscosity} > 0} e(u).
\]

- Newtonian \( \iff F \equiv \text{const.} \)

- Many non-Newtonian fluids are applied in science and engineering. Typical ones are:

  - **Shear thinning** fluid: \( F \) is \( \searrow \) in \( |e(u)| | \)
    “Fluid exhibits shear thinning when dilute polymer melt is added” (Toms effect 1948). This is applied to e.g., automobile engine oil, pipelines for crude oil.

  - **Shear thickening** fluid: \( F \) is \( \nearrow \) in \( |e(u)| | \)
    hydrocarbon fluids, suspension of starch, applications: bullet proof vests, automobile 4WD systems,..).

- **PDE results for non-Newtonian fluids:**
  [ Málek, Nečas, Rokyta, Růžička, 1996 ]
  \( \exists \) sol. and \( \exists 1 \) sol. in some cases.
Plan for the rest of the talk:

1. The stochastic power law fluid (SPLF)
2. The existence thm for a weak solution
3. The proof of the existence thm
4. Future works
1. The stochastic power law fluid (SPLF)

- **Container of the fluid:**
  
  \[ T^d = (\mathbb{R}/\mathbb{Z})^d \cong [0, 1]^d \]

Given the velocity field \( u = (u_j)_{j=1}^d \),

\[ \tau(u) = 2(1 + |e(u)|^2)^{\frac{p-2}{2}} e(u) : T^d \to \mathbb{R}^d \otimes \mathbb{R}^d, \]

with \( p \in (1, \infty) \).

**Note**

\[
\begin{array}{ll}
p \
< 2 & \text{“shear thinning”} \\
& \text{e.g., engine oil (polymer melts), ...} \\
= 2 & \text{“Newtonian”} \\
& \text{e.g., air, water,...} \\
> 2 & \text{“shear thickening”} \\
& \text{e.g., hydrocarbon fluids,} \\
& \text{ suspension of starch,...}
\end{array}
\]
SPDE for Stochastic Power Law Fluids:

- \( u = (u_i(t, x))_{i=1}^d \): velocity of the fluid,
- \( \Pi = \Pi(t, x) \): pressure,
- \( W = (W_i(t, x))_{i=1}^d \): BM in \( L^2(\mathbb{T}^d \to \mathbb{R}^d) \) with the trace class cov., \( \text{div } W = 0 \).

- \((\text{SPLF})_p:\)

2) \( \text{div } u = 0 \) (imcompressible);

2') \( \underbrace{\partial_t u + (u \cdot \nabla) u}_{\text{"acceleration"}} = -\nabla \Pi + \underbrace{\text{div } \tau(u)}_{\text{"friction"}} + \partial_t W. \)

where \( \text{div } \tau(u) = \left( \sum_{j=1}^d \partial_j \tau_{ij}(u) \right)_{i=1}^d \)

Rem. \((\text{SPLF})_2 = (\text{SNS})\), since:

\[ p = 2 \Rightarrow \text{div } \tau(u) = \Delta u. \]
2. The existence thm for a weak sol.

• **Test Functions:**
  
  \[ \mathcal{V} = \text{“div-free smooth vect. fields”} \]
  
  \[ = \{ v : \mathbb{T}^d \to \mathbb{R}^d ; \text{trigo. polyn., div } v = 0 \}. \]

• **Spaces of the solutions:**
  
  \[ V_{p,\alpha} = \text{“Sobolev sp. of div-free vect. fields”} \]
  
  \[ = \| \cdot \|_{p,\alpha}-\text{completion of } \mathcal{V}, \]

  where \( p \in [1, \infty) \), \( \alpha \in \mathbb{R} \) and:

  \[
  \| v \|_{p,\alpha}^p = \int_{\mathbb{T}^d} |(1 - \Delta)^{\alpha/2} v|^p. 
  \]
• **The weak solution:**

  ▶ \( \mu_0 \in \mathcal{P}(V_{2,0}) = \text{prob.'s on } V_{2,0} \).

  ▶ \( (u, W) = ((u_t, W_t))_{t \geq 0} \) : a process s.t.

\[
    u \in L_p,_{\text{loc}}(\mathbb{R}_+ \to V_{p,1}) \bigcap L_\infty,_{\text{loc}}(\mathbb{R}_+ \to V_{2,0}) \\
    \bigcap C(\mathbb{R}_+ \to V_{p',\wedge 2,-\beta}), \text{ for } \exists \beta > 0, \\
    W \text{: BM in } V_{2,0}, \text{ trace class cov. } \Gamma.
\]

  ▶ \( (u, W) \) is a **w-sol.** to \( (\text{SPLF})_p \) with init. law \( \mu_0 \), if:

  3) \( P(u_0 \in \cdot) = \mu_0 \),

\[
    \langle \varphi, u_t - u_0 \rangle = \int_0^t \langle u_s, (u_s \cdot \nabla)\varphi \rangle ds \\
    - \int_0^t \langle e(\varphi), \tau(u_s) \rangle ds + \langle \varphi, W_t \rangle.
\]

  for \( \forall \varphi \in \mathcal{V} \).

**Rem.**

• 2), 2') \( \xrightarrow{\text{IBP}} 3' \), when \( \Pi \) disappears, since

\[
    \langle \varphi, \nabla \Pi \rangle = -\langle \text{div } \varphi, \Pi \rangle = 0.
\]
**Theorem 1** Suppose:

- \( p \in \exists I_d \)
  
  (e.g., \( I_2 = (3/2, \infty) \), \( I_3 = (9/5, \infty) \), \( I_4 = (2, \infty), \ldots \))

- \( \mu_0 \in \mathcal{P}(V_{2,1}), \int \|v\|_{2,1}^2 \mu_0(dv) < \infty \).

- \( \Delta \Gamma = \Gamma \Delta, \{\Gamma, \Gamma \Delta\} \subset \text{trace class.} \)

Then, \( \exists \) w-sol. \((u, W)\) to \((SPLF)_p\) with init. law \( \mu_0 \). Moreover,

\[
E \left[ \sup_{t \leq T} \|u_t\|_2^2 + \int_0^T \|u_t\|_{p,1}^p dt \right] \leq (1 + T)C < \infty.
\]

**Rem.'s**

- \( d = 2, 3, p = 2 \Rightarrow \) result for SNS cf. [Flandoli 2008] and ref.'s therein.

- \( W \equiv 0 \Rightarrow \) PDE result [ Málek et al. '96 ].

- Pathwise uniqueness: OK for \( p \geq \frac{d+2}{2} \).
  
  (looks VERY hard for \( p < \frac{d+2}{2} \), e.g. 3D NS.)
Technical difference: (S)PLF $\leftrightarrow$ (S)NS

• (S)PLF is $L_p$ (Banach sp.)-theory as opposed to $L_2$ (Hilbert sp.)-theory for (S)NS.

• Extra non-linearity in the friction term: the proofs of some a priori bounds are much more difficult to get.
3. The proof of the existence thm.

**Step 1 (Galerkin approximation)**

Set up:

- Subsp.'s $V^{(n)} \to V$, dim $V^{(n)} < \infty$.
- An approx. eq. $\Rightarrow$ unique sol. $u^{(n)} \in V^{(n)}$.

**Step 2 (A priori bds)**

- Establish some a priori bds for $u^{(n)}$ unif in $n$, e.g.,
  $$E \left[ \sup_{t \leq T} \left\| u_t^{(n)} \right\|_2^2 + \int_0^T \left\| u_t^{(n)} \right\|_{p,1}^p dt \right] \leq (1+T)C < \infty.$$  
  
  Technique:
  
  Itô calculus, Martingale ineq.'s (e.g., B-D-G), Sobolev imbedding.
Step 3 (Tightness)

• Prove the tightness of \( u^{(n)}, n \geq 1 \) in Sobolev sp.'s of the form:

\[
X = L_{p_1,\alpha_1}([0, T] \rightarrow V_{p_2,\alpha_2})
\]

so that

\[
u^{(n)} \xrightarrow{n\to\infty} \exists u \text{ in law along a subseq.}
\]

To this end, we choose \( X_1 \subset X \) s.t.

a) \( X_1 \hookrightarrow X \) compactly .

(cpt imbedding thm’s for Sobolev sp’s).

b) \( \exists \delta > 0, \sup_n E[\|u^{(n)}\|_{X_1}^\delta] \leq C_T \)

(A priori bds used here).

Then, the desired tightness in \( X \) can be seen as follows:

\[
\{ u \in X_1 ; \|u\|_{X_1} \leq R \} \overset{a)}{\subset} X
\]

and

\[
\sup_n P(\|u^{(n)}\|_{X_1} > R) \leq \frac{\sup_n E[\|u^{(n)}\|_{X_1}^\delta]}{R^\delta} \overset{b)}{\leq} \frac{C_T}{R^\delta} \xrightarrow{R \to \infty} 0.
\]
**Step 4 (Verification of SPDE)**

- By Step 3,

\[ u^{(n)} \xrightarrow{n \to \infty} \exists u \quad \text{in law along a subseq.} \]

Then, \( \exists \) BM \( W \) s.t. \( (u, W) \) is a w-sol. to \( (SPLF)_p \).
4. Future works

- Invariant measure
  (an example of “non-equilibrium steady state”)

- Ergodicity
  (a starting point to discuss the turbulence)

- (In the distant future ??)
  Approach to Kolmogorov’s K41 theory,
  Onsager conjecture