The transition operator of a random walk perturbated by sparse potentials[1]

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Abstract

We consider an operator $P_V = (1+V)P$ on $\ell^2(\mathbb{Z}^d)$, where P is the transition operator of a symmetric irreducible random walk, and V is a "sparse" potential. We first characterize the essential spectra of this operator. Secondly, we prove that all the eigenfunctions which correspond to discrete spectra decay exponentially fast. Thirdly, we give a sufficient condition for this operator to have an absolute spectral gap at the right edge of the spectra. Finally, as an application of the absolute spectral gap and the exponential decay of the eigenfunctions, we prove a limit theorem for the random walk under the Gibbs measure associated to the potential V.

1 Introduction

In this article, we investigate the spectral properties of an operator $P_V = (1+V)P$ on $\ell^2(\mathbb{Z}^d)$, where P is the transition operator of a symmetric irreducible random walk, and V is a nonnegative bounded function. Here, V is supposed to be a so called "sparse" potential, of which a typical example is that with the property

$$\min\{|x-y| \ ; \ x, y \in \mathrm{supp}V, \ x \neq y, \ |x| \geq r, \ |y| \geq r\} \stackrel{r\to\infty}{\longrightarrow} \infty, \tag{1.1}$$

Before introducing the contents of this article, we start by explaining the probabilistic backgrounds which brought us to the study of the spectra of P_V.

Let $(S_n)_{n\in\mathbb{N}}$ be a random walk on a probability space $(\Omega, \mathcal{F}, P)$ which is associated with the transition operator P. We suppose that $S_0 = 0$. Then, the semigroup P_V^n $(n \in \mathbb{N})$ is expressed by the discrete version of the Feynmann-Kac formula.

$$P_V^n f(x) = E\left[f(x+S_n)\prod_{j=0}^{n-1}(1+V(x+S_j))\right], \quad f \in \ell^2(\mathbb{Z}^d), \quad x \in \mathbb{Z}^d. \tag{1.2}$$

Similarly, if we restrict P_V to $\ell^2(\mathbb{N}^d)$ by imposing Dirichlet boundary condition, and denote the restriction by $P_V^+ \in \mathcal{B}(\ell^2(\mathbb{N}^d))$, then, we obtain

$$\begin{aligned}(P_V^+)^n f(x) &= E\left[f(x+S_n)\mathbf{1}\{S_j \in \mathbb{N}^d, \ j=1,\ldots,n\}\prod_{j=0}^{n-1}(1+V(x+S_j))\right], \\ &\quad f \in \ell^2(\mathbb{N}^d), \quad x \in \mathbb{N}^d.\end{aligned} \tag{1.3}$$

Let $\partial\mathbb{N}^d = \bigcup_{\alpha=1}^d \{x \in \mathbb{N}^d \ ; \ x_\alpha = 0\}$, and $\beta > 0$ be a positive parameter. In connection with statistical physics, the following sequence μ_N, $N \geq 1$ of measures on $(\Omega, \mathcal{F})$ are studied.

$$\mu_N(d\omega) = \frac{1}{Z_N} E\left[\mathbf{1}\{S_j \in \mathbb{N}^d, \ j=1,\ldots,N\}\exp\left(\beta\sum_{j=1}^{N-1}\mathbf{1}_{\partial\mathbb{N}^d}(S_j)\right) : d\omega\right], \tag{1.4}$$

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where Z_N is the normalizing constant. The above measure is related with the operator P_V^+ with $V = \exp(\beta \mathbf{1}_{\partial \mathbb{N}^d}) - 1$ via the formula (1.3). Under the measure μ_N, two competing effects coexist: The paths of the random walk is attracted to $\partial \mathbb{N}^d$ by the potential to lower the eneregy:

$$-\beta \sum_{j=1}^{N-1} \mathbf{1}_{\partial \mathbb{N}^d}(S_j).$$

On the other hand, paths of the random walk are pushed away from $\partial \mathbb{N}^d$ by the entropic repulsion. These competing effects cause the phase transition, called wetting transition, which is expressed as follows, cf. [3, 7]. There exists $\beta_c \in (0, \infty)$ such that

$$\begin{aligned} \lim_{N \to \infty} \quad Z_N^{1/N} &= \max\{|\lambda| \; ; \; \lambda \in \sigma(P_V^+)\} \\ &\begin{cases} = 1, & \text{if } \beta \leq \beta_c \text{ (delocalized phase)}, \\ > 1, & \text{if } \beta > \beta_c \text{ (localized phase)}. \end{cases} \end{aligned} \tag{1.5}$$

Intuitively, if N is large, the random walk under the measure μ_N behaves as if there were no potentials in the delocalized phase, while it is localized near $\partial \mathbb{N}^d$ in the localized phase. In fact, for $d = 1$, these intuitive pictures are justified mathematically in [3]. In particular, it was shown there that, in the localized phase, the measure μ_N converges as $N \to \infty$ to the law of a positively recurrent Markov chain.

Technically, a crutial step in the proof of the limit theorem in [3] referred to above is the existence of the absolute spectral gap of P_V^+ (cf. (1.9) below), and it is here that the particularity of one dimension comes into play. Indeed, if $d = 1$, the operator P_V^+ is a compact perturbation of P^+, from which the existence of the absolute spectral gap follows via Weyl's essential spectrum theorem. Unfortunately, $P_V^+ - P^+$ is no longer a compact in higher dimensions. The present article comes out as a partial progress in the effort to carry the results in [3] over to higher dimensions and therefore, to noncompact perturbation cases.

We now explain contents of this paper a little more in detail. For simplicity, we consider the whole lattice $\mathbb{Z}^d$, rather than its first quadrant $\mathbb{N}^d$.

Let $p : \mathbb{Z}^d \to [0, \infty)$ be a transition probability of a symmetric irreducible random walk. We define $P : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ by

$$Pf(x) = \sum_{y \in \mathbb{Z}^d} p(x - y) f(y), \quad f \in \ell^2(\mathbb{Z}^d), \quad x \in \mathbb{Z}^d. \tag{1.6}$$

We consider a perturbation $P_V : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d)$ of the operator P of the form

$$P_V = (1 + V)P, \tag{1.7}$$

where $V : \mathbb{Z}^d \to [0, \infty)$ is a bounded function. Here, as usual, V is regarded as the multiplication operator. In the context of probability theory, this type of perturbation is quite natural as we have already seen.

In this article, we investigate the structure of spectra $\sigma(P_V)$, as well as its consequence on the long time behavior of the semigruop (1.2). In this context, an important quantity is the right edge

$$r(P_V) = \max \sigma(P_V)$$

of the spectra. More precisely, we are interested in the following properties

$$r(P_V) > \sup\{\lambda\ ;\ \lambda \in \sigma(P_V),\ \lambda \neq r(P_V)\}\ \textbf{(spectral gap)}, \tag{1.8}$$
$$r(P_V) > \sup\{|\lambda|\ ;\ \lambda \in \sigma(P_V),\ |\lambda| \neq r(P_V)\}\ \textbf{(absolute spectral gap)}. \tag{1.9}$$

As it turns out in the sequel, the following quantity plays an important role in studying these properties.

$$v_0 = \inf_{n\in\mathbb{N}} \sup_{|x|\geq n} V(x). \tag{1.10}$$

For example, as is discussed earlier in [3], properties (1.8) and (1.9) are relatively easy to obtain when $v_0 = 0$, i.e., V decays at infinity. Indeed, the multiplication operator V is compact in this case, and hence, by Weyl's essential spectrum theorem, cf. [2, p.358, Proposition 4.2 (e)], [6, p.112, Theorem XIII.14], the set of essential spectra $\sigma_{\mathrm{ess}}(P_V)$ is the same as $\sigma_{\mathrm{ess}}(P) = \sigma(P) = [\ell(P), 1]$ $(-1 \leq \ell(P) < 1)$. Thus, one immediately obtains (1.8) as soon as one knows that $r(P_V) > 1$. Then, it can be improved to (1.9) under reasonable additional assumptions on the transition function p.

In this article, we are mainly interested in the case of $v_0 > 0$, where the multiplication operator V is no longer compact. To compensate the lack of the decay of V at infinity, we will assume that the support of V is sparse enough, to ensure that the perturbation is not too large to control, see (1.24) below for the precise formulation of the sparseness. To the best of our knowledge, research in this direction was initiated in [5], where the Schrödinger operator $-\frac{d^2}{dx^2} + V$ on the real line is discussed.

Firstly, we characterize in Theorem 1.2.1 the set $\sigma_{\mathrm{ess}}(P_V)$ of essential spectra of P_V. Here, we adapt the method in [4] to the present setting. Theorem 1.2.1 has a corollary, which tells us that the excess $\sigma_{\mathrm{ess}}(P_V)\backslash\sigma(P)$ is nonempty if, e.g., $d \leq 2$ and $v_0 > 0$, cf. (1.10). For $d \geq 3$, we have the same conclusion if v_0 is sufficiently large. Thus, the spectral aspect of the operator P_V is indeed different from the compact perturbation case.

Secondly, we prove in Theorem 1.2.3 that that all the eigenfunctions which correspond to discrete spectra decay exponentially fast.

Thirdly, we establish the spectral gap (1.8) in Theorem 1.2.4, and then, improve it to the absolute spectral gap (1.9) in Corollary 1.2.5.

Finally, as an application of the absolute spectral gap (1.9) obtained in Corollary 1.2.5 and the exponential decay of the eigenfunctions (Theorem 1.2.3), we prove a limit theorem for the random walk in the potential V. More precisely, we consider a Gibbs measures μ_N $(N \in \mathbb{N})$ on $(\Omega, \mathcal{F})$ given by

$$\mu_N(d\omega) = \frac{1}{Z_N} E\left[\prod_{j=0}^{N-1}(1 + V(S_j)) : d\omega\right], \tag{1.11}$$

where Z_N is the normalizing constant. Then, we will prove that the measure μ_N converges as $N \to \infty$ to a positively recurrent Markov chain on $\mathbb{Z}^d$, cf. Theorem 1.3.1 below.

The rest of this article is organized as follows.

Contents

1.1 General notations

• For a normed vector space X, we denote the totality of bounded linear operators $T : X \to X$ by $\mathcal{B}(X)$. For $T \in \mathcal{B}(X)$, $\|T\|$, $\sigma(T)$ and $\rho(T)$ stands respectively for its operator norm, the totality of its spectra, and that of its resolvents. If X is a Hilbert space and $T \in \mathcal{B}(X)$ is self-adjoint, we write

$$r(T) = \max \sigma(T), \quad \ell(T) = \min \sigma(T). \tag{1.12}$$

• The Banach space $\ell^p(\mathbb{Z}^d)$ will be abbreviated by ℓ^p. For a subset $S \subset \mathbb{Z}^d$, $\ell^p(S)$ is identified with the totality of $f \in \ell^p$ which vanish outside S.
• For $T \in \mathcal{B}(\ell^2)$, we write its kernel by $T(x,y) \stackrel{\text{def}}{=} \langle\, \delta_x, T\delta_y \,\rangle$ $(x, y \in \mathbb{Z}^d)$.
• For $u \in \ell^p$ and $u \in \ell^q$ $(p, q \in [1,\infty], \frac{1}{p} + \frac{1}{q} = 1)$ let $\langle\, u, v \,\rangle = \sum_{x \in \mathbb{Z}^d} u(x)v(x)^*$, where c^* denotes the complex conjugate of $c \in \mathbb{C}$.

1.2 Spectra of the operator P_V

Let $p : \mathbb{Z}^d \to [0,\infty)$ be a transition probability of a random walk, i.e., $\sum_{x\in\mathbb{Z}^d} p(x) = 1$. Additionally, we assume that

$$(\textbf{symmetry})\ p(x) = p(-x) \text{ for all } x \in \mathbb{Z}^d; \tag{1.13}$$

$$(\textbf{finite range})\ p \text{ is supported on a finite set}; \tag{1.14}$$

$$(\textbf{irreducibility}) \text{ for all } x \in \mathbb{Z}^d, \text{ there exists } n \in \mathbb{N} \text{ such that } p_n(x) > 0, \tag{1.15}$$

where p_n denotes the n-fold convolution. Then, we define $P \in \mathcal{B}(\ell^2)$ and $P_V \in \mathcal{B}(\ell^2)$ respectively by (1.6) and (1.7).

Let $\widehat{p}(\theta) = \sum_{x\in\mathbb{Z}^d} p(x) \exp(\mathbf{i}\theta \cdot x)$, $\theta \in [-\pi,\pi]^d$. Then, the set $\sigma(P)$ of the spectra of P is the interval $[\ell(P), 1]$, where $\ell(P) = \min \widehat{p} \in [2p(0) - 1, 1)$. For $\lambda \in \mathbb{C}\backslash\sigma(P)$, let $G_\lambda \in \mathcal{B}(\ell^2)$ be the resolvent operator

$$G_\lambda = (\lambda - P)^{-1}. \tag{1.16}$$

We then define

$$g_\lambda(x) \stackrel{\text{def}}{=} \lambda G_\lambda(0,x), \ \ \lambda \in \mathbb{R}\backslash\sigma(P), \ x \in \mathbb{Z}^d. \tag{1.17}$$

In the sequel, we need to deal with essential spectra of operators which are not self-adjoint. To do so, we adopt the following definition via the theory of Fredholm operator. Let X be a Banach space and $T \in \mathcal{B}(X)$. We say that T is a Fredholm operator if

$$\mathrm{Ran}T \text{ is closed, } \dim \mathrm{Ker}T < \infty, \text{ and } \dim (X/\mathrm{Ran}T) < \infty. \tag{1.18}$$

We denote the totality of Fredholm operator by $\mathrm{F}(X)$. We then define the set $\sigma_{\mathrm{ess}}(T) \subset \mathbb{C}$ of essential spectra of T by

$$\lambda \in \sigma_{\mathrm{ess}}(T) \iff \lambda - T \notin \mathrm{F}(X). \tag{1.19}$$

If X is a Hilbert space and T is self-adjoint, then,

$$T \in \mathrm{F}(X) \iff \dim \mathrm{Ker}T < \infty \text{ and } 0 \notin \overline{\sigma(T)\backslash\{0\}}. \tag{1.20}$$

cf. [2, p.359, Proposition 4.6].

In what follows, we will exploit the following characterization of the essential spectra of the multiplication operator V.

$$v \in \sigma_{\mathrm{ess}}(V) \iff \sharp\{x \in \mathbb{Z}^d \ ; \ |V(x) - v| < \varepsilon\} = \infty \ \text{ for all } \varepsilon > 0, \tag{1.21}$$

where $\sharp$ stands for the cardinality. Moreover,

$$\max \sigma_{\mathrm{ess}}(V) = v_0, \tag{1.22}$$

where v_0 is defined by (1.10). Let us consider an inner product and the associated norm.

$$\langle\, f, g \,\rangle_V \stackrel{\mathrm{def}}{=} \langle\, (1+V)^{-1} f, g \,\rangle, \ \ \|f\|_V = \sqrt{\langle\, f, f \,\rangle_V}, \ \ f, g \in \ell^2. \tag{1.23}$$

We note that the norms $\|\cdot\|$ and $\|\cdot\|_V$ are equivalent. The Hilbert space $(\ell^2, \|\cdot\|_V)$ will be denoted by ℓ^2_V. Then, the operator $P_V \in \mathcal{B}(\ell^2_V)$ is self-adjoint. As a consequence, all the spectra of P_V are real numbers.

We first identify the set $\sigma_{\mathrm{ess}}(P_V)$ of the essential spectra.

Theorem 1.2.1 *Suppose that $V : \mathbb{Z}^d \to [0, \infty)$ is a bounded function such that*

$$a_\varepsilon(x) \stackrel{\mathrm{def}}{=} \sum_{\substack{y \in \mathbb{Z}^d \\ y \neq x}} \sqrt{V(x)V(y)} \exp(-\varepsilon|x-y|) \stackrel{|x|\to\infty}{\longrightarrow} 0 \ \text{ for any } \varepsilon > 0. \tag{1.24}$$

Then,

$$\sigma_{\mathrm{ess}}(P_V) = \sigma(P) \cup \Lambda_V, \tag{1.25}$$

where

$$\Lambda_V = \{\lambda \in \mathbb{R}\backslash\sigma(P) \ ; \text{ there exists } v \in \sigma_{\mathrm{ess}}(V)\backslash\{0\} \text{ such that } g_\lambda(0) = 1 + v^{-1}\}. \tag{1.26}$$

Remark Condition (1.24) is satisfied, not only when $V(x) \stackrel{|x|\to\infty}{\longrightarrow} 0$, but also when the support of V is sufficiently sparse. For example, the condition (1.24) follows from (1.1). Indeed, suppose that $|x| \geq 2r$. Then, $|x-y| \geq r$ if $|y| \leq r$. Thus, letting $\mathrm{d}(r)$ denote the minimum on the left-hand side of (1.1), we have

$$\begin{aligned} a_\varepsilon(x) &\leq \|V\|_\infty \left(\sum_{|y|\leq r} + \sum_{\substack{|y|>r \\ y\neq x}} \right) \exp(-\varepsilon|x-y|) \\ &\leq Cr^d \exp(-\varepsilon r) + C\exp\left(-\frac{\varepsilon \mathrm{d}(r)}{2}\right) \sum_{\substack{y\in\mathbb{Z}^d \\ y\neq 0}} \exp\left(-\frac{\varepsilon|y|}{2}\right) \stackrel{r\to\infty}{\longrightarrow} 0. \end{aligned}$$

Theorem 1.2.1 has the following corollary, which says that, unlike the compact perturbation case, we may find essential spectra of P_V outside $\sigma(P)$.

Corollary 1.2.2 **a)** *$\sigma_{\mathrm{ess}}(P_V) \cap (1,\infty) \neq \emptyset$, if and only if*

$$v_0 > 0 \text{ and } 1 + v_0^{-1} = g_{\lambda_0}(0) \text{ for some } \lambda_0 \in (1,\infty), \tag{1.27}$$

where v_0 is defined by (1.10). Moreover, (1.27) implies that $\lambda_0 = \max \sigma_{\mathrm{ess}}(P_V)$.

b) *$\sigma_{\mathrm{ess}}(P_V) \cap (-\infty, \ell(P)) \neq \emptyset$, if and only if*

$$\ell(P) < 0,\ v_0 > 0 \text{ and } 1 + v_0^{-1} = g_{\lambda_0}(0) \text{ for some } \lambda_0 \in (-\infty, \ell(P)). \tag{1.28}$$

Remark It follows from (2.6) that the condition (1.27) is satisfied whenever $\sigma_{\mathrm{ess}}(V)\backslash\{0\} \neq \emptyset$ if $d \leq 2$. Similarly, we see from (2.7) that the condition (1.28) is satisfied whenever $\sigma_{\mathrm{ess}}(V)\backslash\{0\} \neq \emptyset$ if $d \leq 2$ and $\ell(P) < 0$.

The next result (Theorem 1.2.3) deals with discrete spectra of P_V. More precisely, it states that the corresponding eigenfunctions decay exponentially fast. Together with the absolute spectral gap (Corollary 1.2.5), this theorem plays an important role in the proof of Theorem 1.3.1 below.

Theorem 1.2.3 *Suppose that the condition (1.24) holds true, $\lambda \in \sigma(P_V)\backslash\sigma_{\mathrm{ess}}(P_V)$ and that a function $\varphi \in \ell^2$ satisfies $(\lambda - P_V)\varphi = 0$. Then, there exist constants $\alpha, C \in (0,\infty)$ such that*

$$|\varphi(x)| \leq C\exp(-\alpha|x|), \quad \text{for all } x \in \mathbb{Z}^d. \tag{1.29}$$

The next result deals with the right edge $r(P_V)$ of the spectra of P_V, cf. (1.12).

Theorem 1.2.4 *In addition to conditions (1.24) and (1.27), suppose that*

$$V(\mathbb{Z}^d) \cap [v_0, \infty) \neq \emptyset. \tag{1.30}$$

Then, there is a spectral gap at $r(P_V)$, i. e., (1.8) holds. Moreover,

$$\text{there exists a strictly positive, normalized function } \varphi \in \ell^2 \text{ such that } \mathrm{Ker}(r(P_V) - P_V) = \mathbb{C}\varphi. \tag{1.31}$$

In the following corollary to Theorem 1.2.4, we improve the spectral gap (1.8) to the absolute spectral gap (1.9). An operator $T \in \mathcal{B}(\ell^2)$ is said to be *bipartite* w.r.t. $J \in \{-1,1\}^{\mathbb{Z}^d}$ if $T(x,y) = 0$ for all $(x,y) \in (\mathbb{Z}^d)^2$ such that $J(x)J(y) = 1$.

Corollary 1.2.5 *Suppose that (1.8) and (1.31) hold true, which is the case if (1.24), (1.27) and (1.30) are satisfied. Then:*

a) *There is an absolute spectral gap at $r(P_V)$, i. e. (1.9) holds if and only if*

$$-r(P_V) < \ell(P_V) \quad \text{or } P \text{ is bipartite w.r.t. some } J \in \{-1,1\}^{\mathbb{Z}^d}. \tag{1.32}$$

b) *Assume (1.32). Then, there exists $\varepsilon \in (0,1)$ such that*

$$\|r(P_V)^{-n}P_V^n(f - \Pi_V f)\|_V \le \varepsilon^n \|f - \Pi_V f\|_V \quad \text{for all } f \in \ell^2 \text{ and } n \in \mathbb{N}, \tag{1.33}$$

where, with φ from (1.31),

$$\Pi_V f \stackrel{\text{def}}{=} \begin{cases} \langle f, \varphi \rangle_V \varphi, & \text{if } -r(P_V) < \ell(P_V), \\ \langle f, \varphi \rangle_V \varphi + \langle f, J\varphi \rangle_V J\varphi, & \text{if } P \text{ is bipartite w.r.t. } J \in \{-1,1\}^{\mathbb{Z}^d}, \end{cases} \tag{1.34}$$

Finally, we provide a sufficient condition in terms of the transition probability p for the condition (1.32).

Proposition 1.2.6 *Let $A = \{x \in \mathbb{Z}^d \, ; \, \sum_{\alpha \in I} x_\alpha \in 2\mathbb{Z}\}$ with $\emptyset \neq I \subset \{1,\ldots,d\}$. Suppose that either p vanishes on A, or*

$$p(0) > \sum_{x \in A \setminus \{0\}} p(x). \tag{1.35}$$

Then, the condition (1.32) is satisfied for any nonnegative $V \in \ell^\infty(\mathbb{Z}^d)$. More precisely,

If p vanishes on A, then, P is bipartite w.r.t. $J = \mathbf{1}_A - \mathbf{1}_{A^c}$.
If (1.35) holds, then, $-r(P_V) < \ell(P_V)$ for any nonnegative $V \in \ell^\infty(\mathbb{Z}^d)$.

1.3 Limit of the random walk in potential V

We introduce a random walk $(S_n)_{n\in\mathbb{N}}$ on a probability space $(\Omega, \mathcal{F}, P)$ such that $S_0 = 0$, $P(S_1 = \cdot) = p$. We define a probability measure μ_N on $(\Omega, \mathcal{F})$ by (1.11).

We assume (1.24) and (1.27) in what follows. We will have a positively recurrent Markov chain as the limit process of μ_N. To describe the Markov chain obtained as the limit, we introduce an operator $P_{V,\varphi} \in \mathcal{B}(\ell^2)$ by

$$P_{V,\varphi}(x,y) = r(P_V)^{-1}\varphi(x)^{-1}P_V(x,y)\varphi(y), \tag{1.36}$$

where φ is from Theorem 1.2.4. Then, $P_{V,\varphi}$ is a transition probability for a Markov chain in $\mathbb{Z}^d$, which we denote by $(\{S_n\}_{n\in\mathbb{N}}, \{\nu^x\}_{x\in\mathbb{Z}^d})$. It is easy to check that the Markov chain $\{\nu^x\}_{x\in\mathbb{Z}^d}$ is reversible with respect to the probability measure m on $\mathbb{Z}^d$ defined by

$$\langle f, m \rangle = \frac{\langle f, \varphi^2 \rangle_V}{\langle \mathbf{1}, \varphi^2 \rangle_V}, \quad f \in \ell^\infty. \tag{1.37}$$

In particular, $\{\nu^x\}_{x\in\mathbb{Z}}$ is positively recurrent.

Theorem 1.3.1 *Assume the same hypothesis as Corollary 1.2.5. Then, there are constants* $C = C(p, V) > 0$ *and* $\varepsilon = \varepsilon(p, V) \in (0, 1)$ *as follows; if* $n \geq k + C$, $f : \mathbb{Z}^k \to \mathbb{R}$ *is polynomialy bounded and* $F(\omega) = f(S_1(\omega), \ldots, S_k(\omega))$, $\omega \in \Omega$, *then*

$$\left| \int_\Omega F d\mu_n - \int_\Omega F d\nu \right| \leq B(f)\varepsilon^{n-k}, \tag{1.38}$$

where $B(f)$ *is a constant which depends only on* p, V *and* f.

Proof: Given Theorem 1.2.3 and Corollary 1.2.5, the proof of Theorem 1.3.1 is identical to that of [3, Theorem 1.3], hence is omitted. □

1.4 An example in dimension one

We provide a simple example for $d = 1$. We define $p : \mathbb{Z} \to [0, 1)$ such that $p(0) = q \in [0, 1)$ $p(1) = p(-1) = (1-q)/2 \in (0, 1/2]$. Then, for $\lambda \in \mathbb{R}\backslash[2q-1, 1]$, the function $g_\lambda(x)$ is computed explicitly.

$$g_\lambda(x) = \frac{\lambda}{\sqrt{\delta(\lambda)}} \times \begin{cases} \varphi(\lambda)^{|x|}, & \text{if } \lambda > 1, \\ -\varphi(\lambda)^{-|x|}, & \text{if } \lambda < 2q - 1, \end{cases} \tag{1.39}$$

where $\delta(\lambda) = (\lambda - 1)(\lambda - (2q - 1)) > 0$ and

$$\varphi(\lambda) = \frac{\lambda - q - \sqrt{\delta(\lambda)}}{1 - q} \in \begin{cases} (0, 1), & \text{if } \lambda > 1, \\ (-\infty, -1), & \text{if } \lambda < 2q - 1. \end{cases}$$

- $g_\lambda(0)$ is strictly decreasing in $\lambda \in (1, \infty)$, $g_\lambda(0) \stackrel{\lambda \searrow 1}{\longrightarrow} \infty$, $g_\lambda(0) \stackrel{\lambda \nearrow \infty}{\longrightarrow} 1$. (1.40)
- if $q = 0$, then $g_\lambda(0)$ is strictly increasing in $\lambda \in (-\infty, -1)$, $g_\lambda(0) \stackrel{\lambda \searrow -\infty}{\longrightarrow} 1$, $g_\lambda(0) \stackrel{\lambda \nearrow -1}{\longrightarrow} \infty$. (1.41)
- if $0 < q < 1/2$, then, $g_\lambda(0)$ is strictly decreasing in $\lambda \in (-\infty, \frac{2q-1}{q})$, strictly increasing in $\lambda \in (\frac{2q-1}{q}, 2q - 1)$, $g_\lambda(0) \stackrel{\lambda \searrow -\infty}{\longrightarrow} 1$, $g_{\frac{2q-1}{q}}(0) = \frac{\sqrt{1-2q}}{1-q}$, $g_{\frac{2q-1}{2q}}(0) = 1$, $g_\lambda(0) \stackrel{\lambda \nearrow 2q-1}{\longrightarrow} \infty$, (1.42)
- if $1/2 \leq q < 1$, then, $g_\lambda(0)$ is strictly decreasing in $\lambda \in (-\infty, 2q - 1)$, $g_\lambda(0) \stackrel{\lambda \searrow -\infty}{\longrightarrow} 1$, $g_0(0) = 0$, and if $1/2 < q < 1$, $g_\lambda(0) \stackrel{\lambda \nearrow 2q-1}{\longrightarrow} -\infty$. (1.43)

For $v > 0$, we set

$$\lambda_\pm(v) \stackrel{\text{def}}{=} c(v) \left(q \pm \sqrt{q^2 - (2q - 1)c(v)^{-1}} \right), \quad \text{where } c(v) \stackrel{\text{def}}{=} \frac{(v + 1)^2}{2v + 1}. \tag{1.44}$$

Suppose that $V : \mathbb{Z} \to [0, \infty)$ is a bounded function which satisfies (1.24).

$$\Lambda_V = \begin{cases} \{\lambda_\pm(v) \ ; \ v \in \sigma_{\text{ess}}(V)\backslash\{0\}\}, & \text{if } 0 \leq q < 1/2, \\ \{\lambda_+(v) \ ; \ v \in \sigma_{\text{ess}}(V)\backslash\{0\}\}, & \text{if } 1/2 \leq q < 1. \end{cases} \tag{1.45}$$

To see this, we observe that for $v > 0$ and $\lambda \in \mathbb{R}\backslash[2q-1, 1]$,

$$\begin{aligned} g_\lambda(0) = 1 + v^{-1} \quad &\Longrightarrow \quad \frac{\lambda^2}{\delta(\lambda)} = (1+v^{-1})^2 \\ &\Longleftrightarrow \quad \lambda^2 - 2qc(v)\lambda + (2q-1)c(v) = 0 \\ &\Longleftrightarrow \quad \lambda = \lambda_\pm(v). \end{aligned}$$

Moreover,

$$\lambda_-(v) < 2q - 1 < 1 < \lambda_+(v) = \lambda_-(v) + 2c(v)\sqrt{q^2 - (2q-1)c(v)^{-1}}. \tag{1.46}$$

Taking (1.40)–(1.46) into account (In particular, if $0 < q < 1/2$, then, $\frac{2q-1}{2q} < \lambda_-(v) < 2q - 1$ and hence $g_\lambda(0) > 1$ at $\lambda = \lambda_-(v)$ by (1.42)), we have

$$\lambda \in \mathbb{R}\backslash[2q-1, 1], \; g_\lambda(0) = 1 + v^{-1} \iff \lambda = \begin{cases} \lambda_\pm(v), & \text{if } 0 \le q < 1/2, \\ \lambda_+(v), & \text{if } 1/2 \le q < 1. \end{cases}$$

This, together with Theorem 1.2.1, implies (1.45).

2 Proof of Theorem 1.2.1 and Corollary 1.2.2

2.1 Outline

Proof of Theorem 1.2.1 *Step1* We prove that $\sigma_{\text{ess}}(P_V)\backslash\sigma(P) = \Lambda_V$. Here, we adapt the method in [4] to the present setting. Let $\lambda \in \mathbb{R}\backslash\sigma(P)$. Then, We introduce the modified Birman-Schwinger operator $G_{V,\lambda} \in \mathcal{B}(\ell^2)$ by

$$G_{V,\lambda} = V^{1/2}(\lambda G_\lambda - 1)V^{1/2}. \tag{2.1}$$

We first show in Lemma 2.2.3 that

$$\lambda \in \sigma_{\text{ess}}(P_V) \iff 1 \in \sigma_{\text{ess}}(G_{V,\lambda}). \tag{2.2}$$

We then prove in Lemma 2.2.4 that

$$G_{V,\lambda} = (g_\lambda(0) - 1)V + H_{V,\lambda}, \tag{2.3}$$

where $H_{V,\lambda}$ is a compact operator. In fact, this is where the condition (1.24) is used. Then, we see from (2.3) that

$$\sigma_{\text{ess}}(G_{V,\lambda}) = (g_\lambda(0) - 1)\sigma_{\text{ess}}(V), \tag{2.4}$$

via Weyl's essential spectrum theorem, cf. [2, p.358, Proposition 4.2 (e)], [6, p.112, Theorem XIII.14]. By (2.4),

$$1 \in \sigma_{\text{ess}}(G_{V,\lambda}) \iff \exists v \in \sigma_{\text{ess}}(V)\backslash\{0\}, \; g_\lambda(0) - 1 = v^{-1} \iff \lambda \in \Lambda_V. \tag{2.5}$$

Thus, we obtain (1.25) from (2.2) and (2.5).
Step2 We prove that $\sigma(P) \subset \sigma_{\text{ess}}(P_V)$. This step is the subject of Lemma 2.2.6 below. □

Proof of Corollary 1.2.2: a) Suppose that the condition (1.27) holds. Then, $\lambda_0 \in \Lambda_V \cap (1, \infty)$. Since $\sigma_{\text{ess}}(V) \cap (1, \infty) = \Lambda_V \cap (1, \infty)$ by Theorem 1.2.1, it follows that $\sigma_{\text{ess}}(V) \cap (1, \infty) \neq$

$\emptyset$. Moreover, by (2.6), $g_\lambda(0)$ is strictly decreasing in $\lambda \in (1,\infty)$. Therefore. (1.27) implies that $\lambda_0 = \max \sigma_{\rm ess}(P_V)$.
Suppose on the other hand that $\sigma_{\rm ess}(V) \cap (1,\infty) \neq \emptyset$. Then, since $\sigma_{\rm ess}(V) \cap (1,\infty) = \Lambda_V \cap (1,\infty)$ by Theorem 1.2.1, there exist $\lambda \in (1,\infty)$ and $v \in \sigma_{\rm ess}(V)\backslash\{0\}$ such that $g_\lambda(0) = 1 + v^{-1}$. Then, $v_0 \geq v > 0$. Moreover, since $0 < v_0^{-1} \leq v^{-1}$ and by (2.6), $g_\lambda(0) \stackrel{\lambda\to\infty}{\longrightarrow} 1$, it follows from the mean value theorem that there exists $\lambda_0 \in (1,\infty)$ such that $g_{\lambda_0}(0) = 1 + v_0^{-1}$.
b) The proof is similar as above. □

2.2 Lemmas

Properties of the function $\lambda \mapsto g_\lambda(0)$, which we need in this article are summarized in the following

Lemma 2.2.1

- *$g_\lambda(0)$ is strictly decreasing in $\lambda \in (1,\infty)$, $g_\lambda(0) \stackrel{\lambda\to\infty}{\longrightarrow} 1$.*
  $$\text{Moreover, } g_{1+}(0) \begin{cases} = \infty, & \text{if } d \leq 2, \\ \in (0,\infty), & \text{if } d \geq 3. \end{cases} \tag{2.6}$$
- *if $\ell(P) < 0$, then, $g_\lambda(0) > \frac{|\lambda|}{|\lambda|+p(0)}$ for $\lambda \in (-\infty, \ell(P))$.*
  $$g_\lambda(0) \stackrel{\lambda\to-\infty}{\longrightarrow} 1, \text{ Moreover, } g_{\ell(P)-}(0) = \infty \text{ if } d \leq 2, \tag{2.7}$$
- *if $\ell(P) \geq 0$, then, $g_\lambda(0)$ is strictly decreasing in $\lambda \in (-\infty, \ell(P))$,*
  $$g_\lambda(0) \in \begin{cases} (0,1), & \text{for } \lambda \in (-\infty, 0), \\ (-\infty, 0), & \text{for } \lambda \in (0, \ell(P)). \end{cases} \tag{2.8}$$

Proof: The behavior (2.6) of $g_\lambda(0)$ for $\lambda > 1$ is well-known in the context of the random walk. Thus, we omit the proofs.

On the other hand, the following integral formula is well-known.

$$g_\lambda(0) = \frac{\lambda}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{d\theta}{\lambda - \widehat{p}(\theta)}, \quad \lambda \in \mathbb{R}\backslash\sigma(P). \tag{2.9}$$

We derive properties (2.6)–(2.8) from this formula. We take (2.7) for example. Since λ is negative,

$$\begin{aligned} g_\lambda(0) &= \frac{|\lambda|}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{d\theta}{|\lambda| + \widehat{p}(\theta)} \\ &> \frac{|\lambda|}{|\lambda| + (2\pi)^{-d} \int_{[-\pi,\pi]^d} \widehat{p}(\theta) d\theta} = \frac{|\lambda|}{|\lambda| + p(0)}, \end{aligned}$$

where we have used Jensen inequality to the convex function $x \mapsto 1/x$ $(x > 0)$ to proceed from the first line to the second.

To show that $g_{\ell(P)-}(0) = \infty$ if $d \leq 2$, we take $\theta_0 \in [-\pi,\pi]^d$ such that $\ell(P) = \widehat{p}(\theta_0)$. Since $\frac{\partial \widehat{p}}{\partial \theta_\alpha}(\theta_0) = 0$ for all $\alpha = 1, \ldots, d$, we have

$$\begin{aligned} \widehat{p}(\theta_0 + \theta) - \ell(P) &= \widehat{p}(\theta_0 + \theta) - \widehat{p}(\theta_0) \\ &= \frac{1}{2} \sum_{\alpha,\beta=1}^d \theta_\alpha \theta_\beta \int_0^1 (1-t) \frac{\partial^2 \widehat{p}}{\partial \theta_\alpha \partial \theta_\beta}(\theta_0 + t\theta) dt \\ &\leq C|\theta|^2 \quad \text{for } \theta \in [-\pi,\pi]^d. \end{aligned}$$

By applying the monotone convergence theorem to the integral $\int_{[-\pi,\pi]^d} \frac{d\theta}{\widehat{p}(\theta)-\lambda}$ as $\lambda \nearrow \ell(P)$, we have

$$
\begin{aligned}
g_{\ell(P)-}(0) &= \frac{|\ell(P)|}{(2\pi)^d}\int_{[-\pi,\pi]^d} \frac{d\theta}{\widehat{p}(\theta)-\ell(P)} = \frac{|\ell(P)|}{(2\pi)^d}\int_{[-\pi,\pi]^d} \frac{d\theta}{\widehat{p}(\theta_0+\theta)-\ell(P)} \\
&\geq \frac{|\ell(P)|}{(2\pi)^d C}\int_{[-\pi,\pi]^d} \frac{d\theta}{|\theta|^2} = \infty.
\end{aligned}
$$

□

Remark If $\ell(P) < 0$, $g_\lambda(0)$ is not necessarily monotone in $\lambda \in (-\infty, \ell(P))$. See (1.42) for a counterexample.

Lemma 2.2.2 *Let $A, B \in \mathcal{B}(X)$ on a Banach space X. Then,*

a) $1 \in \sigma_{\mathrm{ess}}(BA)$ *if and only if* $1 \in \sigma_{\mathrm{ess}}(AB)$.

b) $1 \in \rho(BA)$ *if and only if* $1 \in \rho(AB)$. *Moreover, if these conditions hold true, then*

$$(1 - BA)^{-1} = 1 + B(1 - AB)^{-1}A. \tag{2.10}$$

Proof: a) Since the roles of A and B are exchangeable, it is enough to verify only the "if" part, which is equivalently stated as $1 - AB \in \mathrm{F}(X) \Rightarrow 1 - BA \in \mathrm{F}(X)$. To prove this, we use Atkinson' s theorem which says the following for $T \in \mathcal{B}(X)$, cf. [1, p.161, Theorem 4.46].

A1) Suppose that $T \in \mathrm{F}(X)$ with $n_1 = \dim \mathrm{Ker} T$ and $n_2 = \dim(X/\mathrm{Ran} T)$. Then, there exist $S, K_1, K_2 \in \mathcal{B}(X)$ with $\mathrm{rank} K_j = n_j$ $(j = 1, 2)$ such that

$$ST = 1 + K_1 \text{ and } TS = 1 + K_2. \tag{2.11}$$

A2) Conversely, suppose that there exist $S, K_1, K_2 \in \mathcal{B}(X)$ of which K_1 and K_2 are compact such that (2.11) holds true. Then, $T \in \mathrm{F}(X)$.

Comming back to the proof of the lemma, suppose that $T \stackrel{\mathrm{def}}{=} 1 - AB \in \mathrm{F}(X)$. Then, by Atkinson's theorem, there exist $S, K_1, K_2 \in \mathcal{B}(X)$ as are stated in A1) above. Then, $1+BSA \in \mathcal{B}(X)$, $\mathrm{rank} BK_jA \leq n_j$ $(j = 1, 2)$ and

$$(1 + BSA)(1 - BA) = 1 + BK_1A, \ \ (1 - BA)(1 + BSA) = 1 + BK_2A. \tag{2.12}$$

Therefore, $1 - BA \in \mathrm{F}(X)$, by A2) above.
b) The equivalence of $1 \in \rho(BA)$ and $1 \in \rho(AB)$ can be regarded as a special case of part a), where $K_1 = K_2 = 0$ in the proof above. Moreover, suppose that $1 \in \rho(BA)$, or equivalently, $1 \in \rho(AB)$. Then, $S = (1 - AB)^{-1}$ in (2.12). Thus, the equality (2.10) follows from (2.12). □

Lemma 2.2.3 *Let $\lambda \in \mathbb{R}\backslash\sigma(P)$, and $G_{V,\lambda}$ be the operator defined by (2.1). Then,*

a) $\lambda \in \sigma_{\mathrm{ess}}(P_V)$ *if and only if* $1 \in \sigma_{\mathrm{ess}}(G_{V,\lambda})$.

b) $\lambda \in \sigma(P_V)$ *if and only if* $1 \in \sigma(G_{V,\lambda})$. *Moreover, if $\lambda \in \rho(P_V)$, or equivalently, $1 \in \rho(G_{V,\lambda})$, then,*

$$(\lambda - P_V)^{-1} = G_\lambda + G_\lambda V^{1/2}(1 - G_{V,\lambda})^{-1}V^{1/2}PG_\lambda. \tag{2.13}$$

Proof: a) We need to verify the equivalence. $\lambda - P_V \in F(\ell^2) \iff 1 - G_{V,\lambda} \in F(\ell^2)$. We decompose this task into the following two steps.

$$\lambda - P_V \in F(\ell^2) \iff 1 - G_\lambda PV \in F(\ell^2), \tag{2.14}$$
$$1 - G_\lambda PV \in F(\ell^2) \iff 1 - G_{V,\lambda} \in F(\ell^2). \tag{2.15}$$

Note that

$$\lambda - P_V = \lambda - P - VP = (\lambda - P)(1 - G_\lambda VP). \tag{2.16}$$

or equivalently,

$$1 - G_\lambda VP = G_\lambda(\lambda - P_V). \tag{2.17}$$

Note that $\lambda - P, G_\lambda \in F(\ell^2)$. Recall also that $F(\ell^2)$ is closed under the composition cf. [1, p.158, Theorem 4.43]. Then, (2.14) follows from (2.16) and (2.17).

The equivalence (2.15) follows from Lemma 2.2.2 a), since for $A \stackrel{\text{def}}{=} G_\lambda V^{1/2}$ and $B \stackrel{\text{def}}{=} V^{1/2}P$,

$$\begin{aligned} AB &= G_\lambda V^{1/2}V^{1/2}P = G_\lambda VP, \\ BA &= V^{1/2}PG_\lambda V^{1/2} = V^{1/2}(\lambda G_\lambda - 1)V^{1/2} = G_{V,\lambda}. \end{aligned}$$

b) The proof is similar as above. We use Lemma 2.2.2 b) instead of Lemma 2.2.2 a). Suppose that $\lambda \in \rho(P_V)$. Then, it follows from (2.17) that

$$(\lambda - P_V)^{-1} = (1 - G_\lambda VP)^{-1}G_\lambda.$$

On the other hand, by plugging $A = V^{1/2}P$ and $B = G_\lambda V^{1/2}$ into (2.10),

$$\begin{aligned} (1 - G_\lambda VP)^{-1} &= 1 + G_\lambda V^{1/2}(1 - V^{1/2}PG_\lambda V^{1/2})^{-1}V^{1/2}P \\ &= 1 + G_\lambda V^{1/2}(1 - G_{V,\lambda})^{-1}V^{1/2}P. \end{aligned}$$

Combining these, we obtain (2.13). □

Lemma 2.2.4 *With $g_\lambda(0)$, $G_{V,\lambda}$ defined respectively by (1.17) and (2.1), the following operator $H_{V,\lambda} \in \mathcal{B}(\ell^2)$ is compact.*

$$H_{V,\lambda} \stackrel{\text{def}}{=} G_{V,\lambda} - (g_\lambda(0) - 1)V.$$

Proof: We decompose $G_{V,\lambda}$ into diagonal and off diagonal components as follows.

$$G_{V,\lambda} = V^{1/2}(\lambda G_\lambda - 1)V^{1/2} = (g_\lambda(0) - 1)V + H_{V,\lambda},$$

where the operator $H_{V,\lambda}$ is given by the kernel

$$H_{V,\lambda}(x,y) = \lambda\sqrt{V(x)V(y)}G_\lambda(x,y)\mathbf{1}_{x\neq y}.$$

We will show that

$$\|H_{V,\lambda} - H^{(N)}_{V,\lambda}\| \stackrel{N\to\infty}{\longrightarrow} 0, \tag{2.18}$$

where K_N is a finite rank operator defined by the kernel

$$H^{(N)}_{V,\lambda}(x,y) = \lambda\sqrt{V(x)V(y)}G_\lambda(x,y)\mathbf{1}_{x\neq y,|x|\leq N}.$$

Thus, (2.18) shows that $H_{V,\lambda}$ is a compact operator.

By a standard estimate,

$$\|H_{V,\lambda} - H_{V,\lambda}^{(N)}\| \le |\lambda|\sqrt{A_N B_N},$$

where

$$A_N = \sup_{|x|\ge N} \sum_{\substack{y\in\mathbb{Z}^d \\ x\ne y}} \sqrt{V(x)V(y)}|G_\lambda(x,y)|, \quad B_N = \sup_{x\in\mathbb{Z}^d} \sum_{\substack{y\in\mathbb{Z}^d \\ y\ne x, |y|\ge N}} \sqrt{V(x)V(y)}|G_\lambda(x,y)|.$$

By (1.14), there exist $C = C(\lambda) \in (0,\infty)$ and $\varepsilon = \varepsilon(\lambda) \in (0,\infty)$ such that

$$|G_\lambda(x,y)| \le C\exp(-\varepsilon|x-y|). \tag{2.19}$$

Thus, (2.18) follows from (1.24) and

$$\sup_{x\in\mathbb{Z}^d} b_{N,\varepsilon}(x) \overset{N\to\infty}{\longrightarrow} 0, \text{ where } b_{N,\varepsilon}(x) = \sum_{\substack{y\in\mathbb{Z}^d \\ y\ne x, |y|>N}} \sqrt{V(x)V(y)}\exp(-\varepsilon|x-y|). \tag{2.20}$$

Therefore, it is enough to show that (1.24) implies (2.20). Moreover, it follows immediately from (1.24) that $\sup_{|x|>\lfloor N/2\rfloor} b_{N,\varepsilon}(x) \overset{N\to\infty}{\longrightarrow} 0$. Hence, it is enough to verify that

$$\sup_{|x|\le\lfloor N/2\rfloor} b_{N,\varepsilon}(x) \overset{N\to\infty}{\longrightarrow} 0. \tag{2.21}$$

If $|x| \le \lfloor N/2\rfloor$ and $|y| \ge N$, then $|y-x| \ge N/2$. Thus,

$$\begin{aligned} \sup_{|x|\le\lfloor N/2\rfloor} b_{N,\varepsilon}(x) &\le \exp(-\varepsilon N/2)\|V\|_\infty \sup_{|x|\le\lfloor N/2\rfloor} \sum_{y\in\mathbb{Z}^d} \exp(-\varepsilon|x-y|/2) \\ &= \exp(-\varepsilon N/2)\|V\|_\infty \sum_{y\in\mathbb{Z}^d} \exp(-\varepsilon|y|/2) \overset{N\to\infty}{\longrightarrow} 0, \end{aligned}$$

which proves (2.21). □

In what follows, we use the following notation. For $x = (x_\alpha)_{\alpha=1}^d \in \mathbb{Z}^d$,

$$|x|_\infty = \max_{1\le\alpha\le d} |x_\alpha|. \tag{2.22}$$

For $c \in \mathbb{Z}^d$ and a positive integer ℓ,

$$Q(c,\ell) = \{x \in \mathbb{Z}^d \ ; \ |x-c|_\infty \le \ell\}. \tag{2.23}$$

Suppose that the condition (1.24) holds true and that $v_0 > 0$. Then the following lemma shows that there are infinitely many disjoint cubes $Q(c,\ell)$ in which V takes the value close to v_0 at the center c, while the value $V(x)$ for the other points of the cube are close to zero.

Lemma 2.2.5 *Suppose that the condition (1.24) holds true and that $v_0 > 0$. Then, for any $L, \ell \in (0,\infty)$ and $\varepsilon \in (0,1)$, there exists $c \in \mathbb{Z}^d$ such that*

$$Q(0,L) \cap Q(c,\ell) = \emptyset, \tag{2.24}$$

$$V(c) > (1-\varepsilon)v_0, \tag{2.25}$$

$$\sum_{x\in Q(c,\ell)\setminus\{c\}} V(x) < \varepsilon. \tag{2.26}$$

Proof: Take $\delta \in (0,1)$ such that $\delta^2 \exp(2\ell) < \varepsilon(1-\varepsilon)v_0$. By assumption $v_0 > 0$, the set

$$H_\varepsilon = \{x \in \mathbb{Z}^d \,;\, V(x) > (1-\varepsilon)v_0\}$$

is unbounded. Therefore, by (1.24), we can find $c \in H_\varepsilon$ such that

$$|c| > L + \ell, \ \text{ and } \ A(c) \stackrel{\text{def}}{=} \sum_{\substack{x\in\mathbb{Z}^d \\ x\neq c}} \sqrt{V(c)V(x)} \exp(-|c-x|_\infty) < \delta. \tag{2.27}$$

The first inequality of (2.27) is equivalent to (2.24), while the second inequality implies (2.26) as follows.

$$\begin{aligned} &\sum_{x\in Q(c,\ell)\setminus\{c\}} V(x) \\ &\le \left(\sum_{x\in Q(c,\ell)\setminus\{c\}} \sqrt{V(x)} \right)^2 \stackrel{(2.25)}{\le} A(c)^2 \frac{\exp(2\ell)}{(1-\varepsilon)v_0} \stackrel{(2.27)}{<} \delta^2 \frac{\exp(2\ell)}{(1-\varepsilon)v_0} < \varepsilon. \end{aligned}$$

□

Lemma 2.2.6 *Suppose that the condition (1.24) holds true. Then, $\sigma(P) \subset \sigma_{\text{ess}}(P_V)$.*

Proof: Let $\lambda \in \sigma(P)$ be arbitrary. We will prove that $\lambda - P_V \notin \mathrm{F}(\ell^2)$, or equivalently, $\lambda - P_V^* \notin \mathrm{F}(\ell^2)$, where P_V^* is the adjoint operator of P_V on ℓ^2, therefore, $P_V^* = P + PV$.

For this purpose, we will use the following criterion for an operator T on a Hilbert space X not to be a Fredholm operator. $T \notin \mathrm{F}(X)$ if there exists a normalized sequence $\{u_n\} \subset X$ such that

$$u_n \stackrel{n\to\infty}{\longrightarrow} 0 \text{ weakly and } Tu_n \stackrel{n\to\infty}{\longrightarrow} 0 \text{ strongly,} \tag{2.28}$$

The converse is also true if T is self-adjoint, cf. [2, p. 350, Theorem 2.3]. A sequence with above property is called a *Weyl sequence*.

We will construct a Weyl sequence $u_n \in \ell^2$ for $\lambda - P_V^*$, that is, $u_n \in \ell^2$ is normalized, weakly convergent to zero, and

$$\|(\lambda - P - PV)u_n\| \stackrel{n\to\infty}{\longrightarrow} 0. \tag{2.29}$$

Step1: Let $r > 0$ be an integer such that $\text{supp}[p] \subset Q(0,r)$, We first show that, there exist $c_n \in \mathbb{Z}^d$, $n \in \mathbb{N}$ such that

$$Q(c_m, m+r) \cap Q(c_n, n+r) = \emptyset \quad \text{if } m \neq n, \tag{2.30}$$

$$\max_{x\in Q(c_n,n+r)} V(x) < (n+2)^{-1} \ \text{ for all } n \in \mathbb{N}. \tag{2.31}$$

Suppose that $v_0 = 0$. Then, for any $\varepsilon > 0$, there exists $R > 0$ such that $V(x) < \varepsilon$ if $|x| > R$. Thus, it is easy to find such c_n by an inductive procedure.

Suppose on the contrary that $v_0 > 0$. We then proceed inductively with the help of Lemma 2.2.5 as follows. We start by taking $L = 1$, $\ell = 3r$ and $\varepsilon = 1/2$ in Lemma 2.2.5, so that we can find $b_0 \in \mathbb{Z}^d$ such that

$$\max_{x\in Q(b_0,3r)\setminus\{b_0\}} V(x) < 1/2.$$

Thus, by choosing c_0 such that $Q(c_0, r) \subset Q(b_0, 3r)\backslash\{b_0\}$, we obtain (2.31) for $n = 0$. Next, we take $L = |c_0| + 1 + r$, $\ell = 3 + 3r$ and $\varepsilon = 1/3$ in Lemma 2.2.5, so that we can find $b_1 \in \mathbb{Z}^d$ such that

$$Q(c_0, r) \cap Q(b_1, 3 + 3r) = \emptyset, \quad \text{and} \quad \max_{x \in Q(b_1, 3+3r)\backslash\{b_1\}} V(x) < 1/3.$$

Thus, by choosing c_1 such that $Q(c_1, 1 + r) \subset Q(b_1, 3 + 3r)\backslash\{b_1\}$, we obtain (2.31) for $n = 1$. By repeating this procedure, we obtain c_n, $n \in \mathbb{N}$ as desired.

Step2: We now construct the Weyl sequence u_n, $n \in \mathbb{N}$. Let $\theta \in [-\pi, \pi]^d$ be such that $\lambda = \widehat{p}(\theta)$ and set $e_\lambda(x) = \exp(\mathbf{i}\theta \cdot x)$. Then by Step1, there exist $c_n \in \mathbb{Z}^d$, $n \in \mathbb{N}$ which satisfy (2.30) and (2.31). We set

$$\varphi_n = e_\lambda \mathbf{1}_{Q(c_n, n+r)} \text{ and } u_n = \varphi_n / \|\varphi_n\|. \tag{2.32}$$

We will show that u_n, $n \in \mathbb{N}$ is the Weyl sequence which we look for. By (2.30), u_n, $n \in \mathbb{N}$ are orthonormal. Moreover, we see from (2.31) that

$$0 \leq V \leq (n + 2)^{-1} \mathbf{1}_{Q(c_n, n+r)},$$

and hence $\|V u_n\| \leq (n + 2)^{-1} \stackrel{n\to\infty}{\longrightarrow} 0$. Thus, it only remains to verify that

$$\|(\lambda - P)u_n\| \stackrel{n\to\infty}{\longrightarrow} 0. \tag{2.33}$$

To see this, we observe that

$$(\lambda - P)e_\lambda = 0. \tag{2.34}$$

Note also that for $f, g : \mathbb{Z}^d \to \mathbb{C}$ and $x \in \mathbb{R}^d$,

$$f = g \text{ on } Q(x, r) \implies Pf(x) = Pg(x).$$

This, together with (2.32) and (2.34), implies that

$$(\lambda - P)\varphi_n = 0 \text{ on } Q(c_n, n). \tag{2.35}$$

If we set $h_n \stackrel{\text{def}}{=} \mathbf{1}_{Q(c_n, n+r)\backslash Q(c_n, n)}$, then

$$\|h_n\|^2 = \sharp(Q(c_n, n + r)\backslash Q(c_n, n)) = (2n + 2r + 1)^d - (2n + 1)^d = O(n^{d-1}),$$

and hence

$$\|(\lambda - P)\varphi_n\| \stackrel{(2.35)}{=} \|h_n(\lambda - P)\varphi_n\| \leq \|h_n\| \|(\lambda - P)\varphi_n\|_\infty = O(n^{(d-1)/2}). \tag{2.36}$$

This implies (2.33), since $\|\varphi_n\| = \|\mathbf{1}_{Q(c_n, n+r)}\| = (2n + 2r + 1)^{d/2}$. □

3 Proof of Theorem 1.2.3

3.1 Outline

For $\alpha > 0$, we denote by $\ell^{\infty,\alpha}$ the Banach space of exponentially decaying functions $u : \mathbb{Z}^d \to \mathbb{C}$ with the exponent at least α, more precisely, the functions which satisfy

$$\|u\|_{\infty,\alpha} \stackrel{\text{def}}{=} \sup_{x \in \mathbb{Z}^d} \exp(\alpha|x|)|u(x)| < \infty. \tag{3.1}$$

We also recall the estimate (2.19) on the exponential decay of the kernel $G_\lambda(x,y)$.

Suppose that the condition (1.24) holds true. Let $\lambda \in \mathbb{C}\backslash\sigma_{\rm ess}(P_V)$ and $\alpha \in (0,\varepsilon)$, where $\varepsilon > 0$ is from (2.19). For $K \subset \mathbb{Z}^d$, we define

$$V_K = V\mathbf{1}_{\mathbb{Z}^d\backslash K}. \tag{3.2}$$

In the sequel, we write $K \Subset \mathbb{Z}^d$, when K is a finite subset of $\mathbb{Z}^d$. We will prove in Lemma 3.2.4 below that there exists $K \Subset \mathbb{Z}^d$ such that $\lambda \in \rho(P_{V_K})$ and $(\lambda - P_{V_K})^{-1} \in \mathcal{B}(\ell^{\infty,\alpha})$.

Suppose additionally that $\lambda \in \sigma(P_V)$ and that a function $\varphi \in \ell^2$ satisfies $(\lambda - P_V)\varphi = 0$. Then, with the set K from Lemma 3.2.4, we rewrite $(\lambda - P_V)\varphi = 0$ as

$$(\lambda - P_{V_K})\varphi = \mathbf{1}_K VP\varphi.$$

Since $\lambda \in \rho(P_{V_K})$ by Lemma 3.2.4, it follows from the above display that

$$\varphi = (\lambda - P_{V_K})^{-1}\mathbf{1}_K VP\varphi. \tag{3.3}$$

Since the function $\mathbf{1}_K VP\varphi$ is supported on the finite set K and $(\lambda - P_{V_K})^{-1} \in \mathcal{B}(\ell^{\infty,\alpha})$ by Lemma 3.2.4, we obtain (1.29) from (3.3). □

3.2 Lemmas

Let $\lambda \in \mathbb{C}\backslash\sigma_{\rm ess}(P_V)$ and $\alpha \in (0,\varepsilon)$, where $\varepsilon > 0$ is from (2.19). As is discussed in section 3.1, it is enough to prove that there exists $K \Subset \mathbb{Z}^d$ such that $\lambda \in \rho(P_{V_K})$ and $(\lambda - P_{V_K})^{-1} \in \mathcal{B}(\ell^{\infty,\alpha})$ (Lemma 3.2.4). We will implement this by dealing with the modified Birman-Schwinger operator with the potential V_K:

$$G_{V_K,\lambda} = V_K^{1/2}(\lambda G_\lambda - 1)V_K^{1/2},$$

cf. (2.1). As in the proof of Lemma 2.2.4, we decompose $G_{V_K,\lambda}$ into diagonal and off diagonal components as follows.

$$G_{V_K,\lambda} = \gamma V_K + H_{V_K,\lambda}, \quad \text{where} \quad \gamma = g_\lambda(0) - 1. \tag{3.4}$$

We first look at the diagonal component γV_K of the above decomposition.

Lemma 3.2.1 *Suppose that $\lambda \notin \sigma(P) \cup \Lambda_V$. Then, there exists $K_0 \Subset \mathbb{Z}^d$ such that if $K_0 \subset K \Subset \mathbb{Z}^d$,*

$$\varepsilon_0 \stackrel{\rm def}{=} \inf_{x\in\mathbb{Z}^d} |1 - \gamma V_K(x)| > 0. \tag{3.5}$$

Proof: It follows from the assumption $\lambda \notin \Lambda_V$ that either

$$\textbf{i)}\ \gamma = 0 \ \text{ or } \ \textbf{ii)}\ \gamma \neq 0 \text{ and } \gamma^{-1} \notin \sigma_{\rm ess}(V).$$

If $\gamma = 0$, then, (3.5) is clearly true with $K = \emptyset$ and $\varepsilon_0 = 1$. Suppose ii) above. Then, there exists $\delta > 0$ such that

$$K_0 \stackrel{\rm def}{=} \{x \in \mathbb{Z}^d \ ; \ |1 - \gamma V(x)| < \gamma\delta\} = \{x \in \mathbb{Z}^d \ ; \ |\gamma^{-1} - V(x)| < \delta\} \Subset \mathbb{Z}^d.$$

Let $K_0 \subset K \Subset \mathbb{Z}^d$. Then, it is clear that

$$|1 - \gamma V_K(x)| = |1 - \gamma V(x)| \geq \gamma\delta \ \text{ for all } x \in \mathbb{Z}^d\backslash K,$$

which implies (3.5). □

The following lemma deals with the off diagonal part $H_{V_K,\lambda}$ of the operator $G_{V_K,\lambda}$, cf. (3.4) which is given by the kernel.

$$H_{V_K,\lambda}(x,y) = \lambda\sqrt{V_K(x)V_K(y)}G_\lambda(x,y)\mathbf{1}_{x\neq y}. \tag{3.6}$$

Lemma 3.2.2 *Suppose that the condition (1.24) holds true. Then, for all $\alpha \in (0,\varepsilon)$ and $\beta \in (0,\infty)$, where ε is from (2.19), there exists $K_1 \Subset \mathbb{Z}^d$ such that if $K_1 \subset K \Subset \mathbb{Z}^d$, then*

$$\|H_{V_K,\lambda}\|_{\mathcal{B}(\ell^2)} \leq \beta, \tag{3.7}$$

$$H_{V_K,\lambda} \in \mathcal{B}(\ell^{\infty,\alpha}) \;\; \textit{with} \;\; \|H_{V_K,\lambda}\|_{\mathcal{B}(\ell^{\infty,\alpha})} \leq \beta. \tag{3.8}$$

Proof: Set $\delta \stackrel{\text{def}}{=} \varepsilon - \alpha > 0$. Then, by (1.24), there exists a $K_1 \Subset \mathbb{Z}^d$ such that

$$\sup_{x\in\mathbb{Z}^d\setminus K_1} \sum_{\substack{y\in\mathbb{Z}^d \\ y\neq x}} \sqrt{V(x)V(y)}\exp(-\delta|x-y|) \leq \frac{\beta}{|\lambda|C}, \tag{3.9}$$

where the constant C is from (2.19). Let $K_1 \subset K \Subset \mathbb{Z}^d$. By (2.19) and (3.6),

$$H_{V_K,\lambda}(x,y) \leq |\lambda|C\sqrt{V_K(x)V_K(y)}\exp(-\varepsilon|x-y|)\mathbf{1}_{x\neq y}. \tag{3.10}$$

Since $H_{V_K,\lambda} : \ell^2 \to \ell^2$ is symmetric, we have by a standard estimate that

$$\begin{aligned} \|H_{V_K,\lambda}\| &\leq \sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} |H_{V_K,\lambda}(x,y)| \\ &\stackrel{(3.10)}{\leq} |\lambda|C \sup_{x\in\mathbb{Z}^d\setminus K} \sum_{\substack{y\in\mathbb{Z}^d \\ y\neq x}} \sqrt{V(x)V(y)}\exp(-\varepsilon|x-y|) \stackrel{(3.9)}{\leq} \beta, \end{aligned}$$

which shows (3.7).

Suppose that $u \in \ell^{\infty,\alpha}$ and $x \in \mathbb{Z}^d$. Then, noting that $|x| \leq |x-y| + |y|$,

$$\begin{aligned} &\exp(\alpha|x|)|(H_{V_K,\lambda}u)(x)| \\ &\leq \exp(\alpha|x|)\sum_{y\in\mathbb{Z}^d} |H_{V_K,\lambda}(x,y)||u(y)| \\ &\stackrel{(3.10)}{\leq} \mathbf{1}_{\mathbb{Z}^d\setminus K}(x)|\lambda|C\sum_{\substack{y\in\mathbb{Z}^d \\ y\neq x}} \sqrt{V(x)V(y)}\exp(-\delta|x-y|)\exp(\alpha|y|)|u(y)| \\ &\leq \mathbf{1}_{\mathbb{Z}^d\setminus K}(x)|\lambda|C\|u\|_{\infty,\alpha}\sum_{\substack{y\in\mathbb{Z}^d \\ y\neq x}} \sqrt{V(x)V(y)}\exp(-\delta|x-y|) \stackrel{(3.9)}{\leq} \beta\|u\|_{\infty,\alpha}, \end{aligned}$$

which shows (3.8). □

Combining Lemma 3.2.1 and Lemma 3.2.2, we obtain the following

Lemma 3.2.3 *Suppose that (1.24) holds true. Then, for $\lambda \in \mathbb{C}\setminus\sigma_{\text{ess}}(P_V)$ and $\alpha \in (0,\varepsilon)$, where ε is from (2.19), there exists $K \Subset \mathbb{Z}^d$ such that $1 \in \rho(G_{V_K,\lambda})$ and that $(1-G_{V_K,\lambda})^{-1} \in \mathcal{B}(\ell^{\infty,\alpha})$.*

Proof: We have $\lambda \notin \sigma_{\rm ess}(P_V) = \sigma(P) \cup \Lambda_V$, by the choice of λ and Theorem 1.2.1. Thus, we may apply Lemma 3.2.1 and take $\varepsilon_0 \in (0,1]$ and $K_0 \Subset \mathbb{Z}^d$ from there. We then apply Lemma 3.2.2 with $\beta < \varepsilon_0$. As a consequence, we can find $K_i \Subset \mathbb{Z}^d$ $(i = 0,1)$ with which (3.5), (3.7) and (3.8) hold true, where $\beta < \varepsilon_0$. We now suppose that $K_0 \cup K_1 \subset K \Subset \mathbb{Z}^d$. Then, it follows from (3.5) that the operator $1 - \gamma V_K$ has its inverse in $\mathcal{B}(\ell^2)$ with the norm at most $1/\varepsilon_0$. Then, setting

$$R_K = (1 - \gamma V_K)^{-1} H_{V_K,\lambda}$$

for simplicity, we have

$$\|R_K\|_{\mathcal{B}(\ell^2)} \le \|(1-\gamma V_K)^{-1}\|_{\mathcal{B}(\ell^2)} \|H_{V_K,\lambda}\|_{\mathcal{B}(\ell^2)} \le \varepsilon_0^{-1} \cdot \beta < 1, \tag{3.11}$$

and therefore, the following Neumann series converges:

$$(1 - R_K)^{-1} = \sum_{n=0}^{\infty} R_K^n. \tag{3.12}$$

Since

$$1 - G_{V_K,\lambda} = 1 - \gamma V_K - H_{V_K,\lambda} = (1 - \gamma V_K)(1 - R_K),$$

the operator $1 - G_{V_K,\lambda}$ is invertible in $\mathcal{B}(\ell^2)$, and herefore, $1 \in \rho(G_{V_K,\lambda})$.

By repeating the same argument as above, with $\mathcal{B}(\ell^2)$ replaced by $\mathcal{B}(\ell^{\infty,\alpha})$, we see that $(1 - G_{V_K,\lambda})^{-1} \in \mathcal{B}(\ell^{\infty,\alpha})$. Indeed, it is clear that $1 - \gamma V_K$ has the inverse in $\mathcal{B}(\ell^{\infty,\alpha})$ with the norm at most ε_0^{-1}. Moreover, by (3.8), $\|H_{V_K,\lambda}\|_{\mathcal{B}(\ell^{\infty,\alpha})} \le \beta < \varepsilon_0$. Consequently, we see that $\|R_K\|_{\mathcal{B}(\ell^{\infty,\alpha})} < 1$ and hence that the Neumann series (3.12) converges in $\mathcal{B}(\ell^{\infty,\alpha})$. This implies that $(1 - G_{V_K,\lambda})^{-1} \in \mathcal{B}(\ell^{\infty,\alpha})$. □

As is discussed in section 3.1, Theorem 1.2.3 follows from the following

Lemma 3.2.4 *Suppose that (1.24) holds true. Then, for $\lambda \in \mathbb{C}\backslash\sigma_{\rm ess}(P_V)$ and $\alpha \in (0,\varepsilon)$ where ε is from (2.19), there exists $K \Subset \mathbb{Z}^d$ such that $\lambda \in \rho(P_{V_K})$ and that $(\lambda - P_{V_K})^{-1} \in \mathcal{B}(\ell^{\infty,\alpha})$.*

Proof: The first half of the lemma follows from Lemma 3.2.2, since, $\lambda \in \rho(P_{V_K})$ if and only if $1 \in \rho(G_{V_K,\lambda})$ by Lemma 2.2.3. As for the latter half, by applying (2.13) with V there replaced by V_K, we have

$$(\lambda - P_{V_K})^{-1} = G_\lambda + G_\lambda V_K^{1/2}(1 - G_{V_K,\lambda})^{-1} V_K^{1/2} P G_\lambda.$$

It is clear that $V_K^{1/2}$ and P belong to $\mathcal{B}(\ell^{\infty,\alpha})$. It is also clear from the proof of Lemma 3.2.2 that $G_\lambda \in \mathcal{B}(\ell^{\infty,\alpha})$. Moreover, $(1 - G_{V_K,\lambda})^{-1} \in \mathcal{B}(\ell^{\infty,\alpha})$ by Lemma 3.2.3. Combining these, we conclude that $(\lambda - P_{V_K})^{-1} \in \mathcal{B}(\ell^{\infty,\alpha})$. □

4 Proof of Theorem 1.2.4, Corollary 1.2.5 and Proposition 1.2.6

4.1 Outline

Proof of Theorem 1.2.4: We first prove that

$$1 \le \dim \mathrm{Ker}(r(P_V) - P_V) < \infty \ \text{ and } \ r(P_V) \notin \overline{\sigma(P_V)\backslash\{r(P_V)\}}. \tag{4.1}$$

For this purpose, we prove via a "lower bound by a delta potential" (Lemma 4.2.1) that there exists $u_0 \in \ell^2$ such that

$$\|u_0\|_V = 1 \text{ and } \langle P_V u_0, u_0 \rangle_V > \max \sigma_{\rm ess}(P_V), \tag{4.2}$$

which implies (4.1) as follows. It follows from (4.2) that

$$r(P_V) = \sup_{\|u\|_V=1} \langle P_V u, u \rangle_V \geq \langle P_V u_0, u_0 \rangle_V > \max \sigma_{\rm ess}(P_V).$$

Thus, $r(P_V) \notin \sigma_{\rm ess}(P_V)$, or equivalently,

$$\dim(r(P_V) - P_V) < \infty \ \text{ and } \ r(P_V) \notin \overline{\sigma(P_V) \backslash \{r(P_V)\}},$$

(cf. (1.19) and (1.20)). Since $r(P_V) \in \sigma(P_V)$, $r(P_V)$ is an eigenvalue and therefore, we obtain (4.1).

Now that we have established (4.1), the rest of the proposition follows immediately from an extension of the Perron-Frobenius theorem to infinite dimensional setting (See Lemma 4.2.3 below). Indeed,

$$P_V(x,y) = (1+V(x))P(x,y) \geq P(x,y) \ \text{ for all } x, y \in \mathbb{Z}^d,$$

which shows that P_V is positive and irreducible. Moreover, by Lemma 4.2.2 below, the operator norm of $P_V : \ell^2_V \to \ell^2_V$ equals to $r(P_V)$. □

Proof of Corollary 1.2.5 P_V is positive and irreducible. Moreover, P_V is bipartite w.r.t. $J \in \{-1,1\}^{\mathbb{Z}^d}$ if and only if P is bipartite w.r.t. J. We can pass from Theorem 1.2.4 to Corollary 1.2.5 by applying Lemma 4.2.4 below to $T = P_V$. □

Proof of Proposition 1.2.6: *Case1*: Suppose that p vanishes on A. We prove that P is bipartite w.r.t. $J = \mathbf{1}_A - \mathbf{1}_{A^c}$. Indeed, it follows from assumptions that supp $[p] \subset A^c$, and therefore by definition of A that $(x,y) \notin A^2 \cup (A^c)^2$ if $y - x \in$ supp $[p]$. This implies that P is bipartite w.r.t. $J = \mathbf{1}_A - \mathbf{1}_{A^c}$.

Case2: Suppose that (1.35) holds. We prove that $-r(P_V) < \ell(P_V)$ for any nonnegative $V \in \ell^\infty(\mathbb{Z}^d)$. For this purpose, we use the following inequality, cf. Lemma 4.2.5 below.

$$-r(P_V) + 2\ell(\mathbf{1}_A P \mathbf{1}_A) \leq \ell(P_V). \tag{4.3}$$

On the other hand, we have

$$\ell(\mathbf{1}_A P \mathbf{1}_A) = \min_{\theta \in [-\pi,\pi]^d} \sum_{x \in A} p(x) \exp(\mathbf{i}\theta \cdot x) \geq p(0) - \sum_{x \in A \backslash \{0\}} p(x) > 0.$$

Then, it follows from (4.3) that $-r(P_V) < \ell(P_V)$. □

4.2 Lemmas

We start by explaining the "lower bound by a delta potential" referred to in the proof of Theorem 1.2.4.

Lemma 4.2.1 a) *Suppose that $v_0 > 0$. Suppose also that $\lambda \in (1,\infty)$, $v > 0$, $g_\lambda(0) = 1 + v^{-1}$, and $V(\mathbb{Z}^d) \cap [v, \infty) \neq \emptyset$. Then, there exists a $u_\lambda \in \ell^2$ such that*

$$\|u_\lambda\|_V = 1, \text{ and } \langle P_V u_\lambda, u_\lambda \rangle_V > \lambda. \tag{4.4}$$

b) *Under hypothesis of Theorem 1.2.4, there exists a $u_0 \in \ell^2$ such that (4.2) holds.*

Proof: a) We take $c \in \mathbb{Z}^d$ such that $V(c) \geq v$. Then, by (1.15), $\varphi_\lambda(x) \stackrel{\text{def}}{=} G_\lambda(x,c) > 0$ for all $x \in \mathbb{Z}^d$. We will show that the function $u_\lambda \stackrel{\text{def}}{=} \varphi_\lambda/\|\varphi_\lambda\|_V$ satisfies (4.4). To see this, we make an auxiliary use of the potential $U = v\delta_c$ to give a lower bound for V. We first observe that

$$P_U u_\lambda = \lambda u_\lambda. \tag{4.5}$$

Indeed, noting the equality $PG_\lambda = \lambda G_\lambda - 1$, we have for all $x \in \mathbb{Z}^d$ that

$$\begin{aligned} P_U\varphi_\lambda(x) &= (1 + v\delta_c(x))PG_\lambda\delta_c(x) \\ &= (1 + v\delta_c(x))(\lambda G_\lambda(x,c) - \delta_c(x)) \\ &= \lambda G_\lambda(x,c) - \delta_c(x) + v(\lambda G_\lambda(c,c) - 1)\delta_c(x) \\ &= \lambda G_\lambda(x,c) = \lambda\varphi_\lambda(x), \end{aligned}$$

which implies (4.5).

Now, we use (4.5) to see (4.4) as follows. It follows from $v \leq V(c)$ that $U(x) \leq V(x)$ for all $x \in \mathbb{Z}^d$. Moreover, since $v_0 > 0$, $V(x) > 0$ for infinitely many x's, and hence $U(x) < V(x)$ for at least an x (in fact, for infinitely many x's) . These, together with the strict positivity of u_λ implies that

$$1 = \|u_\lambda\|_V = \|(1+V)^{-1/2}u_\lambda\| < \|(1+U)^{-1/2}u_\lambda\| = \|u_\lambda\|_U. \tag{4.6}$$

Moreover, since $\langle\, Pu_\lambda, u_\lambda \,\rangle > 0$,

$$\langle\, P_V u_\lambda, u_\lambda \,\rangle_V = \langle\, Pu_\lambda, u_\lambda \,\rangle \stackrel{(4.6)}{>} \frac{\langle\, Pu_\lambda, u_\lambda \,\rangle}{\|u_\lambda\|_U} = \frac{\langle\, P_U u_\lambda, u_\lambda \,\rangle_U}{\|u_\lambda\|_U} \stackrel{(4.5)}{=} \lambda.$$

b) Here, by Corollary 1.2.2, $\lambda_0 = \max \sigma_{\text{ess}}(P_V)$ and v_0 are related as

$$g_{\lambda_0}(0) = 1 + v_0^{-1}.$$

Therefore, part b) of the present lemma follows from part a). □

We present a series of lemmas in the following more general framework. Let $(S, \mathcal{A}, m)$ be a σ-finite measure space. We denote the norm and the the inner product of the Hilbert space $L^2(m)$ resectively by $\|\cdot\|$ and $\langle\, \cdot, \cdot \,\rangle$. The totality of $f \in L^2(m)$ such that $f \geq 0$, m-a.e. is denoted by $L^2_+(m)$. An operator $T \in \mathcal{B}(L^2(m))$ is said to be *positive* if $Tf \in L^2_+(m)$ for all $f \in L^2_+(m)$.

Lemma 4.2.2 *Suppose that $T \in \mathcal{B}(L^2(m))$ is positive. Then, for all $f \in L^2(m)$,*

$$T|f| - |Tf| \in L^2_+(m), \quad \text{and thus,} \quad |\langle\, f, Tf \,\rangle| \leq \langle\, |f|, T|f| \,\rangle. \tag{4.7}$$

In particular, if T is positive and self-adjoint, then,

$$|\ell(T)| \leq r(T) = \|T\|.$$

Proof: Let $f \in L^2(m)$ be arbitrary. Since T maps a real-valued function to a real-valued function, we have

$$T(\operatorname{Re} f) = \operatorname{Re}(Tf). \tag{4.8}$$

Moreover, for each $z \in \mathbb{C}$,

$$|z| = \sup_{\theta \in \mathbb{R}} \mathrm{Re}(\exp(\mathbf{i}\theta)z) = \sup_{\theta \in \mathbb{Q}} \mathrm{Re}(\exp(\mathbf{i}\theta)z). \tag{4.9}$$

Let $x \in S$ and $\theta \in \mathbb{Q}$ be fixed. Then, by applying (4.9) to $z = f(x)$, and to $z = Tf(x)$, we see that

$$\begin{aligned} g_\theta(x) &\stackrel{\text{def}}{=} |f(x)| - \mathrm{Re}(\exp(\mathbf{i}\theta)f(x)) \stackrel{(4.9)}{\geq} 0, & (4.10) \\ T|f|(x) - |Tf(x)| &\stackrel{(4.9)}{=} T|f|(x) - \sup_{\theta \in \mathbb{Q}} \mathrm{Re}(\exp(\mathbf{i}\theta)Tf(x)) \\ &= \inf_{\theta \in \mathbb{Q}} (T|f|(x) - \mathrm{Re}(\exp(\mathbf{i}\theta)Tf(x))) \\ &\stackrel{(4.8)}{=} \inf_{\theta \in \mathbb{Q}} Tg_\theta(x). & (4.11) \end{aligned}$$

By (4.10), we have $Tg_\theta \in L^2_+(m)$. Then, combining this with (4.11), we have $T|f| - |Tf| \in L^2_+(m)$. In particular, if T is positive and self-adjoint, then,

$$-\langle f, Tf \rangle \stackrel{(4.7)}{\leq} \langle |f|, T|f| \rangle \leq r(T)\|f\|^2.$$

Taking the supremum of the left-hand side of the above inequality over all normalized functions f in $L^2(m)$, we obtain $-\ell(T) \leq r(T)$, which, together with the obvious inequality $\ell(T) \leq r(T)$, implies that $|\ell(T)| \leq r(T)$. As a consequence, we obtain the equality $r(T) = \|T\|$. □

Let $T \in \mathcal{B}(L^2(m))$. T is said to be *irreducible* if, for all functions $f, g \in L^2_+(m)$ which are not identically zero m-a.e., there exists $n \in \mathbb{N}$ such that $\langle f, T^n g \rangle > 0$. If there exists a measurable function $J : S \to \{-1, 1\}$ such that $JT = -TJ$, i.e., $JTf = -T(Jf)$ for all $f \in L^2(m)$, then, T is said to be *bipartite* and J is called the *sign* of T. T is simply said to be bipartite if T is bipartite w.r.t. some sign J.

We consider the following conditions for an operator $T \in \mathcal{B}(L^2(m))$, in connection with its irreducibility and bipartiteness.

$$\mathrm{Ker}(\|T\| + T) \neq \{0\}. \tag{4.12}$$

$$\mathrm{Ker}(\|T\| - T) \neq \{0\}. \tag{4.13}$$

There exists a normalized function $\varphi \in L^2(m)$ such that

$$\varphi > 0 \text{ a.e. and } \mathrm{Ker}(\|T\| - T) = \mathbb{C}\varphi. \tag{4.14}$$

Lemma 4.2.3 *Suppose that $T \in \mathcal{B}(L^2(m))$ is positive. Then,*

$$(4.14) \iff (4.13) \text{ and } T \text{ is irreducible.} \tag{4.15}$$

Suppose in addition that T is irreducible. Then,

$$(4.12) \iff (4.14) \text{ and that } T \text{ is bipartite.} \tag{4.16}$$

Moreover, the converse part of the above equivalence entails the following relation. If T is bipartite w.r.t. the sign J, then, (4.12) holds with

$$\mathrm{Ker}(\|T\| + T) = \mathbb{C}J\varphi. \tag{4.17}$$

Proof: Let us make a preliminary observation which applies to all $T \in \mathcal{B}(L^2(m))$. Suppose that $u \in L^2(m)$ is normalized and that $\langle u, Tu \rangle = \sigma \|T\|$, where σ is either 1 or -1. Then,

$$Tu = T^*u = \sigma \|T\| u. \tag{4.18}$$

Indeed, since $\langle u, Tu \rangle = \sigma \|T\| \in \mathbb{R}$,

$$\|\sigma \|T\| u - Tu\|^2 = \|T\|^2 - 2\sigma \|T\| \langle u, Tu \rangle + \|Tu\|^2 = -\|T\|^2 + \|Tu\|^2 \le 0.$$

Thus, $Tu = \sigma \|T\| u$. On the other hand, $\langle u, Tu \rangle = \sigma \|T\|$ implies that $\langle u, T^*u \rangle = \sigma \|T^*\|$. Hence by letting T^* play the role of T above, we obtain $T^*u = \sigma \|T\| u$ as well.
(4.15): This equivalence is due to [6, p.202, Theorem XIII.43].
(4.16) ($\Rightarrow$): Suppose that $\psi \in L^2(m)$ is a nonzero function which satisfies the equation $T\psi = -\|T\|\psi$. Since the same equation is satisfied by the real and imaginary parts of ψ, we may assume that ψ is real-valued. Moreover, by normalization, we may assume that $\|\psi\| = 1$. Therefore, we see from the preliminary observation (4.18) that

$$T\psi = T^*\psi = -\|T\|\psi. \tag{4.19}$$

On the other hand, applying (4.7) to $f = \psi$, we obtain

$$\|T\| = -\langle \psi, T\psi \rangle \le \langle |\psi|, T|\psi| \rangle \le \|T\|.$$

and hence, $\langle |\psi|, T|\psi| \rangle = \|T\|$. Therefore, we see from the preliminary observation (4.18) that $T|\psi| = \|T\||\psi|$, Thus, we have proved (4.13), which, together with the irreducibility of T, implies (4.14). Then, it follows from (4.14) that $|\psi| = \varphi$, and by applying (4.18), we have

$$T\varphi = T^*\varphi = \|T\|\varphi. \tag{4.20}$$

Since $|\psi| = \varphi$, the function $J \stackrel{\text{def}}{=} \psi/\varphi$ is defined a.e. and takes values in $\{-1, 1\}$ there. We will prove that $T = -JTJ$. By linearlity, it is enough to prove that $g \stackrel{\text{def}}{=} Tf + JTJf = 0$ a.e. for all $f \in L^2_+(m)$. Since $f \pm Jf \in L^2_+(m)$, we have $Tf \pm TJf \in L^2_+(m)$, and hence $g \in L^2_+(m)$. Therefore, it is enough to show that $\langle \varphi, g \rangle = 0$, which can be done by noting that $\varphi J = \psi$, and that $\psi J = \varphi$ as follows.

$$\begin{aligned}\langle \varphi, g \rangle &= \langle \varphi, Tf \rangle + \langle \psi, TJf \rangle = \langle T^*\varphi, f \rangle + \langle T^*\psi, Jf \rangle \\ &\stackrel{(4.19),\ (4.20)}{=} \|T\|\langle \varphi, f \rangle - \|T\|\langle \psi, Jf \rangle = \|T\|\langle \varphi, f \rangle - \|T\|\langle \varphi, f \rangle = 0.\end{aligned}$$

(4.16) ($\Leftarrow$): By the relation $TJ = -JT$, the eigenspaces $\mathrm{Ker}(\|T\| \pm T)$ are mapped to each other bijectively by J. Thus, (4.14) implies (4.17). □

Lemma 4.2.4 *Suppose that an operator $T \in \mathcal{B}(L^2(m))$ is positive, self-adjoint, and satisfies (4.14) and*

$$r(T) > \sup\{\lambda \,;\ \lambda \in \sigma(T),\ \lambda \neq r(T)\}. \tag{4.21}$$

Then,

a) *The following conditions are equivalent.*

$$r(T) > \sup\{|\lambda| \ ; \ \lambda \in \sigma(T), \ |\lambda| \neq r(T)\}. \tag{4.22}$$

$$\textit{Either } -r(T) < \ell(T) \textit{ or } T \textit{ is bipartite.} \tag{4.23}$$

b) *Suppose that one of the conditions (4.22) and (4.23) holds true (therefore that both do). Then, there exists $\varepsilon \in (0,1)$ such that*

$$\|r(T)^{-n}T^n(f - \Pi f)\| \leq \varepsilon^n \|f - \Pi f\| \quad \textit{for all } f \in L^2(m) \textit{ and } n \in \mathbb{N}, \tag{4.24}$$

where, with φ from (4.14),

$$\Pi f \stackrel{\rm def}{=} \begin{cases} \langle f, \varphi \rangle \varphi, & \textit{if } -r(T) < \ell(T), \\ \langle f, \varphi \rangle \varphi + \langle f, J\varphi \rangle J\varphi, & \textit{if } T \textit{ is bipartite w.r.t. the sign } J. \end{cases} \tag{4.25}$$

Proof: a) (4.22) $\Rightarrow$ (4.23): Suppose that (4.23) fails. Then, it follows from Lemma 4.2.3 a) that $\ell(T) = -r(T)$ is not an eigenvalue, and therefore by (1.20), $-r(T) \in \overline{\sigma(T)\backslash\{-r(T)\}}$. Hence (4.22) fails.
(4.23) $\Rightarrow$ (4.22): If $-r(T) < \ell(T)$, then (4.22) holds, since

$$\sigma(T) \cap [-r(T), \ell(T)) = \emptyset. \tag{4.26}$$

On the other hand, if T is bipartite, then, (4.22) follows from (4.21) and the symmety of $\sigma(T)$ with respect to the origin.
b) Let $T = \int_{\sigma(T)} \lambda dE_\lambda$ denote the spectral decomposition of the self-adjoint operator T on $\ell^2(m)$. We first verify that

$$\Pi = \int_{\substack{\sigma(T) \\ |\lambda| = r(T)}} dE_\lambda. \tag{4.27}$$

We treat each of the cases in condition (4.23) separately. Let us temporarily denote the orthogonal projection on the right-hand side of (4.27) by $\Pi^{\rm RHS}$.

Case1: Suppose that $-r(T) < \ell(T)$. Then, it follows from (4.26) that

$$\{\lambda \in \sigma(T) \ ; \ |\lambda| = r(T)\} = \{r(T)\}.$$

We see from this and (4.14) that

$${\rm Ran}(\Pi^{\rm RHS}) = {\rm Ker}(r(T) - T) = \mathbb{C}\varphi,$$

which implies (4.27).

Case2: Suppoes that T is bipartite the sign J. Then, it follows from the symmetry of $\sigma(T)$ with respect to the origin that

$$\{\lambda \in \sigma(T) \ ; \ |\lambda| = r(T)\} = \{\pm r(T)\}.$$

Putting this, (4.14) and (4.17) together, we have

$${\rm Ran}(\Pi^{\rm RHS}) = {\rm Ker}(r(T) - T) \oplus {\rm Ker}(-r(T) - T) = \mathbb{C}\varphi \oplus \mathbb{C}J\varphi,$$

which implies (4.27).

Now, (4.24) follows easily from condition (4.22). Let

$$r = \sup\{|\lambda| \ ; \ \lambda \in \sigma(T), \ |\lambda| \neq r(T)\}$$

and let $f^\perp = f - \Pi f$. Then, it follows from (4.27) that

$$\|T^n f^\perp\|_V^2 = \int_{\substack{\sigma(T) \\ |\lambda| \le r}} \lambda^{2n} d\langle E_\lambda f^\perp, f^\perp \rangle \le r^{2n} \|f^\perp\|^2,$$

which proves (4.24). □

Lemma 4.2.5 *Suppose that $p : \mathbb{Z}^d \to [0, \infty)$ is a transition probability of a symmetric random walk, and that $A = \{x \in \mathbb{Z}^d \ ; \ \sum_{\alpha \in I} x_\alpha \in 2\mathbb{Z}\}$ with $\emptyset \neq I \subset \{1, \ldots, d\}$. Then,*

$$-r(P_V) + 2\ell(\mathbf{1}_A P \mathbf{1}_A) \le \ell(P_V). \tag{4.28}$$

Proof: We already know from Lemma 4.2.2 that $-r(P_V) \le \ell(P_V)$. Thus, we may assume that $\ell(\mathbf{1}_A P \mathbf{1}_A) \ge 0$. Since

$$-\ell(P_V) = \sup\{-\langle P_V u, u \rangle_V \ ; \ \|u\|_V = 1\},$$

it is enough to show that

$$-\langle P_V u, u \rangle_V \le (-2\ell(\mathbf{1}_A P \mathbf{1}_A) + r(P_V))\|u\|_V^2 \ \text{ for all } u \in \ell^2. \tag{4.29}$$

To see this, let $u \in \ell^2$ be arbitrary and $J = \mathbf{1}_A - \mathbf{1}_{A^c}$. We first verify that

$$-\langle Pu, u \rangle \le -2\ell(\mathbf{1}_A P \mathbf{1}_A)\|u\|_2^2 + \langle PJu, Ju \rangle. \tag{4.30}$$

Let $P_+ = \mathbf{1}_A P \mathbf{1}_A + \mathbf{1}_{A^c} P \mathbf{1}_{A^c}$ and $P_- = \mathbf{1}_A P \mathbf{1}_{A^c} + \mathbf{1}_{A^c} P \mathbf{1}_A$. Then,

$$P_\pm J = \pm J P_\pm, \tag{4.31}$$

as is easily be seen. Moreover, for fixed $c \in A^c$, the map $x \mapsto x + c$ $(A \longrightarrow A^c)$ is a bijection. Hence

$$\begin{aligned} \langle P_+ u, u \rangle &= \left(\sum_{x,y \in A} + \sum_{x,y \notin A} \right) p(y-x) u(x) u(y)^* \\ &= \sum_{x,y \in A} p(y-x) u(x) u(y)^* \\ &\quad + \sum_{x',y' \in A} p(y'-x') u(x'+c) u(y'+c)^* \\ &\ge \ell(\mathbf{1}_A P \mathbf{1}_A) \sum_{x \in A} \left(|u(x)|^2 + |u(x+c)|^2 \right) = \ell(\mathbf{1}_A P \mathbf{1}_A) \|u\|^2. \end{aligned} \tag{4.32}$$

Thus, we obtain (4.30) as follows.

$$\begin{aligned} \langle PJu, Ju \rangle &= \langle P_+ Ju, Ju \rangle + \langle P_- Ju, Ju \rangle \\ &\overset{(4.31)}{=} \langle JP_+ u, Ju \rangle - \langle JP_- u, Ju \rangle \\ &= \langle P_+ u, u \rangle - \langle P_- u, u \rangle = 2\langle P_+ u, u \rangle - \langle Pu, u \rangle \\ &\overset{(4.32)}{\ge} 2\ell(\mathbf{1}_A P \mathbf{1}_A)\|u\|_2^2 - \langle Pu, u \rangle. \end{aligned}$$

We then use (4.30) to prove (4.29) as follows. Note that $\ell(\mathbf{1}_A P \mathbf{1}_A)\|u\|^2 \geq \ell(\mathbf{1}_A P \mathbf{1}_A)\|u\|_V^2$, since $\|u\| \geq \|u\|_V$ and we have assumed that $\ell(\mathbf{1}_A P \mathbf{1}_A) \geq 0$. Therefore,

$$\begin{aligned} -\langle P_V u, u \rangle_V &= -\langle Pu, u \rangle \\ &\stackrel{(4.30)}{\leq} -2\ell(\mathbf{1}_A P \mathbf{1}_A)\|u\|^2 + \langle PJu, Ju \rangle \\ &= -2\ell(\mathbf{1}_A P \mathbf{1}_A)\|u\|^2 + \langle P_V Ju, Ju \rangle_V \\ &\leq (-2\ell(\mathbf{1}_A P \mathbf{1}_A) + r(P_V))\|u\|_V^2. \end{aligned}$$

□

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