Phase Transitions for the Growth Rate of Linear Stochastic Evolutions[1]

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Abstract

We consider a discrete-time stochastic growth model on d-dimensional lattice. The growth model describes various interesting examples such as oriented site/bond percolation, directed polymers in random environment, time discretizations of binary contact path process and the voter model. We study the phase transition for the growth rate of the "total number of particles" in this framework. The main results are roughly as follows: If $d \geq 3$ and the system is "not too random", then, with positive probability, the growth rate of the total number of particles is of the same order as its expectation. If on the other hand, $d = 1, 2$, or the system is "random enough", then the growth rate is slower than its expectation. We also discuss the above phase transition for the dual processes and its connection to the structure of invariant measures for the model with proper normalization.

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Contents

1 Introduction

We write $\mathbb{N} = \{0, 1, 2, ...\}$, $\mathbb{N}^* = \{1, 2, ...\}$ and $\mathbb{Z} = \{\pm x \; ; \; x \in \mathbb{N}\}$. For $x = (x_1, .., x_d) \in \mathbb{R}^d$, $|x|$ stands for the ℓ^1-norm: $|x| = \sum_{i=1}^d |x_i|$. For $\xi = (\xi_x)_{x \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$, $|\xi| = \sum_{x \in \mathbb{Z}^d} |\xi_x|$. Let $(\Omega, \mathcal{F}, P)$ be a probability space. We write $P[X] = \int X \, dP$ and $P[X : A] = \int_A X \, dP$ for a random variable X and an event A.

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1.1 The oriented site percolation (OSP)

We start by discussing the *oriented site percolation* as a motivating example. Let $\eta_{t,y}$, $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be $\{0,1\}$-valued i.i.d. random variables with $P(\eta_{t,y} = 1) = p \in (0,1)$. The site (t,y) with $\eta_{t,y} = 1$ and $\eta_{t,y} = 0$ are referred to as respectively as *open* and *closed*. An *open oriented path* from $(0,0)$ to $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ is a sequence $\{(s,x_s)\}_{s=0}^t$ in $\mathbb{N} \times \mathbb{Z}^d$ such that $x_0 = 0$, $x_t = y$, $|x_s - x_{s-1}| = 1$, $\eta_{s,x_s} = 1$ for all $s = 1,..,t$. A common physical interpretation of OSP is the percolation of water through porus rock. Due to gravity, the water flows only downwards and it is blocked at some locations inside the rock. A variant of OSP is also used to explain the formation of galaxies, where a site (t,x) being open is interpreted as the birth of a star at time-space (t,x) [14].

For oriented site percolation, it is traditional to discuss the presence/absence of the open oriented paths to certain time-space location. On the other hand, we will see that the model exhibits a new type of phase transition, if we look at not only the presence/absence of the open oriented paths, but also their number. Let $N_{t,y}$ be the number of open oriented paths from $(0,0)$ to (t,y) and let $|N_t| = \sum_{y\in\mathbb{Z}^d} N_{t,y}$ be the total number of the open oriented paths from $(0,0)$ to the "level" t. Then, $|\overline{N}_t| \stackrel{\text{def.}}{=} (2dp)^{-t}|N_t|$ is a martingale (Each open oriented path from $(0,0)$ to (t,y) branches and survives to the next level via $2d$ neighbors of y, each of which is open with probability p). Thus, by the martingale convergence theorem the following limit exists almost surely:

$$|\overline{N}_\infty| \stackrel{\text{def}}{=} \lim_{t\to\infty} |\overline{N}_t|$$

As applications of results in this paper, we see the following phase transition.

i) If $d \geq 3$ and p is large enough, then, $|\overline{N}_\infty| > 0$ with positive probability.

ii) For $d = 1,2$, $|\overline{N}_\infty| = 0$, almost surely for all $p \in (0,1)$. Moreover, the convergence is exponentially fast for $d = 1$.

This phase transition was predicted by T. Shiga in late 1990's. The proof however, seems to have been open since then.

We note that $N_{t,y}$ is obtained by successive multiplications of i.i.d. random matrices. Let $A_t = (A_{t,x,y})_{x,y\in\mathbb{Z}^d}$, $t \in \mathbb{N}^*$, where $A_{t,x,y} = \mathbf{1}_{\{|x-y|=1\}}\eta_{t,y}$. Then,

$$\sum_{x\in\mathbb{Z}^d} N_{t-1,x}A_{t,x,y} = N_{t,y}, \quad t \in \mathbb{N}^*. \tag{1.1}$$

We also prove the following phase transition in terms of the invariant measure for the Markov process $\overline{N}_t \stackrel{\text{def.}}{=} ((2dp)^{-t}N_{t,y})_{y\in\mathbb{Z}^d}$. Note that we can take any $N_0 \in [0,\infty)^{\mathbb{Z}^d}$ as the initial state of $(\overline{N}_t)$ via (1.1).

iii) Suppose that $d \geq 3$ and p is large enough. Then, for each $\alpha \in (0,\infty)$, $(\overline{N}_t)$ has an invariant distribution ν_α, which is also invariant with respect to the lattice shift, such that $\int_{[0,\infty)^{\mathbb{Z}^d}} \eta_0 d\nu_\alpha(\eta) = \alpha$.

iv) Suppose that $d = 1,2$ and $p \in (0,1)$ is arbitrary. Then, the only shift-invariant, invariant distribution ν for $(\overline{N}_t)$ such that $\int_{[0,\infty)^{\mathbb{Z}^d}} \eta_0 d\nu(\eta) < \infty$ is the trivial one, that is the point mass at all zero configuration.

We will discuss the above phase transitions i)–iv) in a more general framework.

In this paper, we point out that many other models beside OSP have similar random matrix representations to (1.1), and that the phase transitions i)–iv) are universal for these models.

1.2 The linear stochastic evolution

We now introduce the framework in this article. Let $A_t = (A_{t,x,y})_{x,y \in \mathbb{Z}^d}$, $t \in \mathbb{N}^*$ be a sequence of random matrices on a probability space $(\Omega, \mathcal{F}, P)$ such that

$$A_1, A_2, \ldots \text{ are i.i.d.} \tag{1.2}$$

Here are the set of assumptions we assume for A_1:

$$A_{1,x,y} \geq 0 \text{ for all } x, y \in \mathbb{Z}^d. \tag{1.3}$$

$$\text{The columns } \{A_{1,\cdot,y}\}_{y \in \mathbb{Z}^d} \text{ are independent.} \tag{1.4}$$

$$P[A_{1,x,y}^2] < \infty \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{1.5}$$

$$A_{1,x,y} = 0 \text{ a.s. if } |x - y| > r_A \text{ for some non-random } r_A \in \mathbb{N}. \tag{1.6}$$

$$(A_{1,x+z,y+z})_{x,y \in \mathbb{Z}^d} \stackrel{\text{law}}{=} A_1 \text{ for all } z \in \mathbb{Z}^d. \tag{1.7}$$

$$\begin{array}{l}\text{The set } \{x \in \mathbb{Z}^d \; ; \; \sum_{y \in \mathbb{Z}^d} a_{x+y} a_y \neq 0\} \text{ contains a linear basis of } \mathbb{R}^d, \\ \text{where } a_y = P[A_{1,0,y}].\end{array} \tag{1.8}$$

Depending on the results we prove in the sequel, some of these conditions can be relaxed. However, we choose not to bother ourselves with the pursuit of the minimum assumptions for each result.

We define a Markov chain $(N_t)_{t \in \mathbb{N}}$ with values in $[0, \infty)^{\mathbb{Z}^d}$ by

$$\sum_{x \in \mathbb{Z}^d} N_{t-1,x} A_{t,x,y} = N_{t,y}, \quad t \in \mathbb{N}^*. \tag{1.9}$$

Here and in the sequel (with only exception in Theorem 4.1.3 below), we suppose that the initial state N_0 is non-random and *finite* in the sense that

$$\text{the set } \{x \in \mathbb{Z}^d \; ; \; N_{0,x} > 0\} \text{ is finite and non-empty.} \tag{1.10}$$

If we regard $N_t \in [0, \infty)^{\mathbb{Z}^d}$ as a row vector, (1.9) can be interpreted as

$$N_t = N_0 A_1 A_2 \cdots A_t, \quad t = 1, 2, \ldots$$

The Markov chain defined above can be thought of as the time discretization of the linear particle system considered in the last Chapter in T. Liggett's book [10, Chapter IX]. Thanks to the time discretization, the definition is considerably simpler here. Though we *do not* assume in general that $(N_t)_{t \in \mathbb{N}}$ takes values in $\mathbb{N}^{\mathbb{Z}^d}$, we refer $N_{t,y}$ as the "number of particles" at time-space (t, y), and $|N_t|$ as "total number of particles" at time t.

We now see that various interesting examples are included in this simple framework. In what follows, $\delta_{x,y} = \mathbf{1}_{\{x=y\}}$ for $x, y \in \mathbb{Z}^d$. Recall also the notation a_y from (1.8).

• **Generalized oriented site percolation (GOSP):** We generalize OSP as follows. Let $\eta_{t,y}$, $(t, y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be $\{0, 1\}$-valued i.i.d. random variables with $P(\eta_{t,y} = 1) = p \in [0, 1]$ and let $\zeta_{t,y}$, $(t, y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be another $\{0, 1\}$-valued i.i.d. random variables with $P(\zeta_{t,y} = 1) = q \in [0, 1]$, which are independent of $\eta_{t,y}$'s. To exclude trivialities, we assume that either p or q is in $(0, 1)$. We refer to the process $(N_t)_{t \in \mathbb{N}}$ defined by (1.9) with

$$A_{t,x,y} = \mathbf{1}_{\{|x-y|=1\}} \eta_{t,y} + \delta_{x,y} \zeta_{t,y}$$

as the *generalized oriented site percolation* (GOSP). Thus, the OSP is the special case ($q = 0$) of GOSP. The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = p\mathbf{1}_{\{|y|=1\}} + q\delta_{y,0}, \quad P[A_{t,x,y}A_{t,\widetilde{x},y}] = \begin{cases} q & \text{if } x = \widetilde{x} = y, \\ p & \text{if } |x-y| = |\widetilde{x}-y| = 1, \\ a_{y-x}a_{y-\widetilde{x}} & \text{if otherwise.} \end{cases} \tag{1.11}$$

In particular, we have $|a| = 2dp + q$.

• **Generalized oriented bond percolation (GOBP):** Let $\eta_{t,x,y}$, $(t,x,y) \in \mathbb{N}^* \times \mathbb{Z}^d \times \mathbb{Z}^d$ be $\{0,1\}$-valued i.i.d.random variables with $P(\eta_{t,x,y} = 1) = p \in [0,1]$ and let $\zeta_{t,y}$, $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be another $\{0,1\}$-valued i.i.d. random variables with $P(\zeta_{t,y} = 1) = q \in [0,1]$, which are independent of $\eta_{t,y}$'s. Let us call the pair of time-space points $\langle\ (t-1,x),(t,y)\ \rangle$ a *bond* if $|x-y| \leq 1$, $(t,x,y) \in \mathbb{N}^* \times \mathbb{Z}^d \times \mathbb{Z}^d$. A bond $\langle\ (t-1,x),(t,y)\ \rangle$ with $|x-y| = 1$ is said to be *open* if $\eta_{t,x,y} = 1$, and a bond $\langle\ (t-1,y),(t,y)\ \rangle$ is said to be *open* if $\zeta_{t,y} = 1$. We refer to this model as the *generalized oriented bond percolation* (GOBP). We call the special case $q = 0$ *oriented bond percolation* (OBP). A variant of OBP is used to describe the electric current in non-crystalline semiconductors (silicon, germanium, etc.) at low temperature and subject to strong electric field [16]. There, the electrons, which are almost localized around the impurities, hop discontinuously from one impurity to another in the direction opposite to the electric field (hopping conduction). A bond $\langle\ (t-1,x),(t,y)\ \rangle$ with $x \neq y$ being open is interpreted that an electron hops from $(t-1,x)$ to (t,y).

For GOBP, an *open oriented path* from $(0,0)$ to $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ is a sequence $\{(s,x_s)\}_{s=0}^t$ in $\mathbb{N}\times\mathbb{Z}^d$ such that $x_0 = 0$, $x_t = y$ and bonds $\langle (s-1,x_{s-1}),(s,x_s)\rangle$ are open for all $s = 1,..,t$. If $N_0 = (\delta_{0,y})_{y\in\mathbb{Z}^d}$, then, the number $N_{t,y}$ of open oriented paths from $(0,0)$ to $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ is given by (1.9) with

$$A_{t,x,y} = \mathbf{1}_{\{|x-y|=1\}}\eta_{t,x,y} + \delta_{x,y}\zeta_{t,y}.$$

The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = p\mathbf{1}_{\{|y|=1\}} + q\delta_{y,0}, \quad P[A_{t,x,y}A_{t,\widetilde{x},y}] = \begin{cases} a_{y-x} & \text{if } x = \widetilde{x}, \\ a_{y-x}a_{y-\widetilde{x}} & \text{if otherwise.} \end{cases} \tag{1.12}$$

In particular, we have $|a| = 2dp + q$.

• **Directed polymers in random environment (DPRE):** Let $\{\eta_{t,y}\ ;\ (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\}$ be i.i.d. with $\exp(\lambda(\beta)) \stackrel{\text{def.}}{=} P[\exp(\beta\eta_{t,y})] < \infty$ for any $\beta \in (0,\infty)$. The following expectation is called the partition function of the *directed polymers in random environment*:

$$N_{t,y} = P_S^0\left[\exp\left(\beta\sum_{u=1}^t \eta_{u,S_u}\right) : S_t = y\right], \quad (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d,$$

where $((S_t)_{t\in\mathbb{N}}, P_S^x)$ is the simple random walk on $\mathbb{Z}^d$. We refer the reader to a review paper [4] and the references therein for more information. Starting from $N_0 = (\delta_{0,x})_{x\in\mathbb{Z}^d}$, the above expectation can be obtained inductively by (1.9) with

$$A_{t,x,y} = \frac{\mathbf{1}_{\{|x-y|=1\}}}{2d}\exp(\beta\eta_{t,y}).$$

The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = \frac{e^{\lambda(\beta)}\mathbf{1}_{\{|y|=1\}}}{2d}, \quad P[A_{t,x,y}A_{t,\widetilde{x},y}] = e^{\lambda(2\beta)-2\lambda(\beta)}a_{y-x}a_{y-\widetilde{x}} \tag{1.13}$$

In particular, we have $|a| = e^{\lambda(\beta)}$.

• **The binary contact path process (BCPP):** The binary contact path process is a continuous-time Markov process with values in $\mathbb{N}^{\mathbb{Z}^d}$, originally introduced by D. Griffeath [8]. In this article, we consider a discrete-time variant as follows. Let

$$\{\eta_{t,y} = 0, 1 \; ; \; (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\}, \quad \{\zeta_{t,y} = 0, 1 \; ; \; (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\},$$
$$\{e_{t,y} \; ; \; (t,y) \in \mathbb{N}^* \times \mathbb{Z}^d\}$$

be families of i.i.d. random variables with $P(\eta_{t,y} = 1) = p \in [0,1]$, $P(\zeta_{t,y} = 1) = q \in [0,1]$, and $P(e_{t,y} = e) = \frac{1}{2d}$ for each $e \in \mathbb{Z}^d$ with $|e| = 1$. We suppose that these three families are independent of each other and that either p or q in $(0,1)$ Starting from an $N_0 \in \mathbb{N}^{\mathbb{Z}^d}$, we define a Markov chain $(N_t)_{t \in \mathbb{N}}$ with values in $\mathbb{N}^{\mathbb{Z}^d}$ by

$$N_{t+1,y} = \eta_{t+1,y} N_{t,y-e_{t+1,y}} + \zeta_{t+1,y} N_{t,y}, \quad t \in \mathbb{N}.$$

We interpret the process as the spread of an infection, with $N_{t,y}$ infected individuals at time t at the site y. The $\zeta_{t+1,y} N_{t,y}$ term above means that these individuals remain infected at time $t+1$ with probability q, and they recover with probability $1-q$. On the other hand, the $\eta_{t+1,y} N_{t,y-e_{t+1,y}}$ term means that, with probability p, a neighboring site $y - e_{t+1,y}$ is picked at random (say, the wind blows from that direction), and $N_{t,y-e_{t+1,y}}$ individuals at site y are infected anew at time $t+1$. This Markov chain is obtained by (1.9) with

$$A_{t,x,y} = \eta_{t,y} \mathbf{1}_{\{e_{t,y} = y-x\}} + \zeta_{t,y} \delta_{x,y}.$$

The covariances of $(A_{t,x,y})_{x,y \in \mathbb{Z}^d}$ can be seen from:

$$a_y = \frac{p \mathbf{1}_{\{|y|=1\}}}{2d} + q\delta_{0,y}, \quad P[A_{t,x,y} A_{t,\widetilde{x},y}] = \begin{cases} a_{y-x} & \text{if } x = \widetilde{x}, \\ \delta_{x,y} q a_{y-\widetilde{x}} + \delta_{\widetilde{x},y} q a_{y-x} & \text{if } x \neq \widetilde{x}. \end{cases} \qquad (1.14)$$

In particular, we have $|a| = p + q$.

• **Voter model (VM):** Let $e_{t,y}$, $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be $\mathbb{Z}^d$-valued i.i.d. random variables with $P(e_{t,y} = 0) = 1 - p$ $(p \in (0,1])$ and $P(e_{t,y} = e) = \frac{p}{2d}$ for each $e \in \mathbb{Z}^d$ with $|e| = 1$. We then refer to the process $(N_t)_{t \in \mathbb{N}}$ defined by (1.9) with

$$A_{t,x,y} = \delta_{x, y+e_{t,y}}$$

as the *voter model* (VM). Let us suppose that $N_0 \in \mathbb{N}^{\mathbb{Z}^d}$ for simplicity. This process describes the behavior of voters in a certain election. At time 0, a voter at $y \in \mathbb{Z}^d$ supports the candidate $N_{0,y}$. Then, at time $t = 1$, the voter makes a decision in a random way. With probability $1-p$, the voter still supports the same candidate, and with probability $p/(2d)$, he/she finds the candidate supported by his/her neighbor at $y + e_{1,y}$ ($|e_{1,y}| = 1$) more attractive, and starts to support $N_{0,y+e_{1,y}}$, instead of $N_{0,y}$. The covariances of $(A_{t,x,y})_{x,y \in \mathbb{Z}^d}$ can be seen from:

$$a_y = p \frac{\mathbf{1}_{\{|y|=1\}}}{2d} + (1-p)\delta_{y,0}, \quad P[A_{t,x,y} A_{t,\widetilde{x},y}] = \delta_{x,\widetilde{x}} a_{y-x}. \qquad (1.15)$$

In particular, we have $|a| = 1$.

Remarks: 1) The branching random walk in random environment considered in [9, 17] can also be considered as a "close relative" to the models considered here, although it does not exactly fall into our framework.
2) After the first version of this paper was submitted, the author learned that there is a work

by R. W. R. Darling [6], in which the dual process of $(N_t)_{t\geq 0}$ (cf. section 4) was considered and the duals of OSP and OBP are discussed as examples.

Here are the summary of what are discussed in the rest of this paper. We look at the growth rate of the "total number" of particles:

$$|N_t| = \sum_{y\in\mathbb{Z}^d} N_{t,y} \quad t = 1, 2, ...$$

which will be kept finite for all t by our assumptions. We first show that $|N_t|$ has the expected value $|N_0||a|^t$, where $|a|$ is a positive number (cf. (1.8) and Lemma 1.3.1), so that $|a|^t$ can be considered as the mean growth rate of $|N_t|$. The main purpose of this paper is to investigate whether the limit:

$$|\overline{N}_\infty| \stackrel{\text{def}}{=} \lim_{t\to\infty} |N_t|/|a|^t$$

vanishes almost surely or not. Our results can be summarized as follows:

i) If $d \geq 3$ and the matrix A_t is not "too random", then, $|\overline{N}_\infty| > 0$ with positive probability (Lemma 2.1.1).

ii) In any dimension d, if the matrix A_t is "random enough", then, $|\overline{N}_\infty| = 0$, almost surely (Theorem 3.1.1). Moreover, the convergence is exponentially fast.

iii) For $d = 1, 2$, $|\overline{N}_\infty| = 0$, almost surely, under mild assumptions on A_t (Theorem 3.2.1. The assumptions are so mild that, for many examples, they merely amount to saying that A_t is "random at all". Moreover, the convergence is exponentially fast for $d = 1$.

We will refer i) as *regular growth phase*, and ii)—iii) as *slow growth phase*. In the regular growth phase, $|N_t|$ grows as fast as its mean growth rate with positive probability, whereas in the slow growth phase, the growth of $|N_t|$ is slower than its mean growth rate almost surely. There is a close connection between the growth rate of $|N_t|$ and the spacial distribution of the particles:

$$\rho_{t,x} = \frac{N_{t,x}}{|N_t|}\mathbf{1}_{\{|N_t|>0\}}, \quad x \in \mathbb{Z}^d \tag{1.16}$$

as $t \nearrow \infty$. The connection is roughly as follows. The regular growth implies that, conditionally on the event $\{|\overline{N}_\infty| > 0\}$, the spacial distribution has a Gaussian scaling limit [12]. In contrast to this, slow growth triggers the path localization on the event $\{|N_t| > 0$ for all $t \geq 1\}$ [18]. We remark that the exponential decay of $|N_t|/|a|^t$, mentioned in ii)–iii) above are interpreted as the positivity of the Lyapunov exponents.

The phenomena i)–iii) mentioned above have been observed for various models; for continuous-time linear interacting particle systems [10, Chapter IX], for DPRE [2, 3, 4], and for branching random walks in random environment [9, 17]. Here, we capture phenomena i)–iii) above by a simple discrete-time Markov chain, which however includes various, old and new examples. Here, "old examples" means that some of our results are known for them, such as DPRE, whereas "new examples" means that our results are new for them, such as GOSP and GOBP.

In section 4, we discuss the phase transition i)–iii) for the dual processes and its connection to the structure of invariant measures for the Markov chain $\overline{N}_t \stackrel{\text{def.}}{=} (N_{t,y}/|a|^t)_{y\in\mathbb{Z}^d}$. There, we will prove the following phase transition (Theorem 4.1.3):

iv) Suppose that the dual process is in the regular growth phase. Then, for each $\alpha \in (0,\infty)$, $(\overline{N}_t)$ has an invariant distribution ν_α, which is also invariant with respect to the lattice shift, such that $\int_{[0,\infty)^{\mathbb{Z}^d}} \eta_0 d\nu_\alpha(\eta) = \alpha$.

v) Suppose that the dual process is in the slow growth phase. Then, the only shift-invariant, invariant distribution ν for $(\overline{N}_t)$ such that $\int_{[0,\infty)^{\mathbb{Z}^d}} \eta_0 d\nu(\eta) < \infty$ is the trivial one, that is the point mass at all zero configuration.

The above iv)–v) is known for the continuous-time linear systems [10, Chapter IX]. Therefore, it would not be surprising at all that the same is true for the discrete-time model. However, iv)–v) seem to be new, even for well-studied models like OSP and DPRE.

As is mentioned before, the framework in this paper can be thought of as the time discretization of that in the last Chapter in T. Liggett's book [10, Chapter IX]. The author believes that the time discretization makes sense in some respect. First, it enables us to capture the phenomena as discussed above without much less technical complexity as compared with the continuous time case (e.g., construction of the model). Second, it allows us to discuss many different discrete models, which are conventionally treated separately, in a simple unified framework. In particular, it is nice that many techniques used in the context of DPRE [1, 2, 3, 4, 5] are applicable to many other models.

1.3 Some basic properties

In this subsection, we lay basis to study the growth of $|N_t|$ as $t \nearrow \infty$. We denote by $\mathcal{F}_t$, $t \in \mathbb{N}^*$ the σ-field generated by $A_1, ..., A_t$.

First of all, we identify the mean growth rate of $|N_t|$ with $|a|^t$.

Lemma 1.3.1

$$P[N_{t,y}] = |a|^t \sum_{x \in \mathbb{Z}^d} N_{0,x} P_S^x(S_t = y),$$

where $((S_t)_{t\in\mathbb{N}}, P_S^x)$ *is the random walk on* $\mathbb{Z}^d$ *such that*

$$P_S^x(S_0 = x) = 1 \text{ and } P_S^x(S_1 = y) = \overline{a}_{y-x} \stackrel{\text{def.}}{=} a_{y-x}/|a|.$$

Moreover, $(|\overline{N}_t|, \mathcal{F}_t)_{t\in\mathbb{N}}$ *is a martingale, where we have defined* $\overline{N}_t = \left(\overline{N}_{t,x}\right)_{x\in\mathbb{Z}^d}$ *by*

$$\overline{N}_{t,x} = |a|^{-t} N_{t,x}. \tag{1.17}$$

Proof: The first equality is obtained by averaging the identity:

$$N_{t,y} = \sum_{x_0,,..,x_{t-1}} N_{0,x_0} A_{1,x_0,x_1} A_{2,x_1,x_2} \cdots A_{t,x_{t-1},y}. \tag{1.18}$$

It is also easy to see from the above identity that $(|\overline{N}_t|, \mathcal{F}_t)_{t\in\mathbb{N}}$ is a martingale. □

We next compare $|N_t|$ and its mean growth rate $|a|^t$.

Lemma 1.3.2 *Referring to Lemma 1.3.1, the limit*

$$|\overline{N}_\infty| = \lim_{t\to\infty} |\overline{N}_t| \tag{1.19}$$

exists a.s. and

$$P[|\overline{N}_\infty|] = |N_0| \text{ or } 0. \tag{1.20}$$

Moreover, $P[|\overline{N}_\infty|] = |N_0|$ *if and only if the limit (1.19) is convergent in* $\mathbb{L}^1(P)$.

Before we prove Lemma 1.3.2, we introduce some notation and definitions. For $(s,z) \in \mathbb{N}\times\mathbb{Z}^d$, we define $N_t^{s,z} = (N_{t,y}^{s,z})_{y\in\mathbb{Z}^d}$ and $\overline{N}_t^{s,z} = (\overline{N}_{t,y}^{s,z})_{y\in\mathbb{Z}^d}$, $t \in \mathbb{N}$ respectively by

$$\begin{aligned} &N_{0,y}^{s,z} = \delta_{z,y}, \quad N_{t+1,y}^{s,z} = \sum_{x\in\mathbb{Z}^d} N_{t,x}^{s,z} A_{s+t+1,x,y}, \\ &\text{and} \quad \overline{N}_{t,y}^{s,z} = |a|^{-t} N_{t,y}^{s,z}. \end{aligned} \tag{1.21}$$

In particular, $(N_t^{0,z})_{t\in\mathbb{N}}$ is the Markov chain (1.9) with the initial state $N_0^{0,z} = (\delta_{z,y})_{y\in\mathbb{Z}^d}$. Moreover, we have

$$N_{t,y} = \sum_{z\in\mathbb{Z}^d} N_{0,z} N_{t,y}^{0,z} \quad \text{for any initial state } N_0. \tag{1.22}$$

Now, it follows from Lemma 1.3.2 that

$$P[|\overline{N}_\infty^{0,0}|] = 1, \text{ or } = 0.$$

We will refer to the former case as *regular growth phase* and the latter as *slow growth phase*. By (1.22) and the shift invariance, $P[|\overline{N}_\infty|] = |N_0|$ for all N_0 in the regular growth phase and $P[|\overline{N}_\infty|] = 0$ for all N_0 in the slow growth phase. The regular growth means that, at least with positive probability, the growth of the "total number" $|N_t|$ of the particles is of the same order as its expectation $|a|^t|N_0|$. On the other hand, the slow growth means that, almost surely, the growth of $|N_t|$ is slower than its expectation.

Proof of Lemma 1.3.2: Because of (1.22) and the shift-invariance, it is enough to assume that $N_t = N_t^{0,0}$. The limit (1.19) exists by the martingale convergence theorem, and $\ell \stackrel{\text{def.}}{=} P[|\overline{N}_\infty|] \le 1$ by Fatou's lemma. To show (1.20), we will prove that $\ell = \ell^2$, following the argument in [10, page 433, Theorem 2.4(a)]. Using the notation (1.21), we write

(1) $|\overline{N}_{s+t}| = \sum_y \overline{N}_{s,y} |\overline{N}_t^{s,y}|$.

Since $|\overline{N}_t^{s,y}| \stackrel{\text{law}}{=} |\overline{N}_t|$, the limit

$$|\overline{N}_\infty^{s,y}| = \lim_{t\to\infty} |\overline{N}_t^{s,y}|$$

exists a.s. and is equally distributed as $|\overline{N}_\infty|$. Moreover, by letting $t \nearrow \infty$ in (1), we have that

$$|\overline{N}_\infty| = \sum_y \overline{N}_{s,y} |\overline{N}_\infty^{s,y}|.$$

and hence by Jensen's inequality that

$$P[\exp(-|\overline{N}_\infty|)|\mathcal{F}_s] \ge \exp\left(-P[|\overline{N}_\infty||\mathcal{F}_s]\right) = \exp\left(-|\overline{N}_s|\ell\right) \ge \exp\left(-|\overline{N}_s|\right).$$

By letting $s \nearrow \infty$ in the above inequality, we obtain

$$\exp(-|\overline{N}_\infty|) \stackrel{\text{a.s.}}{\ge} \exp\left(-|\overline{N}_\infty|\ell\right) \ge \exp\left(-|\overline{N}_\infty|\right),$$

and thus, $|\overline{N}_\infty| \stackrel{a.s.}{=} |\overline{N}_\infty|\ell$. By taking expectation, we get $\ell = \ell^2$. Once we know (1.20), the final statement of the lemma is standard([7, page 257–258, (5.2)], for example). □

Let us now take a brief look at the condition for the extinction: $\lim_{t\to\infty} |N_t| = 0$ a.s., although our main objective in this article is to study $|\overline{N}_\infty| = \lim_{t\to\infty} |\overline{N}_t|$.

If $|a| < 1$, we have

$$\lim_{t \to \infty} |N_t| = \lim_{t \to \infty} |a|^t |\overline{N}_t| = 0.$$

For $|a| = 1$, we will present an argument below (Lemma 1.3.3), which applies when $(N_t)_{t \in \mathbb{N}}$ is $\mathbb{N}^{\mathbb{Z}^d}$-valued. Consequently, we will see that $\lim_{t \to \infty} |N_t| = 0$ for GOSP ans GOBP with $(1-p)(1-q) \neq 0$ and for VM with $p \in (0, 1]$. For GOSP and GOBP, we apply Lemma 1.3.3 directly. For VM, we slightly modify the argument (See the remark after the lemma).

It follows from the observations above that $\lim_{t \to \infty} |N_t| = 0$ a.s. if

$$\begin{cases} 2dp + q \leq 1 \text{ and } (1-p)(1-q) \neq 0 & \text{for GOSP and GOBP,} \\ \lambda(\beta) < 0 & \text{for DPRE,} \\ p + q \leq 1 \text{ and } (1-p)(1-q) \neq 0 & \text{for BCPP,} \\ p \in (0, 1] & \text{for VM.} \end{cases} \tag{1.23}$$

Lemma 1.3.3 *Let $\mathcal{O}_t$ be the set of occupied sites at time t,*

$$\mathcal{O}_t = \{x \in \mathbb{Z}^d \; ; \; N_{t,x} > 0\}$$

and $|\mathcal{O}_t|$ be its cardinality. Suppose that

$$\delta \stackrel{\text{def.}}{=} P\left(\bigcap_{x \in \mathbb{Z}^d} \{A_{1,x,0} = 0\}\right) > 0. \tag{1.24}$$

Then,

$$P(\lim_{t \to \infty} |\mathcal{O}_t| \in \{0, \infty\}) = 1.$$

Proof: We will see that

(1) $\{|\mathcal{O}_t| \leq m \text{ i.o.}\} \stackrel{\text{a.s.}}{=} \{|\mathcal{O}_t| = 0 \text{ i.o.}\}$ for any $m \in \mathbb{N}$,

which immediately implies the lemma:

$$\{|\mathcal{O}_t| \not\longrightarrow \infty\} = \bigcup_{m \in \mathbb{N}} \{|\mathcal{O}_t| \leq m \text{ i.o.}\} \stackrel{\text{a.s.}}{=} \{|\mathcal{O}_t| = 0 \text{ i.o.}\}.$$

For (1), we have only to show the $\stackrel{\text{a.s.}}{\subset}$ part. We write $\widetilde{\mathcal{O}}_{t-1} = \bigcup_{x \in \mathcal{O}_{t-1}} \{y \in \mathbb{Z}^d \; ; \; |x-y| \leq r_A\}$ (cf. (1.6)). Since

$$|\mathcal{O}_t| = 0 \iff |N_t| = \sum_{x,y \in \mathbb{Z}^d} N_{t-1,x} A_{t,x,y} = 0,$$

we have

$$\begin{aligned} P(|\mathcal{O}_t| = 0 | \mathcal{F}_{t-1}) &= P\left(\bigcap_{y \in \widetilde{\mathcal{O}}_{t-1}} \{\sum_{x \in \mathbb{Z}^d} N_{t-1,x} A_{t,x,y} = 0\} \middle| \mathcal{F}_{t-1}\right) \\ &\geq P\left(\bigcap_{y \in \widetilde{\mathcal{O}}_{t-1}} \bigcap_{x \in \mathbb{Z}^d} \{A_{t,x,y} = 0\} \middle| \mathcal{F}_{t-1}\right) \\ &= \prod_{y \in \widetilde{\mathcal{O}}_{t-1}} P\left(\bigcap_{x \in \mathbb{Z}^d} \{A_{1,x,y} = 0\}\right) = \delta^{|\widetilde{\mathcal{O}}_{t-1}|}. \end{aligned}$$

This, together with the generalized second Borel-Cantelli lemma ([7, page 237]) implies that

$$\{|\mathcal{O}_t| \le m \ \text{ i.o.}\} \subset \left\{ \sum_{t=1}^{\infty} P(|\mathcal{O}_t| = 0 | \mathcal{F}_{t-1}) = \infty \right\} \stackrel{\text{a.s.}}{=} \{|\mathcal{O}_t| = 0 \ \text{ i.o.}\}.$$

□

Remark: For VM, we argue as follows. Since $|a| = 1$, $|N_t|$ is a martingale and hence converges a.s. Since $|N_t|$ is $\mathbb{N}$-valued, we have $|N_{t-1}| = |N_t|$ for large t, a.s. On the other hand, for some $c = c(p, d) > 0$, we have

$$\{1 \le |\mathcal{O}_{t-1}| \le m\} \subset \{P\left(|N_{t-1}| > |N_t| | \mathcal{F}_{t-1}\right) \ge c^m\} \quad \text{for all } m \in \mathbb{N}^*.$$

(Replace $N_{t-1,y}$ on all y on the interior boundaries of $\mathcal{O}_{t-1}$ with 0, while keeping all the other $N_{t-1,y}$ unchanged.) This implies that $\lim_{t\to\infty} |N_t| = 0$, via a similar argument as in Lemma 1.3.3.

2 Regular growth phase

2.1 Regular growth via Feynman-Kac formula

The purpose of this subsection is to give a sufficient condition for the regular growth phase (Lemma 2.1.1 below) and discuss its application to some examples (section 2.2). The sufficient condition is given by expressing the two-point function

$$P[N_{t,y} N_{t,\widetilde{y}}]$$

in terms of a Feynman-Kac type expectation with respect to the independent product of the random walks in Lemma 1.3.1. We let $(S, \widetilde{S}) = ((S_t, \widetilde{S}_t)_{t\in\mathbb{N}}, P^{x,\widetilde{x}}_{S,\widetilde{S}})$ denote the independent product of the random walks in Lemma 1.3.1. We have the following Feynman-Kac formula.

Lemma 2.1.1 *Define*

$$e_t = \prod_{u=1}^{t} w(S_{u-1}, \widetilde{S}_{u-1}, S_u, \widetilde{S}_u), \quad t \ge 1, \tag{2.1}$$

where

$$w(x, \widetilde{x}, y, \widetilde{y}) = \begin{cases} \dfrac{P[A_{1,x,y} A_{1,\widetilde{x},\widetilde{y}}]}{a_{y-x} a_{\widetilde{y}-\widetilde{x}}} & \text{if } a_{y-x} a_{\widetilde{y}-\widetilde{x}} \ne 0, \\ 0, & \text{if } a_{y-x} a_{\widetilde{y}-\widetilde{x}} = 0. \end{cases} \tag{2.2}$$

Then,

$$P[N_{t,y} N_{t,\widetilde{y}}] = |a|^{2t} \sum_{x,\widetilde{x}\in\mathbb{Z}^d} N_{0,x} N_{0,\widetilde{x}} P^{x,\widetilde{x}}_{S,\widetilde{S}}[e_t : (S_t, \widetilde{S}_t) = (y, \widetilde{y})] \tag{2.3}$$

for all $t \in \mathbb{N}$, $y, \widetilde{y} \in \mathbb{Z}^d$. Consequently,

$$P[|\overline{N}_t|^2] = \sum_{x,\widetilde{x}\in\mathbb{Z}^d} N_{0,x} N_{0,\widetilde{x}} P^{x,\widetilde{x}}_{S,\widetilde{S}}\left[e_t\right], \tag{2.4}$$

and

$$\sup_{t\in\mathbb{N}} P[|\overline{N}_t|^2] < \infty \quad \Longleftrightarrow \quad \sup_{t\in\mathbb{N}} P^{0,0}_{S,\widetilde{S}}\left[e_t\right] < \infty \tag{2.5}$$

$$\Longrightarrow \quad P[|\overline{N}_\infty|] = |N_0|. \tag{2.6}$$

Proof: By (1.18) and the independence, we have

$$\textbf{(1)} \qquad P[N_{t,y}N_{t,\widetilde{y}}] = \sum_{x_0,,..,x_{t-1}} \sum_{\widetilde{x}_0,,..,\widetilde{x}_{t-1}} N_{0,x_0}N_{0,\widetilde{x}_0}\prod_{s=1}^t P[A_{1,x_{s-1},x_s}A_{1,\widetilde{x}_{s-1},\widetilde{x}_s}],$$

with the convention that $x_t = y$, $\widetilde{x}_t = \widetilde{y}$. We have on the other hand that

$$P[A_{1,x_{s-1},x_s}A_{1,\widetilde{x}_{s-1},\widetilde{x}_s}] = |a|^2 w(x_{s-1},\widetilde{x}_{s-1},x_s,\widetilde{x}_s)\overline{a}_{x_s-x_{s-1}}\overline{a}_{\widetilde{x}_s-\widetilde{x}_{s-1}}.$$

Plugging this into (1), we get (2.3). (2.4) is an immediate consequence of (2.3). We now recall (1.22) and that $|N_t^{0,z}| \stackrel{\text{law}}{=} |N_t^{0,0}|$ for all $t \in \mathbb{N}$ and $z \in \mathbb{Z}^d$. Therefore, it is enough to prove (2.5) for $N_t = N_t^{0,0}$. But this follows immediately from (2.4). (2.6) is a consequence of Lemma 1.3.2. □

Remarks: 1) The criterion (2.5)–(2.6) generalizes what is known as the "L^2-condition" for DPRE [1, 4, 13]. It can also be thought of as a discrete-time analogue of [10, page 445, Theorem 3.12], where however, more analytical approach (in terms of the existence of a certain harmonic function) is adopted.
2) The second moment method discussed here is also useful to prove the central limit theorem for the spacial distribution of the particles [12].

Next, we present more explicit expression for the condition (2.5) and for the covariances of the random variables $(|\overline{N}_\infty^{0,x}|)_{x\in\mathbb{Z}^d}$ (cf. (1.21)). We set

$$\tau_1 = \inf\{t \geq 1 \ ; \ S_t = \widetilde{S}_t\} \ \text{ and } \ \pi_x = P^{x,0}_{S,\widetilde{S}}(\tau_1 < \infty). \tag{2.7}$$

By (1.8), $\pi_x < 1$ if $d \geq 3$.

Lemma 2.1.2 *Let $d \geq 3$. Then, for any $x, \widetilde{x} \in \mathbb{Z}^d$,*

$$\begin{aligned} & \sup_{t\in\mathbb{N}} P[|\overline{N}_t^{0,x}||\overline{N}_t^{0,\widetilde{x}}|] < \infty \\ \iff \quad & P^{0,0}_{S,\widetilde{S}}[e_{\tau_1} : \tau_1 < \infty] < 1 \qquad (2.8) \\ \Longrightarrow \quad & P[|\overline{N}_\infty^{0,x}||\overline{N}_\infty^{0,\widetilde{x}}|] = 1 - \pi_{x-\widetilde{x}} + \frac{P^{x,\widetilde{x}}_{S,\widetilde{S}}[e_{\tau_1} : \tau_1 < \infty]}{1 - P^{0,0}_{S,\widetilde{S}}[e_{\tau_1} : \tau_1 < \infty]}(1 - \pi_0). \qquad (2.9) \end{aligned}$$

Proof: Note that $w(S_{t-1}, \widetilde{S}_{t-1}, S_t, \widetilde{S}_t) = 1$ unless $S_t = \widetilde{S}_t$, which occurs only finitely often a.s. Thus, $e_{t-1} = e_t$ for large enough t's and therefore, $e_\infty = \lim_{t\to\infty} e_t$ exists a.s. On the other hand, let

$$\tau_v = \inf\{t \geq 1 \ ; \ \sum_{u=1}^t \delta_{S_u,\widetilde{S}_u} = v\}.$$

Then, by the strong Markov property,

$$\begin{aligned} P^{x,\widetilde{x}}_{S,\widetilde{S}}[e_\infty] &= P^{x,\widetilde{x}}_{S,\widetilde{S}}(\tau_1 = \infty) + \sum_{v=1}^\infty P^{x,\widetilde{x}}_{S,\widetilde{S}}[e_{\tau_v} : \tau_v < \infty = \tau_{v+1}] \\ &= 1 - \pi_{x-\widetilde{x}} + P^{x,\widetilde{x}}_{S,\widetilde{S}}[e_{\tau_1} : \tau_1 < \infty] \sum_{v=1}^\infty P^{0,0}_{S,\widetilde{S}}[e_{\tau_1} : \tau_1 < \infty]^{v-1}(1 - \pi_0). \qquad (2.10) \end{aligned}$$

Now, by (2.3) and Fatou's lemma, we have that

$$P^{x,\widetilde{x}}_{S,\widetilde{S}}[e_\infty] \le \sup_{t\in\mathbb{N}} P^{x,\widetilde{x}}_{S,\widetilde{S}}[e_t] = \sup_{t\in\mathbb{N}} P[|\overline{N}^{0,x}_t||\overline{N}^{0,\widetilde{x}}_t|].$$

These prove "⇒" part of (2.8) (The argument presented above is due to M. Nakashima [12]). To prove the converse, we start by noting that

$$r(p) = P^{0,0}_{S,\widetilde{S}}[e^p_{\tau_1} : \tau_1 < \infty] \text{ is continuous in } p \in [1,\infty),$$

since $e_{\tau_1} \le \sup w < \infty$. Then, by our assumption that $r(1) < 1$, there exists $p > 1$ such that $r(p) < 1$. We fix such p and prove that

(1) $\quad \sup_{t\in\mathbb{N}} P^{x,\widetilde{x}}_{S,\widetilde{S}}[e^p_t] < \infty$, and thus, $(e_t)_{t\in\mathbb{N}}$ is uniformly integrable.

This implies that

(2) $$\infty \overset{(2.10)}{>} P^{x,\widetilde{x}}_{S,\widetilde{S}}[e_\infty] \overset{(1)}{=} \lim_{t\to\infty} P^{x,\widetilde{x}}_{S,\widetilde{S}}[e_t] \overset{(2.3)}{=} \lim_{t\to\infty} P[|\overline{N}^{0,x}_t||\overline{N}^{0,\widetilde{x}}_t|].$$

Also, (2.9) follows from (2) and (2.10). Finally, we prove (1) as follows:

$$\begin{aligned} P^{x,\widetilde{x}}_{S,\widetilde{S}}[e^p_t] &= P^{x,\widetilde{x}}_{S,\widetilde{S}}[\tau_1 > t] + \sum_{v=1}^{t} P^{x,\widetilde{x}}_{S,\widetilde{S}}[e^p_{\tau_v} : \tau_v \le t < \tau_{v+1}] \\ &\le 1 + \sum_{v=1}^{\infty} P^{x,\widetilde{x}}_{S,\widetilde{S}}[e^p_{\tau_v} : \tau_v < \infty] \\ &= 1 + P^{x,\widetilde{x}}_{S,\widetilde{S}}[e^p_{\tau_1} : \tau_1 < \infty] \sum_{v=1}^{\infty} r(p)^{v-1} < \infty. \end{aligned}$$

□

2.2 Examples

We will discuss application of Lemma 2.1.2 to GOSP, GOBP and DPRE. We assume that $d \ge 3$.

Application of Lemma 2.1.2 to GOSP and DPRE: For OSP and DPRE, we see from (1.11) and (1.13) that

$$P[A_{t,x,y}A_{t,\widetilde{x},\widetilde{y}}] = \gamma^{\delta_{y,\widetilde{y}}} a_{y-x} a_{\widetilde{y}-\widetilde{x}}, \quad \text{with } \gamma = \begin{cases} 1/p & \text{for OSP,} \\ \exp(\lambda(2\beta) - 2\lambda(\beta)) & \text{for DPRE.} \end{cases} \tag{2.11}$$

By (2.11),

$$w(x,\widetilde{x},y,\widetilde{y}) = \begin{cases} \gamma^{\delta_{y,\widetilde{y}}} & \text{if } a_{y-x}a_{\widetilde{y}-\widetilde{x}} \ne 0, \\ 0, & \text{if } a_{y-x}a_{\widetilde{y}-\widetilde{x}} = 0. \end{cases} \tag{2.12}$$

and thus,

$$P^{0,x}[e_{\tau_1} : \tau_1 < \infty] = \gamma\pi_x.$$

Therefore, we see from Lemma 2.1.2 that for DPRE and OSP,

$$\sup_{t\in\mathbb{N}} P[|\overline{N}^{0,0}_t|^2] < \infty \iff \pi_0\gamma < 1 \tag{2.13}$$

$$\implies P[|\overline{N}^{0,0}_\infty||\overline{N}^{0,x}_\infty|] = 1 + \pi_x \frac{\gamma - 1}{1 - \pi_0\gamma}. \tag{2.14}$$

The above covariance was computed by F. Comets for DPRE (private communication). Similar formula for the binary contact path process in continuous time can be found in [8, 11]. Also, it follows from (2.5) and (2.13) that

$$\sup_{t\in\mathbb{N}} P[|\overline{N}_t|^2] < \infty \iff \begin{cases} p > \pi_0 & \text{for OSP,} \\ \lambda(2\beta) - 2\lambda(\beta) < \ln(1/\pi_0) & \text{for DPRE.} \end{cases} \tag{2.15}$$

For GOSP with $q \neq 0$, we have

$$w(x, \widetilde{x}, y, \widetilde{y}) = \begin{cases} 1/q & \text{if } y = \widetilde{y} = x = \widetilde{x}, \\ 1/p & \text{if } y = \widetilde{y},\ |x-y| = |\widetilde{x}-y| = 1, \\ \mathbf{1}_{\{a_{y-x}a_{\widetilde{y}-\widetilde{x}}>0\}} & \text{if otherwise.} \end{cases} \tag{2.16}$$

Thus, similar arguments show that:

$$\sup_{t\in\mathbb{N}} P[|\overline{N}_t|^2] < \infty \iff p \wedge q > \pi_0 \ \text{ for GOSP with } q \neq 0. \tag{2.17}$$

For OSP and DPRE, $(S_t)_{t\in\mathbb{N}}$ is the simple random walks. In this case, π_0 is the same as the return probability of the simple random walk itself, for which we have $1/(2d) < \pi_0 \leq 0.3405...$ for $d \geq 3$ [15, page 103]. (2.15) for DPRE case can be found in [13].

Application of Lemma 2.1.2 to GOBP: For GOBP with $q \neq 0$, we have

$$w(x, \widetilde{x}, y, \widetilde{y}) = \begin{cases} 1/q & \text{if } x = \widetilde{x} = y = \widetilde{y}, \\ 1/p & \text{if } x = \widetilde{x},\ y = \widetilde{y},\ |x-y| = 1, \\ \mathbf{1}_{\{a_{y-x}a_{\widetilde{y}-\widetilde{x}}>0\}} & \text{if otherwise.} \end{cases} \tag{2.18}$$

For OBP, we have the formula for w by ignoring the first line of (2.18). Thus,

$$P^{x,0}_{S,\widetilde{S}}[e_{\tau_1} : \tau_1 < \infty] = P^{x,0}_{S,\widetilde{S}}(\tau_1 < \infty) = \pi_x \quad \text{if } x \neq 0$$

and

$$\begin{aligned} & P^{0,0}_{S,\widetilde{S}}[e_{\tau_1} : \tau_1 < \infty] \\ = \ & P^{0,0}_{S,\widetilde{S}}[e_{\tau_1} : \tau_1 = 1] + P^{0,0}_{S,\widetilde{S}}(2 \leq \tau_1 < \infty) \\ = \ & \frac{1}{p} 2d \left(\frac{p}{2dp+q}\right)^2 + \frac{1}{q}\left(\frac{q}{2dp+q}\right)^2 + \left(\pi_0 - 2d\left(\frac{p}{2dp+q}\right)^2 - \left(\frac{q}{2dp+q}\right)^2\right) \\ = \ & \pi_0 + c, \ \text{ with } \ c = \frac{2dp(1-p) + q(1-q)}{(2dp+q)^2}. \end{aligned}$$

Therefore, with c defined above, we have by Lemma 2.1.2 that

$$\sup_{t\in\mathbb{N}} P[|\overline{N}^{0,0}_t|^2] < \infty \iff \pi_0 + c < 1 \tag{2.19}$$

$$\implies P[|\overline{N}^{0,0}_\infty||\overline{N}^{0,x}_\infty|] = 1 + \begin{cases} \pi_x \dfrac{c}{1-\pi_0-c} & \text{if } x \neq 0, \\ \dfrac{c}{1-\pi_0-c} & \text{if } x = 0. \end{cases} \tag{2.20}$$

Remarks: 1) For OBP, $P[|\overline{N}^{0,0}_\infty|^2]$ is also computed in [6, (3.5)]. Unfortunately, the formula (3.5) in [6] is not correct, due to an error (the law of $J(\infty)$ on page 212).
2) The case of BCPP is discussed in [12].

3 Slow growth phase

3.1 Slow growth in any dimension

We give the following sufficient condition for the slow growth phase in any dimension. The condition is typically applies to the limited regions of parameters, which makes particles "hard to survive" (Remark 1 after Theorem 3.1.1).

Theorem 3.1.1 *Suppose that*

$$\sum_{y\in\mathbb{Z}^d} P\left[A_{1,0,y}\ln A_{1,0,y}\right] > |a|\ln|a|. \tag{3.1}$$

Then, there exists a non-random $c>0$ *such that*

$$|\overline{N}_t| = O(e^{-ct}), \quad \textit{as } t\to\infty, \textit{ a.s.}$$

Remarks: 1) It is easy to see that

$$(3.1) \iff \begin{cases} 2dp+q<1 & \text{for GOSP and GOBP,} \\ \beta\lambda'(\beta)-\lambda(\beta)>\ln(2d) & \text{for DPRE,} \\ p+q<1 & \text{for BCPP.} \end{cases}$$

2) Theorem 3.1.1 generalizes [3, Theorem 1.3(a)], which is obtained in the setting of DPRE. Theorem 3.1.1 can also be thought of as the discrete-time analogue of [10, page 455, Theorem 5.1].

Proof of Theorem 3.1.1: By (1.22) and the shift invariance, it is enough to prove the result for $N_t = N_t^{0,0}$. We write

$$|N_t| = \sum_y A_{1,0,y}|N_{t-1}^{2,y}|.$$

Thus, for $h\in(0,1]$,

$$|N_t|^h \le \sum_y A_{1,0,y}^h |N_{t-1}^{2,y}|^h.$$

Since $|N_{t-1}^{2,y}| \overset{\text{law}}{=} |N_{t-1}|$, we have

$$P[|N_t|^h] \le \sum_y P[A_{1,0,y}^h] P[|N_{t-1}|^h],$$

and hence

$$P[|\overline{N}_t|^h] \le \varphi(h) P[|\overline{N}_{t-1}|^h], \quad \text{with } \varphi(h) = \textstyle\sum_y P\left[\left(\frac{A_{1,0,y}}{|a|}\right)^h\right].$$

Note that $\varphi(1)=1$ and that

$$\varphi'(1-) = \sum_{y\in\mathbb{Z}^d} P\left[\frac{A_{1,0,y}}{|a|}\ln\left(\frac{A_{1,0,y}}{|a|}\right)\right] > 0.$$

(For the differentiability, note that $x^h|\ln x| \le (he)^{-1}$ for $x\in[0,1]$, and $x^h|\ln x|\le x\ln x$ for $x\ge 1$.) These imply that there exists $h_0\in(0,1)$ such that $\varphi(h_0)<1$, and hence that $P[|\overline{N}_t|^{h_0}] \le \varphi(h_0)^t$, $t\in\mathbb{N}$. Finally the theorem follows from the Borel-Cantelli lemma. □

3.2 Slow growth in dimensions one and two

We now state a result (Theorem 3.2.1) for slow growth phase in dimensions one and two. Unlike Theorem 3.1.1, Theorem 3.2.1 is typically applies to the entire region of the parameters in various models (cf. Remarks after Theorem 3.2.1).

For $f, g \in [0,\infty)^{\mathbb{Z}^d}$ with $|f|, |g| < \infty$, we define their convolution $f * g \in [0,\infty)^{\mathbb{Z}^d}$ by

$$(f * g)_x = \sum_{y \in \mathbb{Z}^d} f_{x-y} g_y.$$

The identity : $|(f * g)| = |f||g|$ will often be used in the sequel.

Theorem 3.2.1 *Let $d = 1, 2$. Suppose that $P[A^3_{1,0,y}] < \infty$ for all $y \in \mathbb{Z}^d$ and that there is a constant $\gamma \in (1,\infty)$ such that*

$$\sum_{x,\widetilde{x},y \in \mathbb{Z}^d} \left(P[A_{1,x,y} A_{1,\widetilde{x},y}] - \gamma a_{y-x} a_{y-\widetilde{x}} \right) \xi_x \xi_{\widetilde{x}} \geq 0 \tag{3.2}$$

for all $\xi \in [0,\infty)^{\mathbb{Z}^d}$ such that $|\xi| < \infty$. Then, almost surely,

$$|\overline{N}_t| \begin{cases} = O(\exp(-ct)) & \text{if } d = 1, \\ \longrightarrow 0 & \text{if } d = 2 \end{cases} \quad \text{as } t \longrightarrow \infty, \tag{3.3}$$

where c is a non-random constant.

Theorem 3.2.1 is a generalization of [2, Theorem 1.1], [3, Theorem 1.3(b)] and [5, Theorem 1.1], which are obtained in the setting of DPRE. The proof of Theorem 3.2.1 will be built on ideas and techniques developed there. Theorem 3.2.1 can also be thought of as a discrete-time analogue of [10, page 451, Theorem 4.5]. Before we present the proof of Theorem 3.2.1, we check condition (3.2) for GOSP, GOBP, DPRE and BCPP.

Condition (3.2) for OSP and DPRE: By (2.11), (3.2) holds for OSP for all $p \in (0,1)$ and for DPRE for all $\beta \in (0,\infty)$.

Condition (3.2) for GOSP and GOBP: We introduce

$$b_x = \sum_{y \in \mathbb{Z}^d} a_y a_{y-x} \quad \text{and} \quad b^A_x = \sum_{y \in \mathbb{Z}^d} P[A_{1,0,y} A_{1,x,y}] \quad \text{for } x \in \mathbb{Z}^d. \tag{3.4}$$

Then, (3.2) is equivalent to

$$\sum_{x,\widetilde{x} \in \mathbb{Z}^d} \left(b^A_{x-\widetilde{x}} - b_{x-\widetilde{x}} \right) \xi_x \xi_{\widetilde{x}} \geq (\gamma - 1)|(a * \xi)^2|.$$

Note that $|(a * \xi)^2| \leq |a|^2 |\xi^2|$. Thus, if there exists $c \in (0,\infty)$ such that

$$b^A_x \geq b_x + c\delta_{0,x} \quad \text{for all } x \in \mathbb{Z}^d, \tag{3.5}$$

then, we have (3.2) with $\gamma = 1 + (c/|a|^2)$. For GOSP, we have by (1.11) that

$$b_x \begin{cases} = 2dp^2 + q^2, & \text{if } x = 0, \\ = 2pq & \text{if } |x| = 1, \\ > 0 & \text{if } |x| = 2, \end{cases} \qquad b^A_x = \begin{cases} 2dp + q, & \text{if } x = 0, \\ 2pq & \text{if } |x| = 1, \\ p^{-1} b_x & \text{if } |x| = 2, \end{cases} \qquad b_x = b^A_x = 0 \text{ if } |x| \geq 3.$$

The above are the same for GOBP, except that $b_x^A = b_x$ for $|x| = 2$ for GOBP. Thus, (3.5) holds for GOSP and GOBP with $c = 2dp(1-p) + q(1-q)$.

Condition (3.2) for BCPP: For $\xi \in \mathbb{R}^{\mathbb{Z}^d}$ with $|\xi| < \infty$, we denote its Fourier transform by $\widehat{\xi}(\theta) = \sum_{x \in \mathbb{Z}^d} \xi_x \exp(\mathbf{i} x \cdot \theta)$, $\theta \in [-\pi, \pi]^d$. If

$$c_1 \stackrel{\text{def.}}{=} \min_{\theta \in [-\pi,\pi]^d} \left(\widehat{b^A}(\theta) - |\widehat{a}(\theta)|^2 \right) > 0, \tag{3.6}$$

then, (3.2) holds with $\gamma = 1 + (c_1/|a|^2)$. This can be seen as follows. Note that (3.2) can be written as:

$$\sum_{x,\widetilde{x} \in \mathbb{Z}^d} \xi_x \xi_{\widetilde{x}} b^A_{x-\widetilde{x}} \geq \gamma |(a * \xi)^2|.$$

Then, by Plancherel's identity and the fact that $|(a*\xi)^2| \leq |a|^2 |\xi^2|$, we have that

$$\begin{aligned} \sum_{x,\widetilde{x} \in \mathbb{Z}^d} \xi_x \xi_{\widetilde{x}} b^A_{x-\widetilde{x}} &= (2\pi)^{-d} \int_{[-\pi,\pi]^d} \widehat{b^A}(\theta) |\widehat{\xi}(\theta)|^2 d\theta \geq (2\pi)^{-d} \int_{[-\pi,\pi]^d} (|\widehat{a}(\theta)|^2 + c_1) |\widehat{\xi}(\theta)|^2 d\theta \\ &= |(a*\xi)^2| + c_1 |\xi^2| \geq (1 + c_1/|a|^2) |(a*\xi)^2|. \end{aligned}$$

The criterion (3.6) can effectively be used to check (3.2) for BCPP. In fact, we have by (1.14) that

$$b_x \begin{cases} = \frac{p^2}{2d} + q^2, & \text{if } x = 0, \\ = \frac{pq}{d} & \text{if } |x| = 1, \\ > 0 & \text{if } |x| = 2, \\ = 0 & \text{if } |x| \geq 3 \end{cases} \qquad b_x^A = \begin{cases} p + q, & \text{if } x = 0, \\ \frac{pq}{d} & \text{if } |x| = 1, \\ 0 & \text{if } |x| \geq 2, \end{cases}$$

Hence, (3.5) fails in this case. On the other hand,

$$\widehat{a}(\theta) = \frac{p}{d} \sum_{j=1}^d \cos\theta_j + q, \quad \widehat{b^A}(\theta) = p + q + \frac{2pq}{d} \sum_{j=1}^d \cos\theta_j.$$

Thus, (3.6) can be verified as follows:

$$\widehat{b^A}(\theta) - |\widehat{a}(\theta)|^2 = p + q - q^2 - \left(\frac{p}{d} \sum_{j=1}^d \cos\theta_j \right)^2 \geq p(1-p) + q(1-q) > 0.$$

Proof of Theorem 3.2.1: We will first prove that for $h \in (0,1)$,

$$P[|\overline{N}_t|^h] = \begin{cases} O(\exp(-ct^{1/3})) & \text{if } d = 1, \\ O(\exp(-c\sqrt{\ln t})) & \text{if } d = 2 \end{cases} \quad \text{as } t \longrightarrow \infty, \tag{3.7}$$

where $c \in (0,\infty)$ is a constant. This implies that $\lim_{t\to\infty} |\overline{N}_t| = 0$, a.s. by Fatou's lemma. To prove (3.7), we will use the following two lemmas, whose proofs are presented in section 3.3. Recall that the spacial distribution of the particle $\rho_{t,x}$ is defined by (1.16).

Lemma 3.2.2 *For $h \in (0,1)$, there is a constant $c \in (0,\infty)$ such that*

$$P\left[1 - U_t^h | \mathcal{F}_{t-1}\right] \geq c |(a * \rho_{t-1})^2| \quad \textit{for all } t \in \mathbb{N}^*,$$

where $U_t = \frac{1}{|a|} \sum_{x,y \in \mathbb{Z}^d} \rho_{t-1,x} A_{t,x,y}$.

Lemma 3.2.3 *For $h \in (0,1)$ and $\Lambda \subset \mathbb{Z}^d$,*

$$|\Lambda| P\left[|\overline{N}_{t-1}|^h |(\overline{a} * \rho_{t-1})^2|\right] \geq P\left[|\overline{N}_{t-1}|^h\right] - 2P^0_S(S_t \notin \Lambda)^h, \tag{3.8}$$

for all $t \in \mathbb{N}^*$, where $((S_t)_{t\in\mathbb{N}}, P^0_S)$ is the random walk in Lemma 1.3.1.

We have

$$|\overline{N}_t| = \frac{1}{|a|} \sum_{x,y\in\mathbb{Z}^d} \overline{N}_{t-1,x} A_{t,x,y} = |\overline{N}_{t-1}| U_t, \tag{3.9}$$

where U_t is from Lemma 3.2.2. We then see from Lemma 3.2.2 that for $h \in (0,1)$

$$P[|\overline{N}_t|^h | \mathcal{F}_{t-1}] - |\overline{N}_{t-1}|^h = |\overline{N}_{t-1}|^h P\left[U_t^h - 1 | \mathcal{F}_{t-1}\right] \leq -c|\overline{N}_{t-1}|^h |(\rho_{t-1} * \overline{a})^2|.$$

We therefore have by Lemma 3.2.3 that

$$\textbf{(1)} \qquad P[|\overline{N}_t|^h] \leq \left(1 - \frac{c}{|\Lambda|}\right) P[|\overline{N}_{t-1}|^h] + \frac{2c}{|\Lambda|} P^0_S(S_t \notin \Lambda)^h.$$

We set $\Lambda = (-\sqrt{t\ell_t}/2, \sqrt{t\ell_t}/2]^d \cap \mathbb{Z}^d$, where $\ell_t = t^{1/3}$ for $d = 1$, and $\ell_t = \sqrt{\ln t}$ for $d = 2$. Then,

$$P^0_S(S_t \notin \Lambda) = P^0_S\left(\left|S_t/\sqrt{t}\right| \geq \sqrt{\ell_t}/2\right) \leq c_1 \exp(-c_2 \ell_t),$$

so that (1) reads,

$$P[|\overline{N}_t|^h] \leq \left(1 - \frac{c}{(t\ell_t)^{d/2}}\right) P[|\overline{N}_{t-1}|^h] + \frac{c_3}{(t\ell_t)^{d/2}} \exp(-c_2\ell_t).$$

By iteration, we conclude (3.7).

It remains to prove (3.3) for $d = 1$. For $d = 1$, we will prove that for $h \in (0,1)$,

$$P[|\overline{N}_t|^h] = O(\exp(-ct)), \quad t \longrightarrow \infty,$$

where $c \in (0,\infty)$ is a constant. Then, (3.3) for $d = 1$ follows from the Borel-Cantelli lemma. Since the left-hand-side is non-increasing in t, it is enough to show that for some $s \in \mathbb{N}^*$,

$$\textbf{(2)} \qquad P[|\overline{N}_{ns}|^h] = O(\exp(-cn)), \quad n \longrightarrow \infty.$$

We write

$$|N_{s+t}| = \sum_y N_{s,y} |N_t^{s,y}| \ \text{ with } \ |N_t^{s,y}| = \sum_{x_1,..,x_t} A_{s+1,y,x_1} A_{s+2,x_1,x_2} \cdots A_{s+t,x_{t-1},x_t}.$$

Thus, for $h \in (0,1)$,

$$|N_{s+t}|^h \leq \sum_y N^h_{s,y} |N_t^{s,y}|^h.$$

Since $|N_t^{s,y}| \stackrel{\text{law}}{=} |N_t|$, we have by (3.7) that

$$\textbf{(3)} \qquad P[|\overline{N}_{s+t}|^h] \leq \sum_y P[\overline{N}^h_{s,y}] P[|\overline{N}_t|^h] \leq c_1 s \exp(-c_2 s^{1/3}) P[|\overline{N}_t|^h] \text{ for all } t \in \mathbb{N}^*.$$

We now take $s \in \mathbb{N}^*$ such that $c_1 s \exp(-c_2 s^{1/3}) < 1$. Then, (2) follows from (3). □

3.3 Proofs of Lemma 3.2.2 and Lemma 3.2.3

We first prepare a general lemma.

Lemma 3.3.1 *Suppose that $(U_n)_{n\in\mathbb{N}}$ be non-negative random variables such that*

$$\begin{aligned}
&\mathrm{cov}(U_m, U_n) = 0 \ \ \textit{if } m \neq n, \\
&\sum_{n\geq 0} P[U_n] = 1, \ \ \sum_{n\geq 0} P[U_n^3] < \infty, \\
&P[(U-1)^3] \leq c_1 \sum_{n\geq 0} \mathrm{var}(U_n),
\end{aligned}$$

where $U = \sum_{n\geq 0} U_n$ and c_1 is a constant. Then, for $h \in (0,1)$, there is a constant $c_2 \in (0,\infty)$ such that

$$\frac{1}{2+c_1} \sum_{n\geq 0} \mathrm{var}(U_n) \leq P\left[\frac{(U-1)^2}{U+1}\right] \leq c_2 P\left[1 - U^h\right].$$

Proof: Since (U_n) are uncorrelated, we have that

$$\begin{aligned}
\sum_{n\geq 0} \mathrm{var}(U_n) &= P[(U-1)^2] = P\left[\frac{U-1}{\sqrt{U+1}}(U-1)\sqrt{U+1}\right] \\
&\leq P\left[\frac{(U-1)^2}{U+1}\right]^{1/2} P\left[(U-1)^2(U+1)\right]^{1/2}
\end{aligned}$$

and that

$$P\left[(U-1)^2(U+1)\right] = P\left[(U-1)^3 + 2(U-1)^2\right] \leq (c_1+2) \sum_{n\geq 0} \mathrm{var}(U_n).$$

Combining these, we get the first inequality. To get the second, we define a function:

$$f(u) = 1 + h(u-1) - u^h, \quad u \in [0,\infty).$$

Note that $P[U] = 1$ and that there is a constant $c_2 \in (0,\infty)$ such that

$$f(u) \geq \frac{1}{c_2}\frac{(u-1)^2}{u+1} \quad \text{for all } u \in [0,\infty).$$

We then see that

$$P\left[1 - U^h\right] = P[f(U)] \geq \frac{1}{c_2} P\left[\frac{(U-1)^2}{U+1}\right].$$

□

Proof of Lemma 3.2.2: We may focus on the event $\{|N_{t-1}| > 0\}$, since the inequality to prove is trivially true on $\{|N_{t-1}| = 0\}$. We write

$$U_t = \sum_{y\in\mathbb{Z}^d} U_{t,y} \text{ with } U_{t,y} = \frac{1}{|a|} \sum_{x\in\mathbb{Z}^d} \rho_{t-1,x} A_{t,x,y}.$$

For fixed $t \in \mathbb{N}^*$, $\{U_{t,y}\}_{y\in\mathbb{Z}^d}$ are non-negative random variables, which are conditionally independent given $\mathcal{F}_{t-1}$. We will prove the lemma by applying Lemma 3.3.1 to these random variables under the conditional probability. The (conditional) expectations and the variances of $\{U_{t,y}\}_{y\in\mathbb{Z}^d}$ are computed as follows:

$$\begin{aligned} m_{t,y} &\stackrel{\text{def.}}{=} P[U_{t,y}|\mathcal{F}_{t-1}] = (\rho_{t-1} * \overline{a})_y, \\ v_{t,y} &\stackrel{\text{def.}}{=} P[(U_{t,y} - m_{t,y})^2|\mathcal{F}_{t-1}] \\ &= \frac{1}{|a|^2}\sum_{x_1,x_2\in\mathbb{Z}^d} \rho_{t-1,x_1}\rho_{t-1,x_2}\text{cov}(A_{t,x_1,y}, A_{t,x_2,y}). \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{y\in\mathbb{Z}^d} m_{t,y} &= |\rho_{t-1} * \overline{a}| = 1, \\ \sum_{y\in\mathbb{Z}^d} v_{t,y} &= \frac{1}{|a|^2}\sum_{x_1,x_2,y\in\mathbb{Z}^d} \rho_{t-1,x_1}\rho_{t-1,x_2}\text{cov}(A_{t,x_1,y}, A_{t,x_2,y}) \\ &\stackrel{(3.2)}{\geq} c_0|(\rho_{t-1} * \overline{a})^2|. \end{aligned} \tag{3.10}$$

We will check that there exists $c_1 \in (0,\infty)$ such that

(1) $\quad P[(U_t - 1)^3|\mathcal{F}_{t-1}] \leq c_1 \sum_{y\in\mathbb{Z}^d} v_{t,y}$ for all $t \in \mathbb{N}^*$.

Then, the lemma follows from Lemma 3.3.1 and (3.10). There exists $c_2 \in (0,\infty)$ such that

(2) $\quad P[A^3_{1,0,y}] \leq c_2 a_y^3$ for all $y \in \mathbb{Z}^d$.

This can be seen as follows: Note that $a_y = 0 \Leftrightarrow A_{1,0,y} = 0$, a.s. This implies that, for each $y \in \mathbb{R}^d$, there is $c_y \in [0,\infty)$ such that $P[A^3_{1,0,y}] = c_y a_y^3$. Therefore, we have (2) with $c_2 = \sup_{|y|\leq r_A} c_y$ (cf. (1.6)). By (2), we get

$$\begin{aligned} P[U^3_{t,y}|\mathcal{F}_{t-1}] &= \frac{1}{|a|^3}\sum_{x_1,x_2,x_3\in\mathbb{Z}^d}\left(\prod_{j=1}^3 \rho_{t-1,x_j}\right) P\left[\prod_{j=1}^3 A_{t,x_j,y}\right] \\ &\stackrel{\text{Hölder}}{\leq} c_2 \sum_{x_1,x_2,x_3\in\mathbb{Z}^d}\left(\prod_{j=1}^3 \rho_{t-1,x_j}\overline{a}_{y-x_j}\right) = c_2(\rho_{t-1} * \overline{a})^3_y. \end{aligned} \tag{3.11}$$

Consequently, (1) can be verified as follows:

$$\begin{aligned} P[(U_{t,y} - 1)^3|\mathcal{F}_{t-1}] &= \sum_{y\in\mathbb{Z}^d} P[(U_{t,y} - m_{t,y})^3|\mathcal{F}_{t-1}] \\ &\leq 3\sum_{y\in\mathbb{Z}^d}(P[U^3_{t,y}|\mathcal{F}_{t-1}] + m^3_{t,y}) \\ &\stackrel{(3.11)}{\leq} c_3\sum_{y\in\mathbb{Z}^d}(\rho_{t-1} * \overline{a})^3_y \stackrel{(3.10)}{\leq} \frac{c_3}{c_0}\sum_{y\in\mathbb{Z}^d} v_{t,y}. \end{aligned}$$

□

Proof of Lemma 3.2.3: We have on the event $\{|N_{t-1}| > 0\}$ that

$$|\Lambda||(\rho_{t-1} * \overline{a})^2| \geq |\Lambda|\sum_{z\in\Lambda}(\rho_{t-1} * \overline{a})^2_y \geq \left(\sum_{y\in\Lambda}(\rho_{t-1} * \overline{a})_y\right)^2$$

$$
\begin{aligned}
&= \left(1-\sum_{y\notin\Lambda}(\rho_{t-1}*\overline{a})_y\right)^2 \geq 1-2\sum_{y\notin\Lambda}(\rho_{t-1}*\overline{a})_y \\
&\geq 1-2\left(\sum_{y\notin\Lambda}(\rho_{t-1}*\overline{a})_y\right)^h . \qquad (3.12)
\end{aligned}
$$

Note also that

$$
\begin{aligned}
P\left[\left(|\overline{N}_{t-1}|\sum_{y\notin\Lambda}(\rho_{t-1}*\overline{a})_y\right)^h\right] &\leq P\left[|\overline{N}_{t-1}|\sum_{y\notin\Lambda}(\rho_{t-1}*\overline{a})_y\right]^h \\
&= P\left[\sum_{y\notin\Lambda}(\overline{N}_{t-1}*\overline{a})_y\right]^h = P(S_t\notin\Lambda)^h, \qquad (3.13)
\end{aligned}
$$

where the last equality comes from Lemma 1.3.1. We therefore see that

$$
\begin{aligned}
|\Lambda|P\left[|\overline{N}_{t-1}|^h|(\rho_{t-1}*\overline{a})^2|\right] &\overset{(3.12)}{\geq} P\left[|\overline{N}_{t-1}|^h\right]-2P\left[\left(|\overline{N}_{t-1}|\sum_{y\notin\Lambda}(\rho_{t-1}*\overline{a})_y\right)^h\right] \\
&\overset{(3.13)}{\geq} P\left[|\overline{N}_{t-1}|^h\right]-2P_S^0(S_t\notin\Lambda)^h.
\end{aligned}
$$

□

4 Dual processes

In this section, we associate a dual object to the process $(N_t)_{t\in\mathbb{N}}$ and thereby investigate invariant measures for $(\overline{N}_t)_{t\in\mathbb{N}}$. This can be considered as a discrete analogue of the duality theory for the continuous-time linear systems in the book by T. Liggett [10, Chapter IX].

4.1 Dual processes and invariant measures

We define a Markov chain $(M_t)_{t\in\mathbb{N}}$ with values in $[0,\infty)^{\mathbb{Z}^d}$ by

$$
\sum_{x\in\mathbb{Z}^d} A_{t,y,x}M_{t-1,x} = M_{t,y}, \quad t\in\mathbb{N}, \qquad (4.1)
$$

where the initial state $M_0\in[0,\infty)^{\mathbb{Z}^d}$ is a non-random and finite (cf. (1.10)). We refer $(M_t)_{t\in\mathbb{N}}$ as the *dual process* of $(N_t)_{t\in\mathbb{N}}$ defined by (1.9). Regarding (M_t) as column vectors, we can interpret (4.1) as:

$$
M_t = A_tA_{t-1}\cdots A_1M_0.
$$

The dual process can also be understood as being defined in the same way as (1.9), except that the matrix A_t is replaced by its transpose: $A_t^* = (A_{t,y,x})_{(x,y)\in\mathbb{Z}^d\times\mathbb{Z}^d}$.

By the same proof as Lemma 1.3.1, we have:

Lemma 4.1.1

$$
P[M_{t,y}] = |a|^t\sum_{x\in\mathbb{Z}^d} M_{0,x}P_S^x(-S_t=y),
$$

where $((S_t)_{t\in\mathbb{N}}, P_S^x)$ is the random walk in Lemma 1.3.1. Moreover, $(|\overline{M}_t|, \mathcal{F}_t)_{t\in\mathbb{N}}$ is a martingale, where we have defined $\overline{M}_t = \left(\overline{M}_{t,x}\right)_{x\in\mathbb{Z}^d}$ by

$$\overline{M}_{t,x} = |a|^{-t} M_{t,x}. \tag{4.2}$$

Also, Lemma 1.3.2 holds true with $\overline{N}_t$ replaced by $\overline{M}_t$. Accordingly, we have the definition of *regular/slow growth phase* for the dual process in the same way as for the (N_t)-process. For $(s,z) \in \mathbb{N}\times\mathbb{Z}^d$, we define $M_t^{s,z} = (M_{t,y}^{s,z})_{y\in\mathbb{Z}^d}$ and $\overline{M}_t^{s,z} = (\overline{M}_{t,y}^{s,z})_{y\in\mathbb{Z}^d}$, $t\in\mathbb{N}$ respectively by

$$\begin{aligned} &M_{0,y}^{s,z} = \delta_{z,y}, \quad N_{t+1,y}^{s,z} = \sum_{x\in\mathbb{Z}^d} M_{t,x}^{s,z} A_{s+t+1,y,x}, \\ &\text{and} \quad \overline{M}_{t,y}^{s,z} = |a|^{-t} M_{t,y}^{s,z}. \end{aligned} \tag{4.3}$$

$(N_t)_{t\in\mathbb{N}}$ and $(M_t)_{t\in\mathbb{N}}$ are dual to each other in the following sense:

Lemma 4.1.2 *For each fixed $t\in\mathbb{N}^*$,*

$$\left(N_{t,y}^{0,x}\right)_{(x,y)\in\mathbb{Z}^d\times\mathbb{Z}^d} \stackrel{\text{law}}{=} \left(M_{t,x}^{0,y}\right)_{(x,y)\in\mathbb{Z}^d\times\mathbb{Z}^d}. \tag{4.4}$$

Proof: We have

$$\begin{aligned} M_{t,x}^{0,y} &= \sum_{x_1,\ldots,x_{t-1}\in\mathbb{Z}^d} A_{t,y,x_1} A_{t-1,x_1,x_2}\cdots A_{2,x_{t-2},x_{t-1}} A_{1,x_{t-1},x} \\ &\stackrel{\text{law}}{=} \sum_{x_1,\ldots,x_{t-1}\in\mathbb{Z}^d} A_{1,y,x_1} A_{2,x_1,x_2}\cdots A_{t-1,x_{t-2},x_{t-1}} A_{t,x_{t-1},x} = N_{t,y}^{0,x}. \end{aligned}$$

This shows that the left-hand-side of (4.4) is obtained from the right-hand-side by the measure-preserving transform $(A_1, A_2, .., A_t) \mapsto (A_t, A_{t-1}, .., A_1)$. □

The following result show that the structure of invariant measures of $(\overline{N}_t)$ depends on whether the dual process (M_t) is in the regular or slow growth phase. To state the theorem, it is convenient to introduce the following notation: Let $\mathcal{P}([0,\infty)^{\mathbb{Z}^d})$ be the set of Borel probability measures on $[0,\infty)^{\mathbb{Z}^d}$, and

$$\begin{aligned} \mathcal{I} &= \{\mu\in\mathcal{P}([0,\infty)^{\mathbb{Z}^d})\ ;\ \mu \text{ is invariant for the Markov chain } (\overline{N}_t)\}, \\ \mathcal{S} &= \{\mu\in\mathcal{P}([0,\infty)^{\mathbb{Z}^d})\ ;\ \mu \text{ is invariant with respect to the shift of } \mathbb{Z}^d\}. \end{aligned}$$

Theorem 4.1.3 **a)** *Suppose that $P[|\overline{M}_\infty^{0,0}|] = 1$. Then, for each $\alpha\in[0,\infty)$, there is a $\nu_\alpha \in \mathcal{I}\cap\mathcal{S}$ such that*

$$\int_{[0,\infty)^{\mathbb{Z}^d}} \eta_0 d\nu_\alpha(\eta) = \alpha. \tag{4.5}$$

Moreover, ν_α is extremal in $\mathcal{I}\cap\mathcal{S}$.

b) *Suppose on the contrary that $P[|\overline{M}_\infty^{0,0}|] = 0$. Then,*

$$\left\{\mu\in\mathcal{I}\cap\mathcal{S},\ ;\ \int_{[0,\infty)^{\mathbb{Z}^d}} \eta_0 d\mu(\eta) < \infty\right\} = \{\delta_{\mathbf{0}}\},$$

where $\delta_{\mathbf{0}}$ is the unit point mass on $\mathbf{0} = (0)_{x\in\mathbb{Z}^d}$.

Proof: a): Let $(N^{\mathbf{1}}_t)_{t\in\mathbb{N}}$ be the (N_t)-process such that $N^{\mathbf{1}}_{0,x} \equiv 1$ for all $x \in \mathbb{Z}^d$. We have by Lemma 4.1.2 that

$$\alpha\overline{N}^{\mathbf{1}}_t \stackrel{\text{law}}{=} (\alpha|\overline{M}^{0,y}_t|)_{y\in\mathbb{Z}^d},$$

where $\alpha\overline{N}^{\mathbf{1}}_t = \left(\alpha\overline{N}^{\mathbf{1}}_{t,y}\right)_{y\in\mathbb{Z}^d}$. Since the right-hand-side converges a.s. to $(\alpha|\overline{M}^{0,y}_\infty|)_{y\in\mathbb{Z}^d}$ as $t\to\infty$, we see that the weak limit

$$\nu_\alpha \stackrel{\text{def.}}{=} \lim_{t\to\infty} P\left(\alpha\overline{N}^{\mathbf{1}}_t \in \cdot\right),$$

exists and that

(1) $\nu_\alpha = P\left((\alpha|\overline{M}^{0,y}_\infty|)_{y\in\mathbb{Z}^d} \in \cdot\right)$.

We see $\nu_\alpha \in \mathcal{I}$ from the way ν_α is defined. Also, $\nu_\alpha \in \mathcal{S}$, since $P\left(\alpha\overline{N}^{\mathbf{1}}_t \in \cdot\right) \in \mathcal{S}$ for any $t \in \mathbb{N}^*$ by (1.7). Moreover, (1) implies (4.5). The extremality of ν_α follows from the same argument as in [10, page 437, Corollary 2.1.5].
b): This follows from the same argument as in [10, page 435, Theorem 2.7]. □

4.2 Regular/slow growth for the dual process

In this subsection, we adapt arguments from sections 2 and 3 to obtain sufficient conditions for regular/slow growth phases the dual process. A motivation to investigate these sufficient conditions is explained by Theorem 4.1.3.

We let $(S,\widetilde{S}) = ((S_t,\widetilde{S}_t)_{t\in\mathbb{N}}, P^{x,\widetilde{x}}_{S,\widetilde{S}})$ denote the independent product of the random walks in Lemma 1.3.1. We have the following Feynman-Kac formula for the two-point functions of the dual process. The proof is the same as that of Lemma 2.1.1.

Lemma 4.2.1

$$P[M_{t,y}M_{t,\widetilde{y}}] = |a|^{2t} \sum_{x,\widetilde{x}\in\mathbb{Z}^d} M_{0,x}M_{0,\widetilde{x}} P^{x,\widetilde{x}}_{S,\widetilde{S}}[e^*_t : (-S_t,-\widetilde{S}_t) = (y,\widetilde{y})] \quad \textit{for all } y,\widetilde{y}\in\mathbb{Z}^d, \tag{4.6}$$

where

$$e^*_t = \prod_{u=1}^t w(-S_u,-\widetilde{S}_u,-S_{u-1},-\widetilde{S}_{u-1}), \quad (\textit{cf. (2.2)}). \tag{4.7}$$

Consequently,

$$P[|\overline{N}_t|^2] = \sum_{x,\widetilde{x}\in\mathbb{Z}^d} M_{0,x}M_{0,\widetilde{x}} P^{x,\widetilde{x}}_{S,\widetilde{S}}[e^*_t], \tag{4.8}$$

and

$$\sup_{t\in\mathbb{N}} P[|\overline{M}_t|^2] < \infty \iff \sup_{t\in\mathbb{N}} P^{0,0}_{S,\widetilde{S}}[e^*_t] < \infty \tag{4.9}$$

$$\Longrightarrow P[|\overline{M}_\infty|] = |M_0|. \tag{4.10}$$

Lemma 4.2.1 can be used to obtain the following criteria for slow growth for GOSP, DPRE, GOBP as in (2.15), (2.17) and (2.19):

$$\sup_{t\in\mathbb{N}} P[|\overline{M}_t|^2] < \infty \iff d\geq 3 \text{ and } \begin{cases} p>\pi_0 & \text{for OSP,}\\ \pi_0 + \frac{2dp(1-p)+q(1-q)}{(2dp+q)^2} < 1 & \text{for GOBP,}\\ \lambda(2\beta)-2\lambda(\beta) < \ln(1/\pi_0) & \text{for DPRE.}\end{cases} \tag{4.11}$$

$$\sup_{t\in\mathbb{N}} P[|\overline{M}_t|^2] < \infty \Longleftarrow d\geq 3 \text{ and } p\wedge q > \pi_0 \text{ for GOSP with } q\neq 0. \tag{4.12}$$

Let us now turn to sufficient conditions for the dual process to be in the slow growth phase. We first note that exactly the same statement as Theorem 3.1.1 holds true with $\overline{N}_t$ replaced by $\overline{M}_t$, since the proof works for the dual process without change. In particular,

$$|\overline{M}_t| = O(e^{-ct}), \quad \text{as } t \to \infty, \text{ a.s. if} \begin{cases} 2dp + q < 1 & \text{for GOSP and GOBP,} \\ \beta\lambda'(\beta) - \lambda(\beta) > \ln(2d) & \text{for DPRE,} \\ p + q < 1 & \text{for BCPP.} \end{cases}$$

In analogy with Theorem 3.2.1, we have:

Theorem 4.2.2 *Let $d = 1, 2$. Suppose that $P[A^3_{1,0,y}] < \infty$ for all $y \in \mathbb{Z}^d$ and that*

$$\text{the random variable } \sum_{x \in \mathbb{Z}^d} A_{1,x,0} \text{ is not a constant a.s.} \tag{4.13}$$

Then, almost surely,

$$|\overline{M}_t| \begin{cases} = O(\exp(-ct)) & \text{if } d = 1, \\ \longrightarrow 0 & \text{if } d = 2 \end{cases} \quad \text{as } t \longrightarrow \infty, \tag{4.14}$$

where $c > 0$ is a non-random constant.

To explain the proof of Theorem 4.2.2, we introduce

$$\begin{aligned} V_t &= \frac{1}{|a|} \sum_{x,y \in \mathbb{Z}^d} \rho^*_{t-1,y} A_{t,x,y}, \quad t \in \mathbb{N}^*, \\ \text{where} \quad & \rho^*_{t-1,x} = \mathbf{1}_{\{|M_{t-1}|>0\}} M_{t-1,x}/|M_{t-1}|. \end{aligned} \tag{4.15}$$

We then have $|\overline{M}_t| = V_t|\overline{M}_{t-1}|$, $t \in \mathbb{N}^*$. Using this relation instead of (3.9), we can show Theorem 4.2.2 in the same way as Theorem 3.2.1, except that we replace Lemma 3.2.2 by Lemma 4.2.3 below.

Lemma 4.2.3 *For $h \in (0, 1)$, there is a constant $c \in (0, \infty)$ such that*

$$P\left[1 - V_t^h|\mathcal{F}_{t-1}\right] \geq c|(\rho^*_{t-1})^2| \quad \textit{for all } t \in \mathbb{N}^*.$$

Proof: We may focus on the event $\{|M_{t-1}| > 0\}$, since the inequality to prove is trivially true on $\{|M_{t-1}| = 0\}$. By the last part of the proof of Lemma 3.3.1, we see that there exists a constant $c_1 \in (0, \infty)$ such that

(1) $P\left[1 - V_t^h|\mathcal{F}_{t-1}\right] \geq c_1 P\left[\frac{(V_t - 1)^2}{V_t + 1}|\mathcal{F}_{t-1}\right]$ for all $t \in \mathbb{N}^*$.

We write

$$V_t = \sum_{x \in \mathbb{Z}^d} \rho^*_{t-1,y} V_{t,y} \text{ with } V_{t,y} = \frac{1}{|a|} \sum_{x \in \mathbb{Z}^d} A_{t,x,y}.$$

For fixed $t \in \mathbb{N}^*$, $\{V_{t,y}\}_{y \in \mathbb{Z}^d}$ are non-negative random variables, which are i.i.d. with mean one, given $\mathcal{F}_{t-1}$. Furthermore, $V_{t,y}$ is not a constant a.s., because of (4.13). We therefore see from [3, Lemma 2.1] that there exists a constant $c_2 \in (0, \infty)$ such that

$$P\left[\frac{(V_t - 1)^2}{V_t + 1}|\mathcal{F}_{t-1}\right] \geq c_2|(\rho^*_{t-1})^2| \quad \text{for all } t \in \mathbb{N}^*,$$

which, together with (1), proves the lemma. □

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References

[1] Bolthausen, E.: A note on diffusion of directed polymers in a random environment, Commun. Math. Phys. **123**, 529–534, (1989).

[2] Carmona, P., Hu Y.: On the partition function of a directed polymer in a random environment, Probab.Theory Related Fields **124** (2002), no. 3, 431–457.

[3] Comets, F., Shiga, T., Yoshida, N. Directed Polymers in Random Environment: Path Localization and Strong Disorder, Bernoulli, **9**(3), 2003, 705–723.

[4] Comets, F., Shiga, T., Yoshida, N. Probabilistic analysis of directed polymers in random environment: a review, Advanced Studies in Pure Mathematics, **39**, 115–142, (2004).

[5] Comets, F., Vargas, V.: Majorizing multiplicative cascades for directed polymers in random media. ALEA Lat. Am. J. Probab. Math. Stat. 2 (2006) 267–277.

[6] Darling, R. W. R. : The Lyapunov exponent for products of random matrices. Springer Lecture Notes in Mathematics, **1486**, 206–215, (1990).

[7] Durrett, R. :"Probability–Theory and Examples", 3rd Ed., Brooks/Cole–Thomson Learning, 2005.

[8] Griffeath, D.: The Binary Contact Path Process, Ann. Probab. Volume 11, Number 3 (1983), 692-705.

[9] Hu, Y., Yoshida, N. : Localization for Branching Random Walks in Random Environment, to appear in Stoch. Proc. Appl.,2008, arXiv:0712.0649

[10] Liggett, T. M. : "Interacting Particle Systems", Springer Verlag, Berlin-Heidelberg-Tokyo (1985).

[11] Nagahata, Y., Yoshida, N.: Central Limit Theorem for a Class of Linear Systems, preprint, arXiv:0805.0342 (2008).

[12] Nakashima, M.: The Central Limit Theorem for Linear Stochastic Evolutions, in preparation.

[13] Song, R. and Zhou, X. Y. : A Remark on Diffusion on Directed Polymers in Random Environment, J. Stat. Phys. **85**, Nos.1/2, 277–289, (1996).

[14] Schulman, L. S., Seiden, P. E.: Percolation and galaxies. Science 223, 425–431 (1986).

[15] Spitzer, F.: "Principles of Random Walks", Springer Verlag, New York, Heiderberg, Berlin (1976).

[16] van Lien, N., Shklovskii, B. I. : Hopping conduction in strong electric fields and directed percolation, Solid State Communications **38**, 99–102, (1981).

[17] Yoshida, N.: Central Limit Theorem for Branching Random Walk in Random Environment, Ann. Appl. Proba., Vol. 18, No. 4, 1619–1635, 2008.

[18] Yoshida, N.: Localization for Linear Stochastic Evolutions, preprint, 2008.