Localization transition of d-friendly walkers

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Abstract

Friendly walkers is a stochastic model obtained from independent one-dimensional simple random walks $\{S_j^k\}_{j\geq 0}$, $k = 1, 2, \ldots, d$ by introducing "non-crossing condition": $S_j^1 \leq S_j^2 \leq \ldots \leq S_j^d, j = 1, 2, \ldots, n$ and "reward for collisions" characterized by parameters $\beta_2, \ldots, \beta_d \geq 0$. Here, the reward for collisions is described as follows. If, at a given time n, a site in $\mathbb{Z}$ is occupied by exactly $m \geq 2$ walkers, then the site increases the probabilistic weight for the walkers by multiplicative factor $\exp(\beta_m) \geq 1$. We study the localization transition of this model in terms of the positivity of the free energy and describe the location and the shape of the critical surface in the $(d-1)$-dimensional space for the parameters $(\beta_2, \ldots, \beta_d)$.

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Contents

1 Introduction

1.1 The model

Friendly walkers is a stochastic model studied in connection with the Domany-Kinzel model, directed percolation, wetting and various other models; See [2, 5, 7] and references therein. Roughly speaking, the d-friendly walkers (of the length n) is obtained from independent one-dimensional simple random walks $\{S_j^k\}_{j\geq 0}$, $k = 1, 2, \dots, d$ by introducing the following additional rules:

- Non-crossing condition; the walkers are conditioned to preserve the order $S_j^1 \leq S_j^2 \leq \dots \leq S_j^d$, $j = 1, 2, \dots, n$. This restriction makes the walkers repel each other to avoid violating the order.

- Reward for collisions; We introduce an attractive interaction among the walkers characterized by parameters $\beta_2, \dots, \beta_d \geq 0$ as follows. If, at a given time n, a site in $\mathbb{Z}$ is occupied by exactly $m \geq 2$ walkers, then the site increases the probabilistic weight for the walkers by multiplicative factor $\exp(\beta_m) \geq 1$.

The localization transition we will be discussing in this paper is the consequence of the above two competing effects.

To give a precise definition of this model, we start by introducing a d-dimensional random walk (S_j, P_d^x) such that the coordinates $\{S_j^k\}_{j\geq 0}$, $k = 1, 2, \dots, d$, are independent simple random walks on $\mathbb{Z}$. To be consistent with the non-crossing condition and to ensure the possibility of collisions for $d \geq 2$, we always take the starting point x from the set;

$$\mathbb{Z}^d_{\leq} \stackrel{\text{def.}}{=} \{x = (x^k)_{k=1}^d \in \mathbb{Z}^d \ ; \ \frac{x^{k+1} - x^k}{2} \in \mathbb{N}, \ j = 1, \dots, d\}, \tag{1.1}$$

where $\mathbb{N} = \{0, 1, 2, \dots\}$. For $d = 1$, we agree with the convention: $\mathbb{Z}^1_{\leq} = \mathbb{Z}^1$. We will refer to the number d as the "dimension" of the model.

The reward for collisions is described by a parameter $\beta = (\beta_2, \beta_3, \dots, \beta_d) \in [0, \infty)^{d-1}$ and the parameter comes into play with the random walk (S_j, P_d^x) as follows. We define the *multiplicity* of a site $z \in \mathbb{Z}$ for a state $x \in \mathbb{Z}^d$ by

$$m(x, z) = \sharp\{1 \leq k \leq d : x^k = z\}. \tag{1.2}$$

We then define

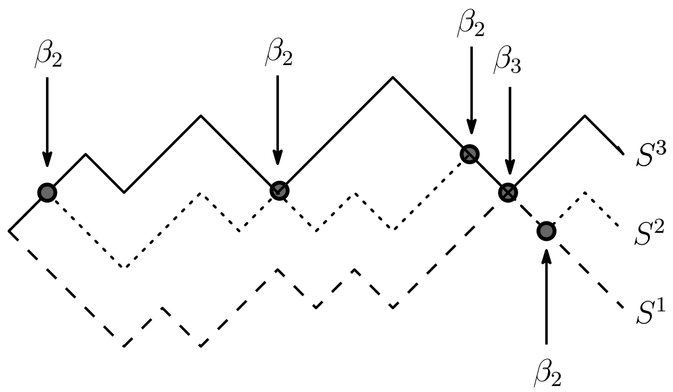

Figure 1: *An example of 3-friendly walkers.*

$$
\begin{aligned}
\chi_j &= \sum_{z \in \mathbb{Z}\,:\, m(S_j,z)\geq 2} \beta_{m(S_j,z)}, && (1.3)\\
L_n &= \sum_{j=0}^{n-1} \chi_j, \quad n \geq 1, && (1.4)\\
z_{n,d}(\beta) &= \exp(L_n) 1\{S_j \in \mathbb{Z}^d_{\leq}, j = 1, 2, \ldots, n\}, \quad n \geq 1, && (1.5)\\
z_{0,d}(\beta) &= 1, && (1.6)
\end{aligned}
$$

where $1\{\cdots\}$ denotes the indicator function.

In this paper, we are concerned with the existence and the the positivity of the *free energy*:

$$\psi_d(\beta) = \lim_{n\to\infty} \frac{1}{n} \ln Z^x_{n,d}(\beta),$$

where $Z^x_{n,d}(\beta)$ is the partition function

$$Z^x_{n,d}(\beta) = P^x_d[z_{n,d}(\beta)]. \tag{1.7}$$

Note that $z_{n,1} \equiv 1$ for $d = 1$ and hence trivially, $\psi_1 \equiv 0$. We sometimes drop parameters d and β from the notations, if it does not generate confusion.

Definition 1.1 The system is said to be *localized* if $\psi_d(\beta) > 0$ and *delocalized* if $\psi_d(\beta) = 0$.

Plausibility of this terminology might be explained as follows. Consider a probability measure μ^x_n, $x \in \mathbb{Z}^d_{\leq}$, defined by

$$\mu^x_n(d\omega) = \frac{1}{Z^x_{n,d}(\beta)} P^x_d[z_{n,d}(\beta) : d\omega]. \tag{1.8}$$

We look at the paths under this probability measure. Then, as is usual the case with models in statistical mechanics, we see competition of energy ($=-L_n$, in this case) and entropy.

- The entropy is maximized when the walkers travel separately as they would do if $\beta_2 = \ldots = \beta_d = 0$, with only a small number of collisions which can be ignored in a macroscopic scale. In this case, the "width" $S^{j+1}_n - S^j_n$, $1 \leq j \leq d-1$, should diverge as $n \nearrow \infty$ (delocalization). On the other hand, this strategy does not let the walkers pick up much reward, and therefore, can be optimal only when β_k's are small so that the gain in entropy makes up the loss in energy.

- The strategy for walkers to minimize the energy (i.e., maximize the reward) is to travel together, so that they can collect as much reward as possible. In this case, the "width" of the group of walkers should remain small as $n \nearrow \infty$ (localization). On the other hand, this strategy lowers the entropy considerably, and therefore, can be optimal only when β_k's are large so that the gain in energy exceeds the loss in entropy.

Then problem now is to determine which strategy becomes "typical" depending on the choice of β_k's. The answer to this question is believed to be given by the positivity of the free energy mentioned above. In fact, it is known [3, 4] for $d = 2$ that

$$\psi_2(\beta_2) = \begin{cases} 0, & \text{if } \beta_2 \le \ln \frac{4}{3}, \\ \ln\{\frac{e^{\beta_2}}{4}(1 + \sqrt{\frac{e^{\beta_2}}{e^{\beta_2}-1}})\} > 0, & \text{if } \beta_2 > \ln \frac{4}{3}. \end{cases} \tag{1.9}$$

The corresponding pathwise descriptions are obtained by Isozaki and Yoshida [4] as follows;

- For $\beta_2 \le \ln(4/3)$, the width $(S_j^2 - S_j^1)_{j=1}^n$ diverges like $\sqrt{n}$, and, if properly scaled (i.e., divided by $\sqrt{n}$), converges to Brownian meander if $\beta_2 < \ln(4/3)$ and to reflecting Brownian motion if $\beta_2 = \ln(4/3)$.

- For $\beta_2 > \ln(4/3)$, the profile of the width $(S_j^2 - S_j^1)_{j=1}^n$ remains bounded and converges to an exponentially mixing Markov chain.

For higher dimensions, we have a set of thermodynamic parameters $(\beta_2, \dots, \beta_d)$, so that we should have a critical surface in $[0, \infty)^{d-1}$ as the boundary between the delocalization and localization region. In this paper, we describe the shape and the location of the critical surface (Theorem 1.2) by studying how the free energy depends on the parameters (Theorem 1.1). In some situation, the information we obtain on the critical surface is good enough to determine exactly when localization occurs, e.g., in $d = 3$ (cf. Figure2) and in Corollary 1.1 below.

Remark 1.1 Consider the mesure μ_n without the reward for collisions, i.e., $\beta_2 = \ldots = \beta_d = 0$. In this setting, Katori and Tanemura [6] recently prove a functional central limit theorem for the process $(S_j)_{1\leq j\leq n}$ for arbitrary $d \geq 2$ with the non-intersecting Brownian motion as the scaling limit. We expect the same limit theorem for all β in the interior of the delocalized region.

Remark 1.2 Our original formulation of the friendly walkers was based on a d-dimensional random walk conditioned to stay above diagonal. We remark that the model can be reformulated in terms of a $(d-1)$-dimensional nearest neighbor random walk conditioned to stay in the first quadrant $\mathbb{N}^{d-1}$. Define a map $\Upsilon_d : \mathbb{R}^d \longrightarrow \mathbb{R}^{d-1}$ by

$$\Upsilon_d(y^1, y^2, \ldots, y^d) = (\frac{y^2 - y^1}{2}, \frac{y^3 - y^2}{2}, \ldots, \frac{y^{d+1} - y^d}{2}).$$

Then $(\Upsilon_d S_j)_{j\geq 1}$ is a $(d-1)$-dimensional nearest neighbor random walk. The non-crossing condition reads:

$$\Upsilon_d S_j \in \mathbb{N}^{d-1}, \quad j = 1, 2, \ldots, n. \tag{1.10}$$

In this way, the friendly walkers model can be translated into a random walk in the first quadrant with attractive interactions with the boundary $\partial\mathbb{N}^{d-1} = \cup_{i=1}^{d-1}\{x \in \mathbb{N}^{d-1} \ ; \ x^i = 0\}$.

1.2 Main Results

For $d \geq 2$ and $n \geq 1$, a vector $\mathbf{k} = (k_u)_{u=1}^n \in \{1, 2, \ldots, d\}^n$ is said to be a *partition* of d with length n, if $\sum_{1\leq u\leq n} k_u = d$. The length of a partition $\mathbf{k}$ is denoted by $n(\mathbf{k})$. In particular, the number d in itself can be considered as an partition of d with $n(d) = 1$. For a partition $\mathbf{k}$ of d, we introduce an event

$$\Delta_{n,\mathbf{k}} = \bigcap_{\alpha=1}^{n(\mathbf{k})}\{S_n^i = S_n^{i'} \text{ if } \sum_{u=1}^{\alpha-1} k_u + 1 \leq i \leq i' \leq \sum_{u=1}^{\alpha} k_u\}. \tag{1.11}$$

In particular, $\Delta_{n,d} = \{S_n^1 = S_n^2 = \ldots = S_n^d\}$.

Theorem 1.1 *Let $d \geq 2$ and $\beta = (\beta_2, \beta_3, \ldots, \beta_d) \in [0, \infty)^{d-1}$.*

(a) *The following limit exists and is independent of a partition* $\mathbf{k}$ *of* d *and an initial configuration* $x \in \mathbb{Z}^d_{\leq}$*;*

$$\psi_d(\beta) = \lim_{n\to\infty} \frac{1}{n} \ln P^x_{n,d}[z_{n,d}(\beta) : \Delta_{n,\mathbf{k}}]. \tag{1.12}$$

In particular,

$$\psi_d(\beta) = \lim_{n\to\infty} \frac{1}{n} \ln Z^x_{n,d}(\beta). \tag{1.13}$$

(b) *It holds for any partition* $\mathbf{k}$ *of* d *that*

$$\psi_d(\beta) \geq \sum_{\alpha=1}^{n(\mathbf{k})} \psi_{k_\alpha}(\beta_2, \beta_3, \ldots, \beta_{k_\alpha}). \tag{1.14}$$

Although ψ_k, $1 \leq k \leq d$ is a function of $(\beta_2, \beta_3, \ldots, \beta_k)$, we often regard them as functions of $\beta = (\beta_2, \beta_3, \ldots, \beta_d)$.

Inequality (1.14) is the main point in this paper. It enables us to describe the shape and location of the critical surface as follows. Let η be the first hitting time to the diagonal set;

$$\eta = \inf\{n \geq 1 \ : \ S_n \in \mathbb{Z}^d_{\text{diag.}}\}, \tag{1.15}$$

where $\mathbb{Z}^d_{\text{diag.}} = \{x \in \mathbb{Z}^d \ \ x^1 = x^2 = \ldots = x^d\}$. We then introduce the following power series in $s \in [0,1]$;

$$\widehat{W}_{s,d}(\beta_2, \ldots, \beta_{d-1}) = \sum_{n\geq 1} s^n P^0_d[z_{n,d}(\beta_2, \ldots, \beta_{d-1}, 0) : \eta = n] \in (0, \infty]. \tag{1.16}$$

Theorem 1.2 *For* $d \geq 2$*, define a concave, decreasing function* $\beta^{\text{crit}}_d : [0,\infty)^{d-2} \longrightarrow [-\infty, \infty)$ *by*

$$\beta^{\text{crit}}_d(\beta_2, \ldots, \beta_{d-1}) = \begin{cases} \ln \frac{4}{3}, & \text{if } d = 2, \\ -\ln\left(\widehat{W}_{1,d}(\beta_2, \ldots, \beta_{d-1})\right) \in [-\infty, \infty), & \text{if } d \geq 3. \end{cases} \tag{1.17}$$

Then, it enjoys the following properties;

(a)

$$\beta^{\text{crit}}_d(\beta^*_2, \ldots, \beta^*_{d-1}) \geq \beta^*_d, \quad \text{where } \ \beta^*_d = \ln \frac{2^d}{d+1} > 0. \tag{1.18}$$

(b) $\psi_d(\beta_2,\dots,\beta_d) > 0$ *if and only if* $\beta_d > \beta_d^{\mathrm{crit}}(\beta_2,\dots,\beta_{d-1})$.

(c) $\psi_d(\beta_2,\dots,\beta_d) > 0$ *if* $\beta_k > \beta_k^{\mathrm{crit}}(\beta_2,\dots,\beta_{k-1})$ *for some* $k = 2,\dots,d$.

Remark 1.3 The important point here is not just the existence of the localization transition, but the information we get on the precise location and the shape of the critical surface. In fact, it is not difficult to prove by a simple perturbative argument that, if *all* β_k's are small (resp. large), then $\psi_d(\beta) = 0$ (resp. $\psi_d(\beta) > 0$). The argument of this kind, however, does not seem to provide any information on the precise location or the shape of the critical surface.

Remark 1.4 The meaning of β_d^* can best be explained by the following identity whose proof is elementary:

$$P_d^{(x^1,\dots,x^d)}\{S_1 \in \mathbb{Z}^d_{\le}\} = \prod_{z\in\mathbb{Z}^d} \frac{m(x,z)+1}{2^{m(x,z)}}. \tag{1.19}$$

We see from (1.19) that, in Z_n with $\beta = (\beta_k^*)_{k=2}^d$, the amount of the mass annihilated by non-crossing restriction is exactly compensated by the creation due to L_n.

Remark 1.5 Part (a) of the above theorem can be made more precise;

$$\beta_d^{\mathrm{crit}}(\beta_2^*,\dots,\beta_{d-1}^*) \begin{cases} = \beta_d^* & \text{if } d = 2,3, \\ > \beta_d^* & \text{if } d \ge 4. \end{cases}$$

The proof is based on the following observation. If $\beta = (\beta_k^*)_{k=2}^d$, then the process $(\Upsilon_d \widetilde{S}_j)_{j\ge1}$ referred to in Remark 1.2 is a reversible Markov chain under the the measure (1.8). It is not difficult to see that the Markov chain is recurrent for $d = 2,3$ and is transient for $d \ge 4$.

Consider now a special case $\beta_k = (k-1)\beta_2$, $k = 2,\dots,d$, in which the reward for a collision is propotional to the multiplicity. This is in fact the "friendly walkers" in the sense of [5] (with $p = \exp(-\beta_2)$ and $\tau = 1$ in notations there), for which we have the following.

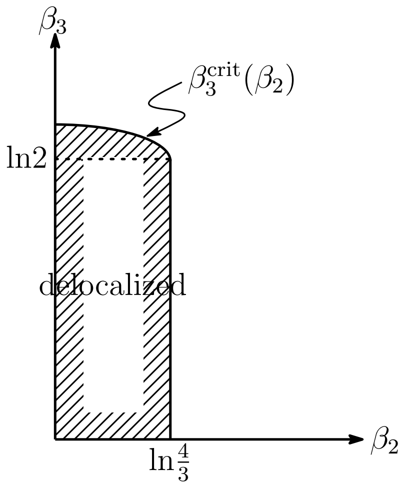

Figure 2: *The delocalized phase for 3-friendly walkers.*

Corollary 1.1

$$\psi_d\,(\beta_2, 2\beta_2, \ldots, (d-1)\beta_2) > 0 \;\; \textit{if and only if} \;\; \beta_2 > \ln(4/3). \tag{1.20}$$

Proof. The "if" part follows immediately from Theorem 1.2 (c). Suppose that $\beta_2 \le \beta_2^*$. Since $\beta_k^* > (k-1)\beta_2^* \ge (k-1)\beta_2$, we see that

$$\begin{aligned} \beta_d^{\rm crit}(\beta_2, 2\beta_2, \ldots, (d-2)\beta_2) &\ge \beta_d^{\rm crit}(\beta_2^*, \ldots, \beta_{d-1}^*) \\ &\ge \beta_d^* \\ &> (d-1)\beta_2, \end{aligned}$$

and hence that $\psi_d\,(\beta_2, 2\beta_2, \ldots, (d-1)\beta_2) = 0$ by Theorem 1.2(b). $\Box$

2 Proof of Theorems

2.1 Proof of Theorem 1.1(a)

Since $1 = \sum_{\mathbf{k}} 1_{\Delta_{n,\mathbf{k}}}$, (1.13) follows from (1.12). To prove (1.12), we will use the following notations;

$$\begin{aligned} Z_{n,d}^x(A) &= P_d^x[z_{n,d} : A], \quad \text{for an event } A, \\ Z_{n,d}^{x,y} &= P_d^x[z_{n,d} : S_n = y] \;\; \text{for } y \in \mathbb{Z}^d_{\le}. \end{aligned}$$

Step1: We first prove that $\lim_{n\to\infty}\frac{1}{n}\ln Z^0_{n,d}(\Delta_{n,d})$ exists. Note first that $Z^x_{n,d}(\Delta_{n,d}) = Z^0_{n,d}(\Delta_{n,d})$ for any $x \in \mathbb{Z}^d_{\text{diag.}}$. We use this to show that $n \mapsto Z^0_{n,d}(\Delta_{n,d})$ is super-multiplicative; for any $m, n \geq 1$,

$$\begin{aligned} Z^0_{m+n,d}(\Delta_{m+n,d}) &\geq \sum_{y\in\mathbb{Z}^d_{\text{diag.}}} Z^{0,y}_{m,d} Z^y_{n,d}(\Delta_{n,d}) \\ &= Z^0_{m,d}(\Delta_{m,d}) Z^0_{n,d}(\Delta_{n,d}). \end{aligned}$$

Step2: We next show that

$$\lim_{n\to\infty}\left(\frac{1}{n}\ln Z^x_{n,d}(\Delta_{n,\mathbf{k}}) - \frac{1}{n}\ln Z^0_{n,d}(\Delta_{n,\mathbf{k}})\right) = 0, \tag{2.1}$$

for any $x \in \mathbb{Z}^d_{\leq}$. Note first that

$$R^x_m \stackrel{\text{def.}}{=} \{z \in \mathbb{Z}^d_{\leq} \ ; \ Z^{x,z}_{m,d} > 0\} \ni 0 \ \text{ for some } m \geq 1.$$

If $n > m$, we have

$$\begin{aligned} Z^x_{n,d}(\Delta_{n,\mathbf{k}}) &= \sum_{z\in R^x_m} Z^{x,z}_m Z^z_{n-m,d}(\Delta_{n-m,\mathbf{k}}) \\ &\geq \left(\min_{z\in R^x_m} Z^{x,z}_{m,d}\right) Z^0_{n-m,d}(\Delta_{n-m,\mathbf{k}}). \end{aligned}$$

This, together with the similar argument with the role of x and 0 exchanged, proves (2.1).

Step3: Lastly, we show that for any partition $\mathbf{k}$ of d,

$$\lim_{n\to\infty}\frac{1}{n}\ln Z^0_{n,d}(\Delta_{n,\mathbf{k}}) = \lim_{n\to\infty}\frac{1}{n}\ln Z^0_{n,d}(\Delta_{n,d}).$$

Since $\Delta_{n,\mathbf{k}} \supset \Delta_{n,d}$, it is enough to prove that

$$\lim_{n\to\infty}\frac{1}{n}\ln Z^0_{n,d} = \lim_{n\to\infty}\frac{1}{n}\ln Z^0_{n,d}(\Delta_{n,d}). \tag{2.2}$$

Clearly, $\liminf_{n\to\infty}\frac{1}{n}\ln Z^0_{n,d} \geq \lim_{n\to\infty}\frac{1}{n}\ln Z^0_{n,d}(\Delta_{n,d})$. We have on the other hand that

$$\begin{aligned} Z^0_{2n,d}(\Delta_{2n,d}) &\geq Z^y_{2n,d}(S_{2n} = 0) \\ &\geq \max_{y\in\mathbb{Z}^d_{\leq}} Z^{0,y}_{n,d} Z^y_{n,d}(S_n = 0) \\ &\geq e^{-\beta_d}\max_{y\in\mathbb{Z}^d_{\leq}} (Z^{0,y}_{n,d})^2 \\ &\geq e^{-\beta_d} n^{-2d} (Z^0_{n,d})^2, \end{aligned}$$

which implies $\limsup_{n\to\infty} \frac{1}{n} \ln Z^0_{n,d} \le \lim_{n\to\infty} \frac{1}{n} \ln Z^0_{n,d}(\Delta_{n,d})$. $\square$

2.2 Proof of Theorem 1.1(b)

We will use the following notation in what follows;

$$\Lambda_{\ell,n}(I, \{a_j\}, \{b_j\}) = \{|S^i_j - a_j| < b_j, \quad \ell \le \forall j \le n, i \in I\},$$

for sequences $\{a_j\}$, $\{b_j\}$, $\ell \ge 1$, and a subset $I \subset I_d \equiv \{1, 2, \ldots, d\}$.

Lemma 2.1 *Let $\{a_j\}_{j\ge1} \subset \mathbb{N}$ be such that $a_0 = 0$ and $a_j - a_{j-1} \in \{0, 1\}$. Then,*

$$\begin{aligned} & P_d[z_{n,d} : \Lambda_{0,n}(I_d, \{a_j\}, \{b_j\}) \cap \Delta_{n,d}] \\ \ge \quad & \left(2^{-d}\right)^{a_n} P_d[z_{n-a_n,d} : \Lambda_{0,n-a_n}(I_d, \{0\}, \{b_j\}) \cap \Delta_{n-a_n,d}], \end{aligned} \tag{2.3}$$

for any increasing positive sequence $\{b_j\}_{j\ge1}$

Proof: Let $\{t_j\}_{j\ge1} = \{1 \le j \le n \ ; \ a_j - a_{j-1} = 0\}$. We define a random walk U by

$$U_j = (S_{t_1} - S_{t_1-1}) + \ldots + (S_{t_j} - S_{t_j-1}).$$

On an event defined by

$$\Xi_n = \{S_j - S_{j-1} = (1, 1, \ldots, 1) \text{ if } 1 \le j \le n \text{ and } a_j - a_{j-1} = 1\},$$

we have

$$S_j = U_{j-a_j} + a_j(1, 1, \ldots, 1), \quad 1 \le j \le n.$$

Therefore,

$$\begin{aligned} z_{n,d}[S] &\ge z_{n-a_n,d}[U] \ \text{ on } \Xi_n, \\ \Delta_{n,d} \cap \Xi_n &= \{U^1_{n-a_n} = U^2_{n-a_n} \cdots = U^d_{n-a_n}\} \cap \Xi_n, \\ \Lambda_{0,n}(I_d, \{a_j\}, \{b_j\}) \cap \Xi_n &= \{|U^i_{j-a_j}| < b_j, \quad 1 \le \forall j \le n, i \in I_d\} \cap \Xi_n \\ &\supset \{|U^i_j| < b_j, \quad 1 \le \forall j \le n - a_n, i \in I_d\} \cap \Xi_n. \end{aligned}$$

Since U is independent of Ξ_n and has the same law as S, (2.3) follows from the observation above. $\square$

We will also use the following lemma whose proof is given in Section 2.4.

Lemma 2.2 *For any $d \geq 2$, $\beta \in [0,\infty)^{d-1}$ and $\gamma > \frac{1}{2}$,*

$$\lim_{\ell\to\infty}\lim_{n\to\infty}\frac{1}{n}\ln P_d^0\left[z_{n,d}(\beta) : \Delta_{n,d}\cap\Lambda_{0,n}\left(I_d,\{0\},\{(j+\ell)^\gamma\}\right)\right] = \varphi_d(\beta). \tag{2.4}$$

Proof of Theorem 1.1(b): We introduce sequences $\{R_j^\alpha\}_{j\geq 0}$, $\alpha = 1,2,\ldots,n(\mathbf{k})$ of $\mathbb{N}$ such that

$$\begin{aligned}
&R_j^1 \equiv 0,\ R_0^\alpha \equiv 0,\ R_{j+1}^\alpha - R_j^\alpha \in \{0,1\},\ \ 1\leq\alpha\leq n(\mathbf{k}),\ \ j\geq 1\\
&R_{j+1}^\alpha - R_j^\alpha \leq R_{j+1}^{\alpha+1} - R_j^{\alpha+1},\ \ 1\leq\alpha\leq n(\mathbf{k})-1,\ \ j\geq 1\\
&2j^\gamma + 2 \leq R_j^{\alpha+1} - R_j^\alpha \leq 2j^\gamma + 3,\ \ j\geq L,\ 1\leq\alpha\leq n(\mathbf{k})-1,
\end{aligned}$$

for some $L \geq 1$ and $\gamma \in (\frac{1}{2},1)$. Note that for $j \geq L$ and $1 \leq \alpha < \alpha' \leq n(\mathbf{k})$,

$$\{z\in\mathbb{Z} : |R_j^\alpha - z| < j^\gamma + 1\}\cap\{z\in\mathbb{Z} : |R_j^{\alpha'} - z| < j^\gamma + 1\} = \emptyset. \tag{2.5}$$

We now introduce index sets $I_{\mathbf{k}}^\alpha = \left\{\sum_{u=1}^{\alpha-1} k_u + j\right\}_{j=1}^{k_\alpha}$, $1 \leq \alpha \leq n(\mathbf{k})$. Then by the Markov proprety, we have

$$\begin{aligned}
&P_d^0[z_{n,d}(\beta) : \Delta_{n,\mathbf{k}}]\\
&\geq P_d^0\left[z_{n,d}(\beta) : \Delta_{n,\mathbf{k}}\cap\bigcap_{\alpha=1}^{n(\mathbf{k})}\Lambda_{\ell,n}\left(I_{\mathbf{k}}^\alpha,\{R_j^\alpha\},\{j^\gamma+1\}\right)\right]\\
&= P_d^0\left[z_{\ell,d}(\beta)P_d^{S_\ell}\left[z_{n-\ell,d}(\beta) : \Delta_{n-\ell,\mathbf{k}}\cap\bigcap_{\alpha=1}^{n(\mathbf{k})}\Lambda_{0,n-\ell}\left(I_{\mathbf{k}}^\alpha,\{R_{j+\ell}^\alpha\},\{(j+\ell)^\gamma+1\}\right)\right]\right].
\end{aligned}$$

By (2.5) and independence of $\{S^i\}$, $i = 1,2,\ldots,d$, for $\ell \geq L$

$$\begin{aligned}
&P_d^x\left[z_{n-\ell,d}(\beta) : \Delta_{n-\ell,\mathbf{k}}\cap\bigcap_{\alpha=1}^{n(\mathbf{k})}\Lambda_{0,n-\ell}\left(I_{\mathbf{k}}^\alpha,\{R_{j+\ell}^\alpha\},\{(j+\ell)^\gamma+1\}\right)\right]\\
&= \prod_{\alpha=1}^{n(\mathbf{k})} P_{k_\alpha}^{\Gamma_{\mathbf{k}}^\alpha x}\left[z_{n-\ell,k_\alpha}(\beta) : \Delta_{n-\ell,k_\alpha}\cap\Lambda_{0,n-\ell}\left(I_{k_\alpha},\{R_{j+\ell}^\alpha\},\{(j+\ell)^\gamma+1\}\right)\right],
\end{aligned}$$

where $\Gamma_{\mathbf{k}}^\alpha x = (x^i)_{i\in I_{\mathbf{k}}^\alpha} \in \mathbb{Z}^{k_\alpha}$. For any $\ell > L$ we can take $y \in \mathbb{Z}_\leq^d$ such that $P_d^0[S_\ell = y] > 0$ and

$$\Gamma_{\mathbf{k}}^\alpha y = (R_\ell^\alpha, R_\ell^\alpha, \ldots, R_\ell^\alpha) \text{ or } (R_\ell^\alpha+1, R_\ell^\alpha+1, \ldots, R_\ell^\alpha+1), \tag{2.6}$$

$1 \leq \alpha \leq n(\mathbf{k})$. Hence

$$
\begin{aligned}
&P_d^0[z_{n,d}(\beta) : \Delta_{n,\mathbf{k}}] \geq P_d^0[z_{\ell,d}(\beta) : S_\ell = y] \\
&\times \prod_{\alpha=1}^{n(\mathbf{k})} P_{k_\alpha}^{\Gamma_{\mathbf{k}}^\alpha y} \left[z_{n-\ell,k_\alpha}(\beta) : \Delta_{n-\ell,k_\alpha} \cap \Lambda_{0,n-\ell} \left(I_{k_\alpha}, \{R_{j+\ell}^\alpha\}, \{(j+\ell)^\gamma + 1\} \right) \right].
\end{aligned}
$$

By shifting the space by $-\Gamma_{\mathbf{k}}^\alpha y^1$, from (2.6) we have

$$
\begin{aligned}
&P_{k_\alpha}^{\Gamma_{\mathbf{k}}^\alpha y} \left[z_{n-\ell,k_\alpha}(\beta) : \Delta_{n-\ell,k_\alpha} \cap \Lambda_{0,n-\ell} \left(I_{k_\alpha}, \{R_{j+\ell}^\alpha\}, \{(j+\ell)^\gamma + 1\} \right) \right] \\
&\geq P_{k_\alpha}^0 \left[z_{n-\ell,k_\alpha}(\beta) : \Delta_{n-\ell,k_\alpha} \cap \Lambda_{0,n-\ell} \left(I_{k_\alpha}, \{R_{\ell+j,\ell}^\alpha\}, \{(j+\ell)^\gamma\} \right) \right], \quad (2.7)
\end{aligned}
$$

where $R_{n,\ell}^\alpha = R_n^\alpha - R_\ell^\alpha$. By (2.3), applied to $a_j = R_{\ell+j,\ell}^\alpha$, we see that the last displayed expectation is bounded from below by

$$
\left(2^{-d}\right)^{R_{n,\ell}^\alpha} P_{k_\alpha}^0 \left[z_{n-\ell-R_{n,\ell}^\alpha}(\beta) : \Delta_{n-\ell-R_{n,\ell}^\alpha,k_\alpha} \cap \Lambda_{0,n-\ell-R_{n,\ell}^\alpha} \left(I_{k_\alpha}, \{0\}, \{(j+\ell)^\gamma\} \right) \right].
$$

Noting that $\gamma < 1$ and $R_{n,\ell}^\alpha / n \to 0$, $n \to \infty$, form the inequalities above we have

$$
\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \ln P_d^0[z_{n,d}(\beta) : \Delta_{n,\mathbf{k}}] \\
&\geq \sum_{\alpha=1}^{n(\mathbf{k})} \liminf_{n\to\infty} \frac{1}{n} \ln P_{k_\alpha}^0 \left[z_{n,k_\alpha}(\beta) : \Delta_{n,k_\alpha} \cap \Lambda_{0,n} \left(I_{k_\alpha}, \{0\}, \{(j+\ell)^\gamma\} \right) \right],
\end{aligned}
$$

for any $\ell > L$. Therefore, (1.14) follows from Lemma 2.2. $\Box$

2.3 Proof of Theorem 1.2

We first show the following expansion formula for the generating function of $P_{n,d}^0[z_{n,d}(\beta) : \Delta_{n,d}]$, $n \geq 1$.

Lemma 2.3

$$
\sum_{n\geq 1} s^n P_{n,d}^0[z_{n,d}(\beta) : \Delta_{n,d}] = \sum_{m\geq 1} [e^{\beta_d} \widehat{W}_{s,d}(\beta)]^m, \tag{2.8}
$$

where $\widehat{W}_{s,d}(\beta)$ *is defined by (1.16). In particular,* $\psi_d(\beta)$ *is characterized by the relation;*

$$
\exp\left(-\psi_d(\beta)\right) = \sup\{s : e^{\beta_d} \widehat{W}_{s,d}(\beta) < 1\}. \tag{2.9}
$$

Proof. We set

$$\eta_0 \equiv 0 \text{ and } \eta_j = \inf\{k > \eta_{j-1} : S_k \in \mathbb{Z}^d_{\text{diag.}}\},\ j \geq 1. \tag{2.10}$$

We then see that if $x \in \mathbb{Z}^d_{\text{diag.}}$, then

$$W_n(\beta) \stackrel{\text{def.}}{=} P^x[z_{n,d}(\beta_2, \beta_3, \ldots, \beta_{d-1}, 0) : \eta_1 = n]$$

does not depend on x. Therefore, by the Markov property,

$$\begin{aligned}
&P^0_{n,d}[z_{n,d}(\beta) : \Delta_{n,d}] \\
&= \sum_{m=1}^{n} \sum_{0=i_0<i_1<\cdots<i_{m-1}<i_m=n} P^0[z_n(\beta) : \eta_1 = i_1, \eta_2 = i_2, \ldots, \eta_m = i_m] \\
&= \sum_{m=1}^{n} e^{\beta_d m} \sum_{0=i_0<i_1<\cdots<i_{m-1}<i_m=n} \prod_{k=1}^{m} P^0[z_{i_k - i_{k-1}}(\beta_2, \ldots, \beta_{d-1}, 0) : \eta_1 = i_k - i_{k-1}] \\
&= \sum_{m=1}^{n} e^{\beta_d m} \sum_{\substack{j_1, j_2, \ldots, j_m \geq 1 \\ j_1 + j_2 + \cdots + j_m = n}} W_{j_1}(\beta) W_{j_2}(\beta) \ldots W_{j_m}(\beta).
\end{aligned}$$

The desired equality (2.8) is now immediately obtained by computing the generating function of the right-hand-side. By Theorem 1.1, $\exp(-\psi_d)$ gives the radius of convergence of the power series on the left-hand-side of (2.8). We therefore see (2.9) from (2.8). □

Proof of Theorem 1.2: **(b):** By (2.9), the positivity of $\psi_d(\beta)$ is equivalent to that

$$\widehat{W}_s(\beta) > \exp(-\beta_d) \text{ for some } s < 1. \tag{2.11}$$

On the other hand, we have by monotone convergence theorem that

$$\lim_{s \nearrow 1} \widehat{W}_s(\beta) = \widehat{W}_1(\beta) = \exp\left(-\beta_d^{\text{crit}}(\beta_2, \ldots, \beta_{d-1})\right). \tag{2.12}$$

If $\beta_d > \beta_d^{\text{crit}}(\beta_2, \ldots, \beta_{d-1})$, then we see from (2.12) that (2.11) holds. Conversely, if (2.11) holds true, then $\widehat{W}_1(\beta) > \exp(-\beta_d)$, and hence $\beta_d > \beta_d^{\text{crit}}(\beta_2, \ldots, \beta_{d-1})$.
(a): It is not difficult to see from (1.19) that

$$Z_n^x(\beta_2^*, \ldots, \beta_d^*) = Z_1^x(\beta_2^*, \ldots, \beta_d^*), \quad n \geq 1$$

and hence that $\psi_d(\beta_2^*, \ldots, \beta_d^*) = 0$. We therefore have $\beta_d^* \leq \beta_d^{\text{crit}}(\beta_2^*, \ldots, \beta_{d-1}^*)$ by part (b).
(c): This follows from part (b) and (1.14). □

2.4 Proof of Lemma 2.2

We first introduce an event

$$\Theta_{n,M} = \Delta_{n,d} \cap \{\eta_m - \eta_{m-1} \le M,\ 0 \le m \le \tau_n\} \tag{2.13}$$

where η_m, $m \ge 0$ are stopping times defined by (2.10) and

$$\tau_n(S) = \max\{m : \eta_m \le n\}. \tag{2.14}$$

We will use the following lemma which relates $\psi_d(\beta)$ with $P_d^0[z_{n,d}(\beta) : \Theta_{n,M}]$.

Lemma 2.4

$$\psi_d(\beta) = \lim_{M\to\infty} \lim_{n\to\infty} \frac{1}{n} \ln P_d^0[z_{n,d}(\beta) : \Theta_{n,M}]. \tag{2.15}$$

Proof. The first limit (in n) on the right-hand-side exists by the superadditivity while the second one (in M) by monotonicity. To identify the limit, take any $c < \psi_d(\beta)$. Then, by Lemma 2.3, we can take a positive integer $M = M(c)$ such that

$$\sum_{n=1}^{M(c)} e^{-cn} P^0[z_{n,d}(\beta_2, \beta_3, \dots, \beta_{d-1}, 0) : \eta_1 = n] e^{\beta_d} \ge 1.$$

By the same procedure to show Lemma 2.3 we have

$$\lim_{n\to\infty} \frac{1}{n} \ln P_d^0[z_{n,d}(\beta) : \Theta_{n,M}] \ge c.$$

This proves (2.15). □

Proof of Lemma 2.2: We write $\Lambda_{0,n,\ell} = \Lambda_{0,n}(I_d, \{0\}, (j+\ell)^\gamma)$ for simplicity. By Lemma 2.4, our task is reduced to proving the following statement: for given $M > 0$, there exists a constant $L(M)$ such that if $\ell > L(M)$, then

$$P^0[z_{n,d}(\beta) : \Delta_n \cap \Lambda_{0,n,\ell}] \ge \frac{1}{2} P_d^0[z_{n,d}(\beta) : \Theta_{n,M}], \quad n \ge 1. \tag{2.16}$$

Proof of (2.16): First we introduce spaces of d-dimensional finite paths $\mathcal{W}(n)$, $\mathcal{W}_+(n)$, $n \ge 1$ defined by

$$\begin{aligned} \mathcal{W}(n) = \{w = \{w_j\}_{j=1}^n : |w_j^i - w_{j-1}^i| = 1,\ 1 \le j \le n, 1 \le i \le d, \\ w_0, w_n \in \mathbb{Z}^d_{\text{diag.}}\}, \\ \mathcal{W}_+(n) = \{w \in \mathcal{W}(n) : w_0 = 0, w_j \notin \mathbb{Z}^d_{\text{diag.}}, 1 \le j \le n-1\}. \end{aligned}$$

For $w' \in \mathcal{W}(n_1)$ and $w'' \in \mathcal{W}(n_2)$, $n_1, n_2 \geq 1$, $w' \cdot w''$ represent the path in $\mathcal{W}(n_1 + n_2)$ defined by

$$(w' \cdot w'')_j = \begin{cases} w'_j, & \text{if } 0 \leq j \leq n_1, \\ w'_{n_1} + w''_{j-n_1} - w''_0, & \text{if } n_1 \leq j \leq n_1 + n_2. \end{cases}$$

Recall that we have defined stopping times η_m by (2.10) and suppose that $S_n = 0$. Then we define $w(m) \in \mathcal{W}_+(\eta_m - \eta_{m-1})$, $1 \leq m \leq \tau_n$ as

$$w(m)_j = S_{\eta_{m-1}+j} - S_{\eta_{m-1}}, \quad 0 \leq j \leq \eta_m - \eta_{m-1}.$$

It is clear that $S_j = (w(1) \cdot w(2) \cdots w(\tau_n))_j$, for $0 \leq j \leq n$. We introduce a map r from $\mathcal{W}_+(n)$ to $\mathcal{W}_+(n)$ defined by

$$(rw)_j = w_{n-j} - w_n, \quad 0 \leq j \leq n, \ w \in \mathcal{W}_+(n).$$

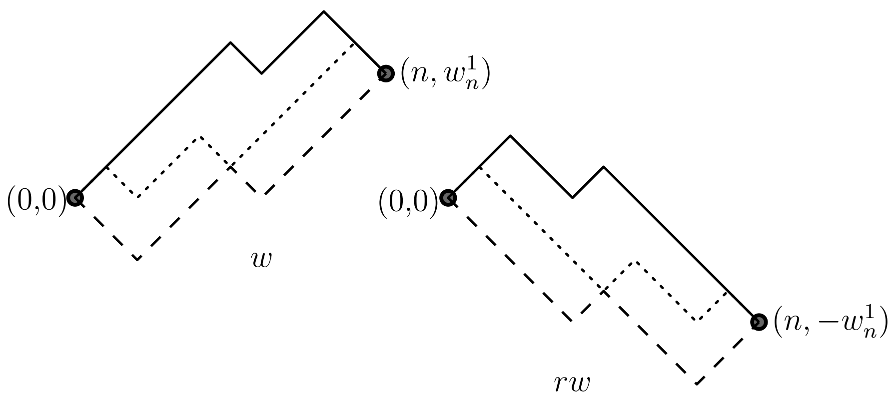

Figure 3: *Examples of w and rw.*

For $\xi = (\xi_m) \in \prod_{m \geq 1} \{-1, +1\}$, we define S^ξ inductively by

$$S_j^\xi = \begin{cases} S^\xi_{\eta_{m-1}} + w(m)_{j-\eta_{m-1}}, & \text{if } \eta_{m-1} \leq j \leq \eta_m, \ \xi_m = 1, \ 1 \leq m \leq \tau_n, \\ S^\xi_{\eta_{m-1}} + (rw(m))_{j-\eta_{m-1}}, & \text{if } \eta_{m-1} \leq j \leq \eta_m, \ \xi_m = -1, \ 1 \leq m \leq \tau_n, \\ S^\xi_n + S_j - S_n, & \text{if } j > n. \end{cases}$$

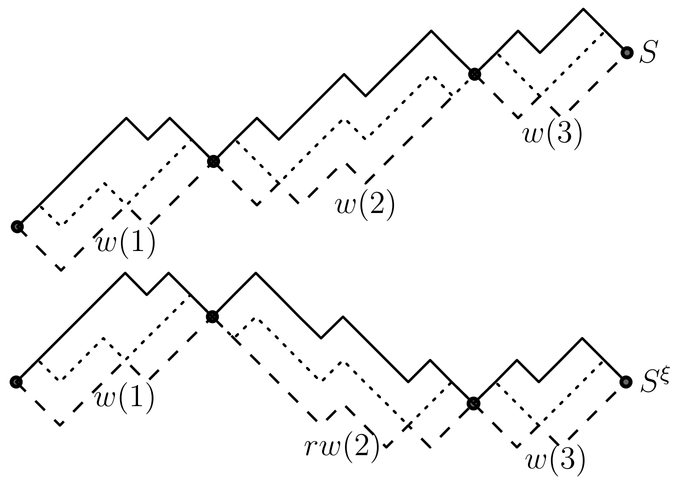

Figure 4: *Examples of S and S^ξ, in the case that $\tau_n = 3$, $\xi_1 = 1$, $\xi_2 = -1$ and $\xi_3 = 1$.*

Now, let $\xi_1, \xi_2, \dots$ be i.i.d. random variable on a probability space $(\Xi, \mathcal{G}, Q)$ such that $Q(\xi_m = \pm 1) = 1/2$. Note that $z_n(\beta)[S_\cdot] = z_n(\beta)[S^\xi_\cdot]$ and that $S \in \Theta_{n,M} \iff S^\xi \in \Theta_{n,M}$. We therefore have that

$$\begin{aligned} P_d^0[z_{n,d}(\beta) : \Delta_n \cap \Lambda_{0,n,\ell}] &\geq P_d^0[z_{n,d}(\beta) : \Theta_{n,M} \cap \Lambda_{0,n,\ell}] \\ &= \int Q(d\xi) P_d^0[z_{n,d}(\beta) : \Theta_{n,M} \cap \{S^\xi \in \Lambda_{0,n,\ell}\}]. \end{aligned}$$

In what follows, we will assume that $\ell^\gamma > 4M$. We define $U_m(S) = S^d_{\eta_m}$ for $m = 1, \dots, \tau_n$. We then see that

$$\Theta_{n,M} \subset \bigcap_{j=1}^{n} \bigcap_{\alpha=1}^{d} \{|(S^\xi)^\alpha_j - U_{\tau_j}(S^\xi)| \leq M\}.$$

and hence that

$$\bigcap_{m=1}^{\tau_n} \{|U_m(S^\xi)| \leq (m+\ell)^\gamma - M\} \cap \Theta_{n,M} \subset \{S^\xi \in \Lambda_{0,n,\ell}\} \cap \Theta_{n,M}.$$

This implies that

$$\int Q(d\xi) P_d^0[z_{n,d}(\beta) : \Theta_{n,M} \cap \{S^\xi \in \Lambda_{0,n,\ell}\}] \geq P_d^0[z_{n,d}(\beta)\rho_\ell(S) : \Theta_{n,M}], \quad (2.17)$$

where

$$\rho_\ell(S) = Q\left(\bigcap_{m=1}^{\tau_n} \{|U_m(S^\xi)| \leq (m+\ell)^\gamma - M\}\right).$$

Since

$$U_m(S^\xi) - U_{m-1}(S^\xi) = (U_m(S) - U_{m-1}(S))\,\xi_i,$$

the process $(U_m(S^\xi))_{m=1}^{\tau_n}$ is of independent increments bounded by M. We can therefore use Azuma's inequality [1, page85] together with observation $(m+\ell)^\gamma - M \geq (m^\gamma \vee \ell^\gamma)/2$ and $|U_m(S^\xi)| \leq mM$ to conclude that

$$\begin{aligned}
1-\rho_\ell(S) &\leq \sum_{m=1}^{\tau_n} Q\{|U_m(S^\xi)| > (m+\ell)^\gamma - M\} \\
&\leq \sum_{m=[\ell^\gamma/2M]}^{\tau_n} Q\{|U_m(S^\xi)| > m^\gamma/2\} \\
&\leq \sum_{m\geq[\ell^\gamma/2M]} \exp\left(-m^{2\gamma-1}/8\right) \\
&\leq 1/2,
\end{aligned}$$

if ℓ is large enough. We now obtain (2.4) by plugging this into (2.17). $\square$

3 Remarks

By the free energies we define the following regions;

$$\begin{aligned}
\mathcal{D}_0 &= \{\beta \in [0,\infty)^{d-1} : \psi_d(\beta) = 0\}, \\
\mathcal{D}_d &= \{\beta \in [0,\infty)^{d-1} : \psi_d(\beta) > \psi_{\mathbf{k}}(\beta), \quad \text{for any partition } \mathbf{k} \neq d \text{ of } d\},
\end{aligned}$$

where $\psi_{\mathbf{k}} = \sum_{i=1}^{n(\mathbf{k})} \psi_{k_i}$. We call $\mathcal{D}_0$ the delocalized phase in accordance with Definition 1.1 and $\mathcal{D}_d$ the completely localized phase. For a partition $\mathbf{k} \neq d$ of d, we define intermediate phase $\mathcal{D}_{\mathbf{k}}$ as the interior of

$$\{\beta \in [0,\infty)^{d-1} \setminus \mathcal{D}_0 : \psi_d(\beta) = \psi_{\mathbf{k}}(\beta)\}.$$

It is a very interesting problem to study the phases $\mathcal{D}_{\mathbf{k}}$, $\mathbf{k} \neq d$.

In the case $d = 3$, there is one intermediate phase $\mathcal{D}_{(2,1)} = \mathcal{D}_{(1,2)}$. We call the region the phase of 2-walkers collision. $\mathcal{D}_{(2,1)} \neq \emptyset$ if and only if

$$\frac{\partial}{\partial\beta_3}\psi_3(\beta_2,\beta_3) = 0, \tag{3.1}$$

for some $\beta_2 > \beta_2^{\rm crit}$ and $\beta_3 \geq 0$. We expect the condition holds for (β_2, β_3) in a small neighborhood of $(\beta_2^{\rm crit}, 0)$.

In the case $d = 4$, there are two intermediate phases $\mathcal{D}_{(2,2)}$, $\mathcal{D}_{(3,1)} = \mathcal{D}_{(1,3)}$. $\mathcal{D}_{(2,2)} \neq \emptyset$ if and only if

$$\frac{\partial}{\partial \beta_3}\psi_4(\beta_2, \beta_3, \beta_4) = \frac{\partial}{\partial \beta_4}\psi_4(\beta_2, \beta_3, \beta_4) = 0, \tag{3.2}$$

for some $\beta_2 > \beta_2^{\rm crit}$ and $\beta_3, \beta_4 \geq 0$. $\mathcal{D}_{(3,1)} \neq \emptyset$ if and only if

$$\frac{\partial}{\partial \beta_2}\psi_4(\beta_2, \beta_3, \beta_4) = \frac{\partial}{\partial \beta_4}\psi_4(\beta_2, \beta_3, \beta_4) = 0, \tag{3.3}$$

for some $(\beta_2, \beta_3, \beta_4)$ with $\psi_3(\beta_2, \beta_3) > 2\psi_2(\beta_2)$ and $\beta_4 \geq 0$. We also expect that the condition (3.2) holds for $(\beta_2, 0, 0) \in \mathcal{D}_{(2,2)}$, for sufficiently large β_2, and the condition (3.3) holds for $(0, \beta_3, \beta_4)$ in a small neighborhood of $(0, \beta_3^{\rm crit}(0), 0)$.

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