Directed Polymers in Random Environment: Path Localization and Strong Disorder

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July 24, 2002

Abstract

We consider directed polymers in random environment. Under some mild assumptions on the environment, we prove here: (i) equivalence between the decay rate of the partition function and some natural localization properties of the path, (ii) some quantitative estimates of the decay of the partition function in dimensions one or two, or at sufficiently low temperature, (iii) the existence of quenched free energy. In particular, we generalize to general environments, the results recently obtained by P. Carmona and Y. Hu for a Gaussian environment. Our approach is based on martingale decomposition and martingale analysis. It leads to a natural, asymptotic relation between the partition function, and the probability that two polymers in the same environment, but independent otherwise, end up at the same point.

Short Title. Directed Polymers in Random Environment
Key words and phrases. Directed polymers, martingales, random environment
AMS 1991 subject classifications. Primary 60K35; secondary 60G42, 82A51, 82D30

1 Introduction and Main Results

1.1 Directed Polymers in Random Environment

The models we consider in this paper are defined in terms of a random walk and of a random environment, which we introduce now:

- *The random walk:* $(\{S_n\}_{n\geq 0}, \{P^x\}_{x\in\mathbb{Z}^d})$ is a simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$. More precisely, let Ω be the path space $\Omega = \{\omega = (\omega_n)_{n\geq 0}; \omega_n \in \mathbb{Z}^d, n \geq 0\}$, let $\mathcal{F}$ be the cylindric σ-field on Ω, and, for all $n \geq 0$, $S_n : \omega \mapsto \omega_n$ the projection map. For any $x \in \mathbb{Z}^d$ we consider the unique probability measure P^x on $(\Omega, \mathcal{F})$ such that $S_1 - S_0, \ldots, S_n - S_{n-1}$ are independent and
$$P^x\{S_0 = x\} = 1, \quad P^x\{S_n - S_{n-1} = \pm\delta_j\} = (2d)^{-1}, \quad j = 1, 2, \ldots, d,$$
where $\delta_j = (\delta_{kj})_{k=1}^d$ is the j-th vector of the canonical basis of $\mathbb{Z}^d$. For $x = 0$ we will write simply P by P^0.

- *The random environment:* $\xi = \{\xi(x, n) : x \in \mathbb{Z}^d, \ n \geq 1\}$ is a sequence of random variables which are real valued, non-constant, and i.i.d. defined on a probability space $(\Xi, \mathcal{E}, Q)$ such that
$$Q[\exp(\beta\xi(x, n))] < \infty \quad \text{for all } \beta \in \mathbb{R}. \tag{1.1}$$
(Throughout, $Q[Y]$ denotes the Q-expectation of a r.v. Y.) Let $\lambda(\beta)$ be the logarithmic moment generating function of $\xi(x, n)$,
$$\lambda(\beta) = \ln Q[\exp(\beta\xi(x, n))], \quad \beta \in \mathbb{R}. \tag{1.2}$$

For any $n > 0$, define the probability measure μ_n on the path space $(\Omega, \mathcal{F})$
$$\mu_n(d\omega) = P[e_n]^{-1} e_n \, P(d\omega), \tag{1.3}$$
where
$$e_n = e_n(\xi, S) = \exp\left(\sum_{1\leq j\leq n} (\beta\xi(S_j, j) - \lambda(\beta))\right) \tag{1.4}$$
with a parameter $\beta \in \mathbb{R}$. Here, the graph $\{(S_j, j)\}_{j\geq 0}$ may be interpreted as a polymer chain living in the $(d+1)$-dimensional space, constrained to stretch in the $(d+1)$-th direction, and governed by the Hamiltonian
$$-\beta \sum_{j\geq 1} \xi(S_j, j) \ ,$$
i.e. the so-called directed polymer in the environment ξ. Note that the term $\lambda(\beta)$, from the exponent in (1.4), cancels out in definition (1.3) of μ_n. The reason why to add it in (1.4), is to normalize $P[e_n]$, which has now expectation equal to 1. If $\beta > 0$, then the parameter $\beta > 0$ plays the role of the inverse temperature in this interpretation. Since this Hamiltonian is parametrized by ξ, the polymer measure μ_n is random. The polymer is attracted to sites where the random environment is large and positive, and repelled by sites where the environment is large and negative. Here are two standard choices for ξ.

Example 1.1 *Gaussian environment* (Carmona and Hu, 2001) This is the case in which $\xi(x,n)$ is a standard normal random variable, so that $\lambda(\beta) = \frac{1}{2}\beta^2$.

Example 1.2 *Bernoulli environment* (Bolthausen 1989, Imbrie and Spencer 1988, Song and Zhou 1996): This is the case in which $\xi(x,n)$ takes two different values a and b with probability $p > 0$ and $1-p > 0$, respectively, so that $\lambda(\beta) = \ln(pe^{\beta a} + (1-p)e^{\beta b})$. As discussed by Johansson (2000, Remark 1.8), directed percolation can be understood as the case of $0 = a > b$ and zero-temperature ($\beta \to \infty$), which however is outside the scope of this paper.

We are interested in the large time behavior of the path $\{S_k\}_{k=1}^n$ under the (sequence of) polymer measures μ_n. As is the case in many other models in statistical mechanics, one of the fundamental questions is the asymptotic behavior of the partition function

$$Z_n = Z_n(\xi) = P[e_n]\ . \tag{1.5}$$

Since Z_n is a positive martingale on $(\Xi, \mathcal{E}, Q)$, the following limit exists Q-a.s.:

$$Z_\infty \stackrel{\text{def.}}{=} \lim_{n \nearrow \infty} Z_n\ . \tag{1.6}$$

The event $\{Z_\infty = 0\}$ is measurable with respect to the tail σ-field

$$\bigcap_{n\geq 1} \sigma[\xi(x,j)\ ;\ j \geq n,\ x \in \mathbb{Z}^d]$$

and therefore by Kolmogorov's 0-1 law

$$Q\{Z_\infty = 0\} = 0 \text{ or } 1. \tag{1.7}$$

We refer to the former case as **weak disorder** and the latter as **strong disorder**. It is known (e.g., Song and Zhou 1996) that for $d \geq 3$,

$$Q\{Z_\infty = 0\} = 0 \ \text{ if } \gamma_1(\beta) \stackrel{\text{def.}}{=} \lambda(2\beta) - 2\lambda(\beta) < -\ln(1-q) \tag{1.8}$$

where $q = P\{S_n \neq 0 \text{ for all } n \geq 1\}$; similar results for weak disorder were obtained by Bolthausen (1989) and Sinai (1995). Note that $\gamma_1(\beta)$ is decreasing on $(-\infty, 0]$, increasing on $[0,\infty)$ and $\gamma_1(0) = 0$ so that the condition in (1.8) does hold if $|\beta|$ is small. In dimension $d \geq 3$, this condition amounts to L^2-convergence in (1.6), and it allows using the so-called second moment method: for small β and $d \geq 3$, Imbrie and Spencer (1988) first, then Bolthausen (1989) with martingales techniques, proved that the polymer is diffusive, i.e., $\mu_n[S_n^2] \sim n$ as $n \nearrow \infty$; more recently Albeverio and Zhou (1996) showed that the invariance principle holds for almost every environment. On the other hand, for the strong disorder, it can be seen that

$$Q\{Z_\infty = 0\} = 1 \ \text{ if } \gamma_2(\beta) \stackrel{\text{def.}}{=} \beta\lambda'(\beta) - \lambda(\beta) \geq \ln(2d). \tag{1.9}$$

This was shown by Kahane and Peyrière (1976) for a different model called Mandelbrot martingale (or, equivalently, multiplicative chaos), where graphs $\{(S_j, j)\}_{j\geq 0}$ are replaced by infinite paths, without loops and starting from the root, on the d-ary tree. Although the directed polymer we are considering here is more intricate due to correlations, the same argument applies as far as to deduce (1.9). Note that $\gamma_2(\beta)$ is decreasing on $(-\infty, 0]$, increasing on $[0, \infty)$ and $\gamma_2(0) = 0$ so that the condition in (1.9) roughly says that $|\beta|$ is large enough. Recently, P. Carmona and Y. Hu (2001) proved for Gaussian environment that for all $\beta \neq 0$,

$$Q\{Z_\infty = 0\} = 1, \quad d = 1, 2, \tag{1.10}$$

which, together with (1.8) and (1.9), displays a non-trivial dependence on the dimension.

In the present paper, we consider general environments and present some results mainly for the strong disorder case: $Q\{Z_\infty = 0\} = 1$, including the extension of (1.10) to non-Gaussian case. Using martingale analysis, we also obtain natural localization properties which characterize the strong disorder regime. More precisely, the decay of the partition function is equivalent to concentration of the path on favourite sites. We present the proofs in a self-contained way, except for that of Proposition 1.4(b).

Among other interesting subjects related to the directed polymer are superdiffusivity and critical exponents. Although we will not discuss about them here, we refer to Johansson (2000), Licea *et al* (1996), Petermann (2000) and Piza (1997) for rigorous results in this direction.

1.2 Results

On the product space $(\Omega^2, \mathcal{F}^{\otimes 2})$, we consider the probability measure $\mu_n^{\otimes 2} = \mu_n^{\otimes 2}(d\omega, d\tilde{\omega})$, that we will view as the distribution of the couple $(S, \widetilde{S})$ with $\widetilde{S} = \{\widetilde{S}_k\}_{k\geq 0}$ an independent copy of $S = \{S_k\}_{k\geq 0}$ with law μ_n. An important role in the analysis is played by the random sequence

$$I_n = \mu_{n-1}^{\otimes 2}(S_n = \widetilde{S}_n) \;, \tag{1.11}$$

which conveys some information on the localization of paths under μ_n, see (1.18) below. Roughly, large values of $I_n \in (0, 1]$ indicate that the polymer concentrates, at time n, on a few significant sites, though small values indicate that it spreads out on a large number of sites. Our basic result relates the partition function Z_n and the expected intersection time $\sum_{j\leq n} I_j$ of two independent polymers in the same environment.

Theorem 1.1 *Let $\beta \neq 0$. Then,*

$$\{Z_\infty = 0\} = \left\{\sum_{n\geq 1} I_n = \infty\right\}, \quad Q\text{-a.s.} \tag{1.12}$$

Moreover, if $Q\{Z_\infty = 0\} = 1$, then there exist $c_1, c_2 \in (0, \infty)$ such that

$$-c_1 \ln Z_n \leq \sum_{1\leq j\leq n} I_j \leq -c_2 \ln Z_n \qquad \text{for large enough } n\text{'s}, \quad Q\text{-a.s.} \tag{1.13}$$

We make a brief comment on the result. On the one hand, we recall the definition – below (1.7) – of weak and strong disorder, which is natural in view of the high temperature behavior (in high dimension) (1.8) and the low temperature behavior (1.9). On the other hand, when the polymer strongly feels the environment, it is strongly attracted to sites with favourable environment and it follows from the definition (1.11) that I_n takes large values. Our result is a rigourous statement of equivalence of these two properties. The decay property of Z_n is reflected in some specific localization property of the path $\{S_n\}_{n\geq 1}$ under the random measure (1.3). The proof of Theorem 1.1 is based on a general estimate for the summation of i.i.d.random variable (Lemma 2.1 below) and martingale analysis.

The most interesting case relative to the following, straightforward corollary is $a_n = n$, $n \geq 1$.

Corollary 1.2 *For $\beta \neq 0$ and a sequence $a_n \nearrow \infty$ of positive numbers, the following properties are equivalent:*

(Z1) *There exists $c > 0$ such that*

$$Q\left\{ \underline{\lim}_{n\nearrow\infty} -\frac{1}{a_n} \ln Z_n \geq c \right\} = 1. \tag{1.14}$$

(I1) *There exists $c > 0$ such that*

$$Q\left\{ \underline{\lim}_{n\nearrow\infty} \frac{1}{a_n} \sum_{1\leq j\leq n} I_j \geq c \right\} = 1. \tag{1.15}$$

Remark 1.1 The equivalence presented in Theorem 1.1 was shown first by Carmona and Hu (2001, Theorem 1.1 and Proposition 5.1) in the Gaussian case.

Some sufficient conditions for (Z1) and (I1) in Corollary 1.2 are provided by the following result.

Theorem 1.3 **(a)** *(Z1) in Corollary 1.2 holds for $a_n = n$ if $\gamma_2(\beta) > \ln(2d)$ (cf. (1.9)).*

(b) *If $\beta \neq 0$, then for Q-a.s.,*

$$Z_n \begin{cases} = \mathcal{O}\left(\exp(-c_1 n^{1/3})\right), & \text{as } n \nearrow \infty \text{ if } d = 1 \\ \longrightarrow 0, & \text{as } n \nearrow \infty \text{ if } d = 2, \end{cases}$$

where c_1 is a positive constant.

The proof of Theorem 1.3 is carried out by estimating fractional moment: $Q[Z_n^\theta]$, $0 < \theta < 1$; see Lemma 3.1 below. Beside the quantitative bound for the rate of decay for $d = 1$ presented above, we also give quantitative bound for the fractional moment for $d = 1, 2$ in the course of the proof.

Remark 1.2 Theorem 1.3(b) generalizes Theorem 1.1 in Carmona and Hu (2001) to non-gaussian environments. Moreover, the proof in this paper shed more light on the decay rate.

We now go on to discuss sufficient conditions for another localization property of the polymer chain, described in terms of I_n;

Proposition 1.4 *Consider the following property:*

(I2) *There is a constant* $c \in (0, \infty)$ *such that*

$$\overline{\lim_{n \nearrow \infty}} I_n \geq c, \quad Q\text{-}a.s. \tag{1.16}$$

Then,

(a) *(I2) holds true if Property (Z1) in Corollary 1.2 holds with* $a_n = n$*, in particular if* $\gamma_2(\beta) > \ln(2d)$ *(cf. (1.9) and Theorem 1.3(a)).*

(b) *(I2) holds true if* $d = 1, 2$.

(c) *If* $Q\{Z_\infty > 0\} = 1$*, then, in contrast with (I2),*

$$\lim_{n \nearrow \infty} I_n = 0, \quad Q\text{-}a.s.$$

Assume moreover that $\gamma_1(\beta) < -\ln(1-q)$ *(cf.(1.8)). Then, there is a constant* $c > 0$ *such that*

$$I_n = O(n^{-c}) \text{ in } Q\text{-probability.} \tag{1.17}$$

A natural quantity of interest here, related to localization phenomenon, is the favorite site for the path at time n. First observe that

$$\max_{x \in \mathbb{Z}^d} \mu_{n-1}(S_n = x)^2 \leq I_n \leq \max_{x \in \mathbb{Z}^d} \mu_{n-1}(S_n = x) \,. \tag{1.18}$$

Therefore, all statements we obtained for I_n can be translated into those for $\max_{x \in \mathbb{Z}^d} \mu_{n-1}(S_n = x)$. In particular, we showed in Proposition 1.4 that the probability of the favorite site vanishes for weak disorder, but does not vanish for strong disorder. In the latter case the polymer localizes (in a set of lattice points depending on the evironment), though in the former it spreads out somewhat similarly to the usual simple random walk.

Remark 1.3 Proposition 1.4(b) generalizes Theorem 1.2 in Carmona and Hu (2001) to non-gaussian environments. To prove this, and only there, we refer the readers to some of the arguments in Carmona and Hu (2001).

Finally, we remark that the "quenched free energy"

$$\lim_{n \nearrow \infty} \frac{1}{n} \ln Z_n$$

exists Q-a.s. under our assumption (1.1).

Proposition 1.5 *The limit*

$$\psi(\beta) = \lim_{n \nearrow \infty} \frac{1}{n} Q[\ln Z_n] \in (-\infty, 0]$$

exists. Moreover, for any $\varepsilon > 0$, there is an $n_0 = n_0(\beta, \varepsilon) < \infty$ such that

$$Q\left\{ \left| \frac{1}{n} \ln Z_n - Q\left[\frac{1}{n} \ln Z_n \right] \right| > \varepsilon \right\} \le \exp\left(-\frac{\varepsilon^{2/3} n^{1/3}}{4} \right), \quad n \ge n_0. \tag{1.19}$$

As a consequence,

$$\lim_{n \nearrow \infty} \frac{1}{n} \ln Z_n = \psi(\beta), \quad Q\text{-a.s.}$$

Remark 1.4 The inequality (1.19) is a concentration inequality with the stretched exponential decay rate. An inspection of our proof reveals that an exponential concentration result can be obtained by a slightly stronger assumption. In fact, if we assume that there is $\delta > 0$ such that

$$Q\left[\exp(\delta |\xi(x,n)|^2) \right] < \infty, \tag{1.20}$$

then, we obtain the following; for any $\varepsilon > 0$, there is an $n_0 = n_0(\beta, \varepsilon) < \infty$ such that

$$Q\left\{ \left| \frac{1}{n} \ln Z_n - Q\left[\frac{1}{n} \ln Z_n \right] \right| > \varepsilon \right\} \le \exp\left(-\frac{\varepsilon^2 n}{c} \right), \quad n \ge n_0. \tag{1.21}$$

where $c = c(\beta) > 0$. See Remark 5.1 below for the proof. Note also that (1.20) is true if $\xi(x,n)$ is a Gaussian or Bernoulli r.v. as in Example 1.1 or Example 1.2.

Remark 1.5 We can define a similar model by considering a Markov chain $(\{S_n\}_{n \ge 0}, \{P^x\}_{x \in \Gamma})$ on a certain state space Γ instead of the random walk on $\mathbb{Z}^d$. The proofs presented in this paper apply without change to this generalization.

2 Proof of Theorem 1.1

We first state some technical estimates.

Lemma 2.1 *Let η_i, $1 \le i \le m$ be positive, non-constant i.i.d. random variables on a probability space $(\Xi, \mathcal{E}, Q)$ such that*

$$Q[\eta_1] = 1, \quad Q[\eta_1^3 + \ln^2 \eta_1] < \infty.$$

For $\{\alpha_i\}_{1 \le i \le m} \subset [0, \infty)$ such that $\sum_{1 \le i \le m} \alpha_i = 1$, define a centered random variable $U > -1$ by $U = \sum_{1 \le i \le m} \alpha_i \eta_i - 1$. Then, there exists a constant $c \in (0, \infty)$, independent of $\{\alpha_i\}_{1 \le i \le m}$ such that

$$\frac{1}{c} \sum_{1 \le i \le m} \alpha_i^2 \le Q\left[\frac{U^2}{2 + U} \right], \tag{2.1}$$

$$\frac{1}{c} \sum_{1 \le i \le m} \alpha_i^2 \le -Q\left[\ln(1 + U) \right] \le c \sum_{1 \le i \le m} \alpha_i^2, \tag{2.2}$$

$$Q\left[\ln^2(1 + U) \right] \le c \sum_{1 \le i \le m} \alpha_i^2. \tag{2.3}$$

Remark 2.1 These estimates are proved in Carmona and Hu (2001) for the Gaussian case with the help of Brownian motion and making use of Itô's formula. Here, we present a simple argument which works in the general case.

We postpone the proof of Lemma 2.1 to the end of the section, and, assuming the lemma, we start to prove Theorem 1.1. To conclude (1.12) and (1.13), it is enough to show the following (2.4) and (2.5):

$$\{Z_\infty = 0\} \subset \left\{ \sum_{n\geq 1} I_n = \infty \right\}, \quad Q\text{-a.s.} \tag{2.4}$$

There are $c_1, c_2 \in (0, \infty)$ such that

$$\left\{ \sum_{n\geq 1} I_n = \infty \right\} \subset \left\{ -c_1 \ln Z_n \leq \sum_{1\leq j\leq n} I_j \leq -c_2 \ln Z_n \text{ for large enough } n\text{'s.} \right\}, \quad Q\text{-a.s.} \tag{2.5}$$

The proof of (2.4) and (2.5) is based on Doob's decomposition for the process $-\ln Z_n$. It is convenient to introduce some more notations. For a sequence $(a_n)_{n\geq 0}$ (random or non-random), we set $\Delta a_n = a_n - a_{n-1}$ for $n \geq 1$. We denote by $\mathcal{E}_n$ the σ-field generated by $\{\xi(x,j)\ ;\ 1 \leq j \leq n,\ x \in \mathbb{Z}^d\}$, and we denote by Q_n^ξ the conditional expectation with respect to Q given $\mathcal{E}_n$. Let us now recall Doob's decomposition in this context; any $(\mathcal{E}_n)$-adapted process $X = \{X_n\}_{n\geq 0} \subset L^1(Q)$ can be decomposed in a unique way as

$$X_n = M_n(X) + A_n(X), \quad n \geq 1,$$

where $M(X)$ is an $(\mathcal{E}_n)$-martingale and

$$A_0 = 0, \ \ \Delta A_n = Q_{n-1}^\xi[\Delta X_n], \quad n \geq 1.$$

$M_n(X)$ and $A_n(X)$ are called respectively, the martingale part and compensator of the process X. If X is a square integrable martingale, then the compensator $A_n(X^2)$ of the process $X^2 = \{(X_n)^2\}_{n\geq 0} \subset L^1(Q)$ is denoted by $\langle\, X \,\rangle_n$ and is given by the following formula;

$$\Delta\langle\, X \,\rangle_n = Q_{n-1}^\xi[(\Delta X_n)^2]$$

Here, we are interested in the Doob's decomposition of $X_n = -\ln Z_n$, whose martingale part and the compensator will be henceforth denoted M_n and A_n respectively;

$$-\ln Z_n = M_n + A_n. \tag{2.6}$$

To compute M_n and A_n, we introduce

$$U_n = \mu_{n-1}[\exp(\beta\xi(S_n, n) - \lambda(\beta))] - 1\ .$$

It is then clear that

$$Z_n / Z_{n-1} = 1 + U_n \tag{2.7}$$

and hence that

$$\Delta A_n = -Q^{\xi}_{n-1} \ln(1+U_n), \quad \Delta M_n = -\ln(1+U_n) + Q^{\xi}_{n-1} \ln(1+U_n). \tag{2.8}$$

In particular,

$$\Delta\langle M \rangle_n \leq Q^{\xi}_{n-1} \ln^2(1+U_n). \tag{2.9}$$

On the other hand, we have that

$$I_n = \sum_{|z|_1 \leq n} \mu_{n-1}(S_n = z)^2.$$

We now claim that there is a constant $c \in (0, \infty)$ such that

$$\frac{1}{c} I_n \leq \Delta A_n \leq c I_n, \tag{2.10}$$

$$\Delta\langle M \rangle_n \leq c I_n. \tag{2.11}$$

Indeed, both follow from (2.8), (2.9) and Lemma 2.1; $\{\eta_i\}$, $\{\alpha_i\}$ and Q in the lemma play the roles of $\{\exp(\beta\xi(z,n) - \lambda(\beta))\}_{|z|_1 \leq n}$, $\{\mu_{n-1}(S_n = z)\}_{|z|_1 \leq n}$ and Q^{ξ}_{n-1}.

We now conclude (2.4) from (2.10), (2.11) as follows (the equalities and the inclusions here being understood as Q-a.s.):

$$\begin{aligned}
\left\{ \sum_{n \geq 1} I_n < \infty \right\} &= \{A_\infty < \infty\} \\
&= \{A_\infty < \infty,\ \langle M \rangle_\infty < \infty\} \\
&\subset \{A_\infty < \infty,\ \lim_{n \nearrow \infty} M_n \text{ exists and is finite}\} \\
&\subset \{Z_\infty > 0\}.
\end{aligned}$$

Here, on the third line, we have used a well-known property for martingales, e.g. (4.9) page 255 in Durrett (1995) or Neveu (1975).

Finally we prove (2.5). By (2.10), it is enough to show that

$$\{A_\infty = \infty\} \subset \left\{ \lim_{n \nearrow \infty} -\frac{\ln Z_n}{A_n} = 1 \right\}, \quad Q\text{-a.s.} \tag{2.12}$$

Thus, let us suppose that $A_\infty = \infty$. If $\langle M \rangle_\infty < \infty$, then again by (4.9) page 255 in Durrett (1995) or Neveu (1975). $\lim_{n \nearrow \infty} M_n$ exists and is finite and therefore (2.12) holds. If, on the contrary, $\langle M \rangle_\infty = \infty$, then

$$-\frac{\ln Z_n}{A_n} = \frac{M_n}{\langle M \rangle_n} \frac{\langle M \rangle_n}{A_n} + 1 \to 1 \quad Q\text{-a.s.}$$

by (2.10), (2.11) and the law of large numbers for martingales, see (4.10) page 255 in Durrett (1995) or Neveu (1975). This completes the proof of Theorem 1.1. □

Proof of Lemma 2.1: In this proof, we let $c_1, c_2, \dots$ stand for constants which are independent of $\{\alpha_i\}_{1\le i\le m}$. We have by direct computations that

$$Q[U^2] = c_1 \sum_{1\le i\le m} \alpha_i^2, \quad Q[U^3] \le c_2 \sum_{1\le i\le m} \alpha_i^2.$$

Then, (2.1) is obtained as follows;

$$\begin{aligned} c_1 \sum_{1\le i\le m} \alpha_i^2 &= Q\left[\frac{U}{\sqrt{2+U}} U\sqrt{2+U}\right] \\ &\le Q\left[\frac{U^2}{2+U}\right]^{1/2} Q\left[2U^2+U^3\right]^{1/2} \\ &\le c_3 Q\left[\frac{U^2}{2+U}\right]^{1/2} \left(\sum_{1\le i\le m} \alpha_i^2\right)^{1/2}. \end{aligned}$$

To prove the other inequalities, it is convenient to define a function $\varphi : (-1,\infty) \to [0,\infty)$ by $\varphi(u) = u - \ln(1+u)$, so that

$$-Q\left[\ln(1+U)\right] = Q\left[\varphi(U)\right].$$

Since $\frac{1}{4}\frac{u^2}{2+u} \le \varphi(u), \quad u > -1$, the left-hand-side inequality of (2.2) follows from (2.1). The right-hand-side inequality can be seen as follows. We have for any $\varepsilon \in (0,1)$,

$$\begin{aligned} Q\left[\varphi(U)\right] &= Q[\varphi(U) : 1+U \ge \varepsilon] + Q[\varphi(U) : 1+U \le \varepsilon] \\ &\le Q[\varphi(U) : 1+U \ge \varepsilon] - Q[\ln(1+U) : 1+U \le \varepsilon]. \end{aligned}$$

Since $\varphi(u) \le \frac{1}{2}(u/\varepsilon)^2$ if $1+u \ge \varepsilon$,

$$\begin{aligned} Q[\varphi(U) : 1+U \ge \varepsilon] &\le \tfrac{1}{2}\varepsilon^{-2} Q[U^2] \\ &= \tfrac{1}{2}\varepsilon^{-2} c_1 \sum_{1\le i\le m} \alpha_i^2. \end{aligned} \tag{2.13}$$

We now set $\gamma = -Q[\ln \eta_1] \ge 0$ and choose $\varepsilon > 0$ so small that $\ln(1/\varepsilon) - \gamma \ge 1$. We introduce another centered random variable $V = \sum_{1\le i\le m} \alpha_i(\ln \eta_i + \gamma)$. We then see from Jensen's inequality that

$$\begin{aligned} \{1+U \le \varepsilon\} &= \{V - \gamma \le \ln(1+U) \le \ln \varepsilon\} \\ &\subset \{-\ln(1+U) \le -V + \gamma\} \cap \{1 \le -V\}. \end{aligned}$$

Hence we have

$$\begin{aligned} -Q[\ln(1+U) : 1+U \le \varepsilon] &\le Q[-V : 1 \le -V] + \gamma Q\{1 \le -V\} \\ &\le (1+\gamma) Q[V^2] \\ &= c_4 \sum_{1\le i\le m} \alpha_i^2. \end{aligned}$$

This, together with (2.13) proves the right-hand-side inequality of (2.2). The proof of (2.3) is similar. Indeed, since $|\ln(1+u)| \le \varepsilon^{-1}\ln(\varepsilon^{-1})|u|$ if $\varepsilon \le 1+u$, we have that

$$Q[\ln^2(1+U) : \varepsilon \le 1+U] \le \varepsilon^{-2}\ln^2(\varepsilon^{-1})Q[U^2].$$

We see on the other hand that

$$\begin{aligned}\{1+U \le \varepsilon\} &= \{V-\gamma \le \ln(1+U) \le \ln\varepsilon\}\\ &\subset \{\ln^2(1+U) \le 2V^2+2\gamma^2\} \cap \{1 \le -V\}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}Q[\ln^2(1+U) : 1+U \le \varepsilon] &\le 2Q[V^2] + 2\gamma^2 Q\{1 \le -V\}\\ &\le c_5 \sum_{1\le i\le m} \alpha_i^2.\end{aligned}$$

□

3 Proof of Theorem 1.3

3.1 A sufficient condition for (Z1) via fractional moment

Lemma 3.1 *Suppose that there exist constants $c \in (0,\infty)$, $\theta \in (0,1)$ and a sequence $a_n \nearrow \infty$ such that*

$$Q[Z_n^\theta] \le c\exp(-a_n), \quad n \ge 1. \tag{3.1}$$

Then $Q\{Z_\infty = 0\} = 1$. If moreover

$$\sum_{n\ge1} \exp(-\delta a_n) < \infty \quad \textit{for some } \delta \in (0,1),$$

then (Z1) in Corollary 1.2 holds true.

Proof: The first statement follows from Fatou's lemma and the second from the Borel-Cantelli lemma. □

3.2 Proof of part (a)

We will check (3.1) with $a_n = cn$ for some $c > 0$. Set $\eta(x,j) = \exp(\beta\xi(x,j) - \lambda(\beta))$ and

$$Z_{n,m}^x = P^x\left[\exp\left(\sum_{1\le j\le m}(\beta\xi(S_j, j+n) - \lambda(\beta))\right)\right], \quad n,m \ge 1. \tag{3.2}$$

For $\theta \in (0,1)$, by the subadditive estimate $(u+v)^\theta \le u^\theta + v^\theta$, $u,v > 0$, we get

$$Z_n^\theta \le (2d)^{-\theta} \sum_{x,|x|_1=1} (\eta(x,1)Z_{1,n-1}^x)^\theta.$$

Since $Z^x_{1,n-1}$ has the same law as Z_{n-1},

$$Q[Z^\theta_n] \le r(\theta) Q[Z^\theta_{n-1}],$$

where $r(\theta) = (2d)^{1-\theta} Q[\eta(x,1)^\theta]$. Note that $\theta \mapsto \ln r(\theta)$ is convex, continuously differentiable, and that $\ln(2d) = \ln r(0) > \ln r(1) = 0$. Therefore $r(\theta) < 1$ for some $\theta \in (0,1)$ if and only if $0 < \left.\frac{d \ln r(\theta)}{d\theta}\right|_{\theta=1}$, which is equivalent to $\gamma_2(\beta) > \ln(2d)$. □

3.3 Proof of part (b)

We will check (3.1) with

$$a_n = \begin{cases} c_1 n^{1/3} & \text{if } d=1 \\ c_2\sqrt{\ln n} & \text{if } d=2 \end{cases} \tag{3.3}$$

where $c_1, c_2 \in (0,\infty)$ are some constants. In this respect, we first prove an auxiliary lemma.

Lemma 3.2 *For $\theta \in [0,1]$ and $\Lambda \subset \mathbb{Z}^d$,*

$$Q\left[Z^\theta_{n-1} I_n\right] \ge \frac{1}{|\Lambda|} Q\left[Z^\theta_{n-1}\right] - \frac{2}{|\Lambda|} P(S_n \not\in \Lambda)^\theta. \tag{3.4}$$

Proof: Repeating the argument in Liggett (1985, page 453), we see that

$$\begin{aligned} I_n &\ge \sum_{z\in\Lambda} \mu_{n-1}(S_n = z)^2 \\ &\ge \frac{1}{|\Lambda|}\mu_{n-1}(S_n \in \Lambda)^2 \\ &= \frac{1}{|\Lambda|}\left(1 - \mu_{n-1}(S_n \not\in \Lambda)\right)^2 \\ &\ge \frac{1}{|\Lambda|}\left(1 - 2\mu_{n-1}(S_n \not\in \Lambda)\right) \\ &\ge \frac{1}{|\Lambda|}\left(1 - 2\mu_{n-1}(S_n \not\in \Lambda)^\theta\right). \end{aligned}$$

Note also that

$$\begin{aligned} Q\left[Z^\theta_{n-1}\mu_{n-1}(S_n \not\in \Lambda)^\theta\right] &\le Q\left[Z_{n-1}\mu_{n-1}(S_n \not\in \Lambda)\right]^\theta \\ &= P(S_n \not\in \Lambda)^\theta. \end{aligned}$$

We therefore see that

$$\begin{aligned} Q\left[Z^\theta_{n-1} I_n\right] &\ge \frac{1}{|\Lambda|} Q\left[Z^\theta_{n-1}\right] - \frac{2}{|\Lambda|} Q\left[Z^\theta_{n-1}\mu_{n-1}(S_n \not\in \Lambda)^\theta\right] \\ &\ge \frac{1}{|\Lambda|} Q\left[Z^\theta_{n-1}\right] - \frac{2}{|\Lambda|} P(S_n \not\in \Lambda)^\theta. \end{aligned}$$

□

Assume now that $\theta \in (0,1)$, and define a function $f : (-1,\infty) \to [0,\infty)$ by

$$f(u) = 1 + \theta u - (1+u)^\theta.$$

It is then clear that there are constants $c_1, c_2 \in (0,\infty)$ such that

$$\frac{c_1 u^2}{2+u} \le f(u) \le c_2 u^2 \quad \text{for all } u \in (-1,\infty). \tag{3.5}$$

We see from (2.7), (3.5) and (2.1) that

$$\begin{aligned} Q^\xi_{n-1} \Delta Z^\theta_n &= Z^\theta_{n-1} Q^\xi_{n-1} \left((1+U_n)^\theta - 1 \right) \\ &= -Z^\theta_{n-1} Q^\xi_{n-1} f(U_n) \\ &\le -c_3 Z^\theta_{n-1} I_n. \end{aligned}$$

We therefore have by (3.4) that

$$QZ^\theta_n \le \left(1 - \frac{c_3}{|\Lambda|}\right) Q\left[Z^\theta_{n-1}\right] + \frac{2c_3}{|\Lambda|} P(S_n \notin \Lambda)^\theta. \tag{3.6}$$

For $d = 1$, set $\Lambda = (-n^{2/3}, n^{2/3}]$. Then,

$$P(S_n \notin \Lambda) = P\left(\left| \frac{S_n}{n^{1/2}} \right| \ge n^{1/6} \right) \le 2\exp(-\frac{n^{1/3}}{2}),$$

so that (3.6) reads,

$$QZ^\theta_n \le \left(1 - \frac{c_3}{2n^{2/3}}\right) Q\left[Z^\theta_{n-1}\right] + 4c_3 \exp(-\frac{n^{1/3}}{2}).$$

It is not difficult to conclude (3.1) with $a_n = c_1 n^{1/3}$ from the above.

For $d = 2$, we set

$$\Lambda = (-n^{1/2} \ln^{1/4} n, n^{1/2} \ln^{1/4} n]^2$$

to get (3.1) with $a_n = c_2 \sqrt{\ln n}$ in a similar way as above. □

4 Proof of Proposition 1.4

4.1 Proof of part (a)

This follows directly from (1.15). □

4.2 Proof of part (b)

We now state the following lemma which corresponds to Lemma 2.2 in Carmona and Hu (2001).

Lemma 4.1 *Let η_i, $1 \leq i \leq m$ be positive, non-constant i.i.d. random variables on a probability space $(\Xi, \mathcal{E}, Q)$ such that*

$$m_\theta \stackrel{\text{def}}{=} Q[\eta_1^\theta] < \infty \quad \text{for } \theta = \pm 4 \text{ and } m_1 = 1.$$

For $\{\alpha_i\}_{1 \leq i \leq m} \subset [0,\infty)^m$ such that $\sum_{1\leq i\leq m} \alpha_i = 1$, define a centered random variable $U > -1$ by $U = \sum_{1\leq i \leq m} \alpha_i \eta_i - 1$. Then,

$$1 - 2(m_2 - 1)(\alpha_1 + \alpha_2) + \frac{1}{C} \sum_{1\leq i\leq m} \alpha_i^2 \;\leq\; Q\left[\frac{\eta_1\eta_2}{(1+U)^2}\right] \leq m_2\sqrt{m_{-4}}, \tag{4.1}$$

$$m_2 - 2(m_3 - m_2)\alpha_1 + \frac{1}{C} \sum_{1\leq i\leq m} \alpha_i^2 \;\leq\; Q\left[\frac{\eta_1^2}{(1+U)^2}\right] \leq \sqrt{m_4 m_{-4}}, \tag{4.2}$$

where $C > 0$ is a constant which depends only on m_4.

Proof: Since the proofs of (4.1) and (4.2) are similar, we present that of (4.1) only.

$$\begin{aligned} Q\left[\eta_1\eta_2(1+U)^{-2}\right]^2 &\leq m_2^2 Q\left[(1+U)^{-4}\right] \\ &\leq m_2^2 Q\left[\sum_{1\leq i\leq m} \alpha_i \eta_i^{-4}\right] \\ &= m_2^2 m_{-4}, \end{aligned}$$

where, on the second line, we have used the Jensen inequality for the measure $\{\alpha_i\}$.

To prove the other inequalities, it is convenient to define a function $\varphi : (-1,\infty) \to [0,\infty)$ by $\varphi(u) = (1+u)^{-2} - 1 + 2u$. By an elementary inequality: $\frac{cu^2}{2+u} \leq \varphi(u), \;\; u > -1$, we have

$$Q\left[\eta_1\eta_2(1+U)^{-2}\right] \geq 1 - 2Q[\eta_1\eta_2 U] + c_1 Q\left[\frac{\eta_1\eta_2 U^2}{2+U}\right]. \tag{4.3}$$

On the other hand, we have by direct computations that

$$\begin{aligned} Q[\eta_1\eta_2 U] &= (m_2 - 1)(\alpha_1 + \alpha_2), \\ \frac{1}{c}\sum_{1\leq i\leq m} \alpha_i^2 \leq Q[\eta_1\eta_2 U^2] &\leq c \sum_{1\leq i\leq m} \alpha_i^2, \\ Q[\eta_1\eta_2 U^3] &\leq c \sum_{1\leq i\leq m} \alpha_i^2. \end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{c}\sum_{1\le i\le m}\alpha_i^2 &\le Q[\eta_1\eta_2U^2]\\
&= Q\left[\frac{\sqrt{\eta_1\eta_2}U}{\sqrt{2+U}}\sqrt{\eta_1\eta_2}U\sqrt{2+U}\right]\\
&\le Q\left[\frac{\eta_1\eta_2U^2}{2+U}\right]^{1/2} Q\left[2\eta_1\eta_2U^2+\eta_1\eta_2U^3\right]^{1/2}\\
&\le cQ\left[\frac{\eta_1\eta_2U^2}{2+U}\right]^{1/2}\left(\sum_{1\le i\le m}\alpha_i^2\right)^{1/2}.
\end{aligned}$$

Putting things together, we get (4.1) □

With Lemma 4.1 established, Proposition 1.4(b) is obtained merely by following the argument in Carmona and Hu (2001, Section 6). In doing so, we use Lemma 4.1 above in place of Lemma 2.2 in that paper. In fact, Carmona and Hu used the specific properties of the Gaussian random variable only in the proof of Lemma 2.2. □

4.3 Proof of part (c)

The first statement is derived using the convergence of I_n to 0. We now prove (1.17). Since $Q\{Z_\infty > 0\} = 1$ in the present case -see (1.8)-,it is enough to show that

$$Z_{n-1}^2 I_n = O(n^{-c}) \tag{4.4}$$

in Q-probability. With $\gamma = \lambda(2\beta) - 2\lambda(\beta) < -\ln(1-q)$, we compute

$$\begin{aligned}
Q\left[Z_{n-1}^2 I_n\right] &= Q\left[P^{\otimes 2}(e_{n-1}(\xi,S)e_{n-1}(\xi,\widetilde{S}) : S_n = \widetilde{S}_n)\right]\\
&= P^{\otimes 2}(Q[e_{n-1}(\xi,S)e_{n-1}(\xi,\widetilde{S})] : S_n = \widetilde{S}_n)\\
&= P^{\otimes 2}\left(\exp\left\{\gamma\sum_{j=1}^{n-1}\mathbf{1}_{S_j=\widetilde{S}_j}\right\} : S_n = \widetilde{S}_n\right)\\
&\le P^{\otimes 2}\left(\exp\left\{\alpha\gamma\sum_{j=1}^{n-1}\mathbf{1}_{S_j=\widetilde{S}_j}\right\}\right)^{1/\alpha} P^{\otimes 2}(S_n = \widetilde{S}_n)^{1/\alpha'} ,
\end{aligned}$$

using Hölder's inequality with the conjugate exponents α, α'. Since $\sum_{j\ge 1}\mathbf{1}_{S_j=\widetilde{S}_j}$ is geometrically distributed with failure probability $1-q \in (0,1)$ with q as in (1.8), the first factor on the right-hand side is bounded for $\alpha\gamma < -\ln(1-q)$. The second factor is $O(n^{-d/(2\alpha')})$. From this we obtain (1.17) for arbitrary $c < d[1+\gamma/\ln(1-q)]/2$. □

5 Proof of Proposition 1.5

Though the first statement is well known, we give a proof here for completeness. Recall the notation $Z^x_{n,m}$ introduced by (3.2) and note that for $m, n \geq 1$,

$$Z_{n+m} = Z_n \sum_x \mu_n\{S_n = x\} Z^x_{n,m} \ ,$$

Since $Z^x_{n,m}$ has the same law as Z_m, we have by Jensen's inequality that

$$\ln Z_{n+m} \geq \ln Z_n + \sum_x \mu_n\{S_n = x\} \ln Z^x_{n,m} \ .$$

Recall also the notation $\mathcal{E}_n$ and Q^ξ_n introduced in the proof of Theorem 1.1. Taking expectation and using independence, we obtain

$$\begin{aligned} Q[\ln Z_{n+m}] &\geq Q[\ln Z_n] + Q[\sum_x \mu_n\{S_n = x\} Q^\xi_n[\ln Z^x_{n,m}]] \\ &= Q[\ln Z_n] + Q[\ln Z_m] \ , \end{aligned}$$

i.e., $Q[\ln Z_n]$ is super-additive. From the super-additive Lemma we see that

$$\lim_{n \nearrow \infty} \frac{1}{n} Q[\ln Z_n] = \sup_n \frac{1}{n} Q[\ln Z_n] = \psi(\beta).$$

In order to prove the second statement (1.19), we write $\ln Z_n - Q[\ln Z_n]$ as a sum of $(\mathcal{E}_j)_{1 \leq j \leq n}$-martingale differences,

$$\ln Z_n - Q[\ln Z_n] = \sum_{j=1}^n V_{n,j}$$

with $V_{n,j} = Q^\xi_j[\ln Z_n] - Q^\xi_{j-1}[\ln Z_n]$. Set

$$\widehat{e}_{n,j} = \exp\left(\sum_{1 \leq k \leq n, k \neq j} (\beta \xi(S_k, k) - \lambda(\beta)) \right) \ , \quad \widehat{Z}_{n,j} = P[\widehat{e}_{n,j}].$$

Clearly $Q^\xi_j[\ln \widehat{Z}_{n,j}] = Q^\xi_{j-1}[\ln \widehat{Z}_{n,j}]$, and hence,

$$V_{n,j} = Q^\xi_j \left[\ln \frac{Z_n}{\widehat{Z}_{n,j}} \right] - Q^\xi_{j-1} \left[\ln \frac{Z_n}{\widehat{Z}_{n,j}} \right] \ .$$

By (2.2) in Lemma 2.1 with $\eta_\cdot = \eta(\cdot, j) = \exp(\beta \xi(\cdot, j) - \lambda(\beta))$ and $\alpha_\cdot = \frac{P[\widehat{e}_{n,j} : S_j = \cdot]}{\widehat{Z}_{n,j}}$, we see that

$$\begin{aligned} -Q^\xi_{j-1} \left[\ln \frac{Z_n}{\widehat{Z}_{n,j}} \right] &= -Q^\xi_{j-1} \left[Q \left[\ln \left(\sum_x \alpha_x \eta(x, j) \right) \middle| \mathcal{E}_{n,j} \right] \right] \\ &\in [0, c] \ , \end{aligned}$$

where $\mathcal{E}_{n,j} = \sigma[\xi(\cdot,k)\ ;\ 1 \le k \le n, k \neq j]$; note that $\eta(x,j)$ is independent of $\mathcal{E}_{n,j}$ and that $\alpha.$ is $\mathcal{E}_{n,j}$-measurable. Therefore, for $\theta \in \mathbb{R}$, we have by Jensen's inequality that

$$\begin{aligned} Q[\exp\theta V_{n,j}|\,\mathcal{E}_{n,j}] &\le e^{c\theta^+} Q\left[\exp\theta Q_j^{\xi}\left[\ln\frac{Z_n}{\widehat{Z}_{n,j}}\right]\middle|\,\mathcal{E}_{n,j}\right] \\ &\le e^{c\theta^+} Q\left[\left(\frac{Z_n}{\widehat{Z}_{n,j}}\right)^{\theta}\middle|\,\mathcal{E}_{n,j}\right] \\ &= e^{c\theta^+} Q\left[\left(\sum_x \alpha_x\eta(x,j)\right)^{\theta}\middle|\,\mathcal{E}_{n,j}\right], \end{aligned}$$

where $\theta^+ = \max\{0,\theta\}$. For $\theta \in (0,1)$, the function $x \mapsto x^{\theta}$ is concave on $(0,\infty)$ and hence

$$Q[\exp\theta V_{n,j}|\,\mathcal{E}_{n,j}] \le e^{c\theta} Q\left[\sum_x \alpha_x\eta(x,j)\middle|\,\mathcal{E}_{n,j}\right]^{\theta} = e^{c\theta}.$$

For $\theta \notin (0,1)$, the function $x \mapsto x^{\theta}$ is convex on $(0,\infty)$ and hence

$$\begin{aligned} Q[\exp\theta V_{n,j}|\,\mathcal{E}_{n,j}] &\le e^{c\theta^+}\sum_x \alpha_x Q\left[\eta(x,j)^{\theta}\middle|\,\mathcal{E}_{n,j}\right] \\ &= e^{c\theta^+}\sum_x \alpha_x Q[\eta(x,1)^{\theta}] \\ &= \exp\{c\theta^+ + \lambda(\theta\beta) - \theta\lambda(\beta)\}, \end{aligned}$$

Finally we conclude that

$$Q[\exp|V_{n,j}|] \le Q[\exp(V_{n,j}) + \exp(-V_{n,j})] \le c_1 < \infty\ .$$

Therefore, the large deviation estimate for sum of martingale-differences of Lesigne and Volný (2001, Theorem 3.2) applies to our case, yielding (1.19). The final statement in Proposition 1.5 is now a simple consequence of the Borel-Cantelli lemma. □

Remark 5.1 Let us remark that the stronger assumption (1.20) implies the exponential concentration (1.21). In what follows, $c_i = c_i(\delta)$ $(i = 1,2)$ and $c_i = c_i(\beta,\delta)$ $(i = 3,4,\dots)$ are positive constants. Note first that (1.20) implies that $\lambda(\beta) \le c_1 + c_2\beta^2$ for all $\beta \in \mathbb{R}$ and hence that

$$Q\left[\exp(\theta V_{n,j})|\mathcal{E}_{n,j}\right] \le \exp\left(c_3 + c_4\theta^2\right), \quad \text{for all } \theta \in \mathbb{R}, \tag{5.1}$$

by the computations in our proof of Proposition 1.5. By expanding the exponential and using the fact $Q[V_{n,j}|\mathcal{E}_{n,j}] = 0$, we can improve (5.1) into the following stronger form;

$$Q\left[\exp(\theta V_{n,j})|\mathcal{E}_{n,j}\right] \le \exp\left(c_5\theta^2\right), \quad \text{for all } \theta \in \mathbb{R}. \tag{5.2}$$

It is then not difficult to conclude (1.21) by (5.2) and a standard Gaussian estimate for a martingale.

Acknowledgements: We would like to thank Philippe Carmona and Yueyun Hu for sending us their preprint (2001) prior to its publication. FC thanks the Graduate School of Science of Kyoto University for its hospitality. FC is partially supported by CNRS, UMR 7599 "Probabilités et Modèles Aléatoires". TS is partially supported by JSPS Grant-in-Aid for Scientific Research, Kiban(C) 13640103. NY is partially supported by JSPS Grant-in-Aid for Scientific Research, Shorei(A) 40240303.

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