The period group of a characteristic function[1]

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Abstract

Let $\widehat{\mu}$ be the Fourier transform of a Borel probability measure μ on $\mathbb{R}^d$. We look at the closed abelian subgruop $\Gamma(\mu)$ of $\mathbb{R}^d$, which consists of the periods of the function $\widehat{\mu}$. We prove the following dichotomy: i) The support of μ is non-degenerate if and only if $\Gamma(\mu)$ is a lattice. ii) The support of μ is degenerate if and only if $\Gamma(\mu)$ contains a linear subspace $\neq \{0\}$ of $\mathbb{R}^d$. A similar dichotomy is also discussed for the period group of the function $|\widehat{\mu}|$.

1 Introduction

1.1 Definitions and background

In this article, we address a fundamental question of understanding the characteristic function of a probability measure, perhaps from slightly different perspective from conventional ones. For a Borel probability measure μ on $\mathbb{R}^d$, we write its Fourier transform by:

$$\widehat{\mu}(\theta) = \int_{\mathbb{R}^d} \exp(\mathbf{i}\theta \cdot x)\mu(dx), \quad \theta \in \mathbb{R}^d, \tag{1.1}$$

where $\theta \cdot x$ denotes the standard inner product of $\mathbb{R}^d$. In the context of probability theory, the above function is referred to as the characteristic function. There, the speed at which the function $\widehat{\mu}(\theta)$ approaches to $\widehat{\mu}(0) = 1$ as $\theta \to 0$ is of great interest, in connection with various limit theorems and recurrence/transience criteria. On the other hand, depending on the measure μ, there exist non-zero θ' s for which $\widehat{\mu}(\theta) = 1$, for example, the law for the each step of the simple symmetric random walk. This brings our attention to the following set:

$$\Gamma(\mu) = \{\theta \in \mathbb{R}^d \; ; \; \widehat{\mu}(\theta) = 1\}. \tag{1.2}$$

In this article, we investigate the structure of this set in connection with the support of μ, denoted by supp $[\mu]$ in the sequel. Clearly, $\theta \in \Gamma(\mu)$ if and only if $\mathrm{Re}\,\widehat{\mu}(\theta) = 1$. In addition,

$$1 - \mathrm{Re}\,\widehat{\mu}(\theta) = \int_{\mathbb{R}^d} (1 - \cos(\theta \cdot x))\, \mu(dx).$$

Therefore,

$$\theta \in \Gamma(\mu) \iff \theta \cdot x \in 2\pi\mathbb{Z},\ \mu\text{-a.e.} \iff \exp(\mathbf{i}\theta \cdot x) = 1,\ \mu\text{-a.e.} \tag{1.3}$$

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This shows that $\Gamma(\mu)$ is an abelian subgroup of $\mathbb{R}^d$, which consists exactly of the periods of the function $\widehat{\mu}$.

To explain the content of this article, let us introduce some definitions. A subset Γ of $\mathbb{R}^d$ is said to be *non-degenerate*, if it contains a linear basis of $\mathbb{R}^d$. Otherwise, the set Γ is said to be *degenerate*. A subset Γ of $\mathbb{R}^d$ is said to be a *lattice*, if there exist linearly independent vectors $\gamma_1, \ldots, \gamma_k \in \mathbb{R}^d$ such that

$$\Gamma = \mathbb{Z}\gamma_1 + \cdots + \mathbb{Z}\gamma_k. \tag{1.4}$$

We prove the following dichotomy:

- The support of μ is non-degenerate if and only if $\Gamma(\mu)$ is a lattice (Proposition 1.2.1); (1.5)
- The support of μ is degenerate if and only if $\Gamma(\mu)$ contains a linear subspace $\neq \{0\}$ of $\mathbb{R}^d$ (Proposition 1.2.2). (1.6)

We are also interested in the following set:

$$\Gamma_2(\mu) = \{\theta \in \mathbb{R}^d \; ; \; |\widehat{\mu}(\theta)| = 1\}, \tag{1.7}$$

for which we obtain a similar dichotomy as follows:

- The set $\mathrm{supp}\,[\mu] - a$ is non-degenerate for all $a \in \mathbb{R}^d$ if and only if $\Gamma_2(\mu)$ is a lattice (Proposition 1.2.3); (1.8)
- There exists an $a \in \mathbb{R}^d$ such that the set $\mathrm{supp}\,[\mu] - a$ is degenerate if and only if $\Gamma_2(\mu)$ contains a linear subspace $\neq \{0\}$ of $\mathbb{R}^d$ (Proposition 1.2.4). (1.9)

These statements seem quite fundamental. Therefore, we naturally believed that these should be somewhere in the literature. To our perplexity, though, we were not able to dig out any systematic study of this kind, except some fragmentary examples, e.g., [1, PROBLEM 26.1 on p.353], [2, T1 on p. 67]. Thus, we finally decided to write them down ourselves, instead of pursuing the effort of searching for them in the literature.

We now look at the set $\Gamma_2(\mu)$ more in detail. For this purpose, it is convenient to introduce a Borel probability measure μ_2 on $\mathbb{R}^d$ by:

$$\mu_2(dx) = \mu^{\otimes 2}((x_1, x_2) \in (\mathbb{R}^d)^2 \; ; \; x_1 - x_2 \in dx), \tag{1.10}$$

where $\mu^{\otimes 2}$ denotes the direct product. Then,

$$\Gamma_2(\mu) = \Gamma(\mu_2), \tag{1.11}$$

since

$$|\widehat{\mu}(\theta)|^2 = \widehat{\mu}(\theta)\widehat{\mu}(-\theta) = \widehat{\mu_2}(\theta) \tag{1.12}$$

for all $\theta \in \mathbb{R}^d$. We see from (1.11) and (1.12) that the set $\Gamma_2(\mu)$ consists of the period of the function $|\widehat{\mu}|$. We write

$$D(\mu) = \mathrm{supp}\,[\mu] - \mathrm{supp}\,[\mu] = \{x_1 - x_2 \; ; \; x_1, x_2 \in \mathrm{supp}\,[\mu]\}. \tag{1.13}$$

Then, we have

$$\mathrm{supp}\,[\mu_2] = \overline{D(\mu)}, \tag{1.14}$$

which shows in particular that

$$\text{the set supp } [\mu_2] \text{ is degenerate if supp } [\mu] \text{ is.} \tag{1.15}$$

To prove (1.14), it is enough to verify that

$$D(\mu) \subset \text{supp } [\mu_2] \subset \overline{D(\mu)}. \tag{1.16}$$

To prove the first inclusion of (1.16), suppose that $x_1, x_2 \in \text{supp } [\mu]$. Then, $\mu(B(x_j, r/2)) > 0$ for any $r > 0$ $(j = 1, 2)$. Thus,

$$\mu_2(B(x_1 - x_2, r)) \geq \mu^{\otimes 2}(B(x_1, r/2) \times B(x_2, r/2)) = \mu(B(x_1, r/2))\mu(B(x_2, r/2)) > 0.$$

Hence $x_1 - x_2 \in \text{supp } [\mu_2]$. To prove the second inclusion of (1.16), suppose that $z \notin \overline{D(\mu)}$. Then, there exists $r \in (0, \infty)$ such that $B(z, r) \subset \mathbb{R}^d \backslash D(\mu)$. This implies that

$$\mu_2(B(z, r)) = \int_{\text{supp } [\mu]^2} \mathbf{1}_{B(z,r)}(x_1 - x_2)\mu^{\otimes 2}(dx_1 dx_2) = 0.$$

Hence $z \notin \text{supp } [\mu_2]$.
The converse to (1.15) is not true in general (for example $\mu = (\delta_{e_1} + \cdots + \delta_{e_d})/d$, where $e_1, \ldots, e_d$ are canonical basis). By applying Lemma 2.3.1 f2) for $A = \text{supp } [\mu]$, we see that the converse to (1.15) is true under the following extra assumption:

$$\int_{\mathbb{R}^d} |x|\mu(dx) < \infty, \quad \int_{\mathbb{R}^d} x\mu(dx) = 0. \tag{1.17}$$

We conclude this introduction with the following simple consequence of the relation (1.3) (cf. [1, PROBLEM 26.1 on p.353] for the case of $d = 1$):

$$\text{If the set } \Gamma(\mu) \text{ is non-degenerate, then the measure } \mu \text{ concentrates on a lattice.} \tag{1.18}$$

This can be seen as follows. Suppose that $\gamma_1, \ldots, \gamma_d \in \Gamma(\mu)$ are linearly independent. We write $\gamma_\alpha = (\gamma_{\alpha,\beta})_{\beta=1}^d$ $(\alpha = 1, \ldots, d)$. Then the matrix $C = (\gamma_{\alpha,\beta})_{1 \leq \alpha,\beta \leq d}$ is invertible and satisfies $Cx \in 2\pi\mathbb{Z}^d$, μ-a.e. Thus, the measure μ concentrates on the lattice $2\pi C^{-1}\mathbb{Z}^d$.

1.2 Results

For $x \in \mathbb{R}^d$ and $r \in (0, \infty)$, we denote by $B(x, r)$ the open ball in $\mathbb{R}^d$ centered at x, with radius r.

Proposition 1.2.1 *The following are equivalent:*

a) *The support of μ is non-degenerate.*

b) *The set $\Gamma(\mu)$ has no accumulation point.*

c) *The set $\Gamma(\mu)$ is a lattice.*

Proposition 1.2.2 *The following are equivalent:*

a') *The support of μ is degenerate.*

b') *The set* $\Gamma(\mu)$ *has an accumulation point.*

c') *The set* $\Gamma(\mu)$ *contains a linear subspace* $\neq \{0\}$ *of* $\mathbb{R}^d$.

Proposition 1.2.3 *The following are equivalent:*

a2) *The set* $\mathrm{supp}\,[\mu] - a$ *is non-degenerate for all* $a \in \mathbb{R}^d$.

b2) *The set* $\Gamma_2(\mu)$ *has no accumulation point.*

c2) *The set* $\Gamma_2(\mu)$ *is a lattice.*

Proposition 1.2.4 *The following are equivalent:*

a2') *There exists an* $a \in \mathbb{R}^d$ μ *such that the set* $\mathrm{supp}\,[\mu] - a$ *is degenerate.*

b2') *The set* $\Gamma_2(\mu)$ *has an accumulation point.*

c2') *The set* $\Gamma_2(\mu)$ *contains a linear subspace* $\neq \{0\}$ *of* $\mathbb{R}^d$.

2 Proofs

2.1 Proof of Proposition 1.2.1

a) $\Rightarrow$ b): We use Lemma 2.1.2 below to prove this part. It follows from (2.4) that there exists $\delta > 0$ such that for all $\gamma \in \Gamma$ and all $\theta \in B(\gamma, \delta) \backslash \{\gamma\}$, $\mathrm{Re}\,\widehat{\mu}(\theta) < 1$. This implies b).
b) $\Rightarrow$ c): This part follows from Lemma 2.1.3 below.
c) $\Rightarrow$ a): We show this implication by showing a') $\Rightarrow$ c') of Proposition 1.2.2. By assumption there exists a unit vector u such that $u \cdot x = 0$ for all $x \in \mathrm{supp}\,[\mu]$. Since the function $x \mapsto u \cdot x$ is continuous, this implies that $u \cdot x = 0$, μ-a.a. x and hence $\widehat{\mu}(tu) = 1$ for all $t \in \mathbb{R}$. Thus, $\mathbb{R}u \subset \Gamma(\mu)$. □

Lemma 2.1.1 *Suppose that* $\mathrm{supp}\,[\mu]$ *is non-degenerate. Then, for any* $n = 1, \ldots, d$,

$$\mu^{\otimes n}(A_n) > 0,$$

where

$$A_n = \{(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \ ; \ x_1, \ldots, x_n \text{ are linearly independent}\}. \tag{2.1}$$

Proof: We proceed by induction on n. $\mu(A_1) > 0$, since $\mu \neq \delta_0$. Let $(x_1, \ldots, x_{n-1}) \in (\mathbb{R}^d)^{n-1}$ be arbitrary. Since the support of μ contains a linear basis of $\mathbb{R}^d$, we have that

$$f(x_1, \ldots, x_{n-1}) \stackrel{\text{def}}{=} \mu\left(\mathbb{R}^d \backslash (\mathbb{R}x_1 + \ldots + \mathbb{R}x_{n-1})\right) > 0.$$

Therefore, if $\mu^{\otimes n-1}(A_{n-1}) > 0$, then,

$$\mu^{\otimes n}(A_n) = \int_{A_{n-1}} f(x_1, \ldots, x_{n-1}) \mu^{\otimes n-1}(dx_1 \cdots dx_{n-1}) > 0.$$

□

Lemma 2.1.2 *The following conditions* a), d), e) *are related as* a) $\Longleftrightarrow$ d) $\Longrightarrow$ e).

a) *The support of μ is non-degenerate.*

d) *There exists a Borel subset B of* $\operatorname{supp}[\mu]$ *with $\mu(B) > 0$ such that*

$$\int_B |x|^2 \mu(dx) < \infty, \tag{2.2}$$

$$\text{The matrix } \left(\int_B x_\alpha x_\beta \mu(dx)\right)_{\alpha,\beta=1,\ldots,d} \text{ is strictly positive definite.} \tag{2.3}$$

e) *There exist positive constants c, r such that for all $\gamma \in \Gamma(\mu)$ and $\theta \in B(\gamma, r)$,*

$$\operatorname{Re} \widehat{\mu}(\theta) \leq 1 - c|\theta - \gamma|^2. \tag{2.4}$$

Proof: a) $\Rightarrow$ d): Let $B = B(0,r) \cap \operatorname{supp}[\mu]$. If $r > 0$ is large enough, then, B contains a linear basis of $\mathbb{R}^d$, $\mu(B) > 0$, and (2.2) holds. To prove (2.3), suppose that $\theta \in \mathbb{R}^d$ satisfies $\int_B (\theta \cdot x)^2 \mu(dx) = 0$. Let μ_B be the measure defined by $\mu_B(A) \stackrel{\text{def}}{=} \frac{1}{\mu(B)} \mu(A \cap B)$. Then, μ_B is non-degenerate by the choice of B. Hence, by applying Lemma 2.1.1 to μ_B, the set $A_d \subset (\mathbb{R}^d)^d$ has positive measure with respect to $\mu_B^{\otimes d}$. Moreover,

$$\int_{B^n} \left((\theta \cdot x_1)^2 + \cdots + (\theta \cdot x_d)^2\right) \mu_B^{\otimes d}(dx_1 \cdots dx_d) = d \int_B (\theta \cdot x)^2 \mu(dx) = 0.$$

In particular, the integrand of the left-hand side integral vanishes $\mu_B^{\otimes d}$-a.e. on the set A_d. This implies that $\theta = 0$.
a) $\Leftarrow$ d): Suppose that d) holds and that the set $\operatorname{supp}[\mu]$ is degenerate. Then, there exists $\theta \in \mathbb{R}^d \backslash \{0\}$ which is orthogonal to all vectors in $\operatorname{supp}[\mu]$. Hence $\int_B (\theta \cdot x)^2 \mu(dx) = 0$, a contradiction.
a) $\Rightarrow$ e): Since $\gamma \in \Gamma(\mu)$ is a period of $\widehat{\mu}$, it is enough to prove (2.4) for $\gamma = 0$. By the proof of a) $\Rightarrow$ d), the condition d) holds for $B = B(0,r) \cap \operatorname{supp}[\mu]$ with large enough $r > 0$. Let $\lambda > 0$ be such that

$$\int_B (\theta \cdot x)^2 \mu(dx) \geq \lambda |\theta|^2, \quad \forall \theta \in \mathbb{R}^d.$$

Note on the other hand that $1 - \cos t \geq \frac{2t^2}{\pi^2}$ for $|t| \leq \pi$. Thus, for $\theta \in B(0, \pi/r)$,

$$\begin{aligned} 1 - \operatorname{Re} \widehat{\mu}(\theta) &\geq \int_B (1 - \cos(\theta \cdot x))\, \mu(dx) \\ &\geq \frac{2}{\pi^2} \int_{B(0,r)} (\theta \cdot x)^2 \mu(dx) \geq \frac{2\lambda}{\pi^2} |\theta|^2. \end{aligned}$$

□

Remark: By the proof of Proposition 1.2.1, the condition e) of Lemma 2.1.2 implies c) of Proposition 1.2.1. Therefore, it is also one of the equivalent conditions to the non-degeneracy of $\operatorname{supp}[\mu]$.

Lemma 2.1.3 *Suppose that Γ is an abelian subgroup of $\mathbb{R}^d$ without accumulation point. Then, Γ is a lattice.*

Proof: Starting from $k = 1$, and then, by a successive procedure explained below, we will find linearly independent vectors $\gamma_1, \ldots, \gamma_k \in \Gamma$ such that

1) $\Gamma \cap (\mathbb{R}\gamma_1 + \ldots + \mathbb{R}\gamma_k) = \mathbb{Z}\gamma_1 + \ldots + \mathbb{Z}\gamma_k$.

By repeating this procedure up to some $k \le d$ for which the set on the left-hand side of 1) coincides with Γ, we conclude that Γ is a lattice.

It follows from the assumption that the origin is an isolated point of Γ, and that $\Gamma\backslash\{0\}$ is closed. Thus, there exists $\gamma_1 \in \Gamma$ such that

$$|\gamma_1| = \min\{|\gamma| \ ; \ \gamma \in \Gamma\backslash\{0\}\}.$$

Similarly, we can find $\gamma_k \in \Gamma$ $(k = 2, 3, \ldots)$ such that

$$|\gamma_k| = \min\{|\gamma| \ ; \ \gamma \in \Gamma\backslash\{0, \pm\gamma_1, \ldots, \pm\gamma_{k-1}\}\}.$$

We write

$$\Delta(\gamma_1, \ldots, \gamma_k) = \{t_1\gamma_1 + \ldots + t_k\gamma_k \ ; \ (t_1, \ldots, t_k) \in [0, 1)^k\}.$$

If $\Gamma = \mathbb{Z}\gamma_1$, we are done. If not, we proceed as follows. Clearly, $\Gamma \cap \Delta(\gamma_1) = \{0\}$. This implies that

2) $\Gamma \cap \mathbb{R}\gamma_1 = \mathbb{Z}\gamma_1$.

Indeed, suppose that $\gamma \in (\Gamma \cap \mathbb{R}\gamma_1)\backslash\mathbb{Z}\gamma_1$. Then there exists $n \in \mathbb{Z}$ such that $\gamma \in n\gamma_1 + \Delta(\gamma_1)$. Then $0 \neq \gamma - n\gamma_1 \in \Gamma \cap \Delta(\gamma_1)$, which is a contradiction.
By 2), we have $\gamma_2 \notin \mathbb{R}\gamma_1$, hence γ_1 and γ_2 is linearly independent. If $\Gamma = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$, we are done. If not, we proceed as follows. Note that

3) $\Gamma \cap \Delta(\gamma_1, \gamma_2) = \{0\}$.

Indeed, suppose that $\gamma \stackrel{\text{def}}{=} t_1\gamma_1 + t_2\gamma_2 \in \Gamma\backslash\{0, \pm\gamma_1, \pm\gamma_2\}$ for some $(t_1, t_2) \in [0, 1)^2$. Suppose first that $t_1 + t_2 \le 1$. Then,

$$|\gamma| < |\gamma_1| \vee |\gamma_2| \le |\gamma_2|,$$

which contradicts the definition of γ_2. Suppose on the other hand that $t_1 + t_2 > 1$. Then, $\gamma_1 - \gamma = (1 - t_1)\gamma_1 + t_2(-\gamma_2) \in \Gamma\backslash\{0, \pm\gamma_1, \pm\gamma_2\}$ and

$$|\gamma_1 - \gamma| < |\gamma_1| \vee |\gamma_2| \le |\gamma_2|,$$

which again contradicts the definition of γ_2.

4) $\Gamma \cap (\mathbb{R}\gamma_1 + \mathbb{R}\gamma_2) = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2$.

Indeed, suppose that $\gamma \in (\Gamma \cap (\mathbb{R}\gamma_1 + \mathbb{R}\gamma_2))\backslash(\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2)$. Then there exist $n_1, n_2 \in \mathbb{Z}$ such that $v \in n_1\gamma_1 + n_2\gamma_2 + \Delta(\gamma_1, \gamma_2)$. Then $0 \neq \gamma - n_1\gamma_1 - n_2\gamma_2 \in \Gamma \cap \Delta(\gamma_1, \gamma_2)$, which is a contradiction.
Repeating this procedure, we obtain linearly independent vectors $\gamma_1, \ldots, \gamma_k \in \Gamma$ for some $k \in \{1, 2, \ldots, d\}$ such that 1) holds true. $\square$

2.2 Proof of Proposition 1.2.2

a') $\Rightarrow$ c'): This part is already shown in the proof of Proposition 1.2.1.
c') $\Rightarrow$ b'): This is obvious.
b') $\Rightarrow$ a'): This follows from the part: a) $\Rightarrow$ b) of Proposition 1.2.1. $\Box$

Remark: The proof of b') $\Rightarrow$ c') can alternatively be given by the following lemma. Note that the set $\Gamma(\mu)$ is closed, since the function $\widehat{\mu}$ is continuous.

Lemma 2.2.1 *Suppose that Γ is a closed abelian subgroup of $\mathbb{R}^d$ with an accumulation point. Then, Γ contains a linear subspace $\neq \{0\}$ of $\mathbb{R}^d$.*

Proof: By translation, the origin is also an accumulation point of Γ. Thus, there exists a sequence $\gamma_n \in \Gamma \backslash \{0\}$ which converges to the origin. Then, by taking a subsequence if necessary, we may assume that $u_n \stackrel{\text{def}}{=} \gamma_n/|\gamma_n|$ converges to a unit vector u. We have for any $t \in \mathbb{R}$ that $t_n \stackrel{\text{def}}{=} |\gamma_n|\lfloor t/|\gamma_n|\rfloor \stackrel{n\to\infty}{\longrightarrow} t$. Hence

$$\Gamma \ni \lfloor t/|\gamma_n|\rfloor \gamma_n = t_n u_n \stackrel{n\to\infty}{\longrightarrow} tu.$$

Since Γ is closed, we see that $tu \in \Gamma$. Thus, $\mathbb{R}u \subset \Gamma$. $\Box$

2.3 Proof of Proposition 1.2.3 and Proposition 1.2.4

Proposition 1.2.3 and Proposition 1.2.4 are reduced to Proposition 1.2.1 and Proposition 1.2.2, thanks to the following

Lemma 2.3.1 *The following are equivalent:*

a2) *The set* supp $[\mu] - a$ *is non-degenerate for all $a \in \mathbb{R}^d$.*

d2) *There exists an $a \in$ supp $[\mu]$ such that set* supp $[\mu] - a$ *is non-degenerate.*

e2) *The set* supp $[\mu_2]$ *is non-degenerate.*

f2) *There exists a closed set $A \subset$ supp $[\mu]$ with $\mu(A) > 0$ such that*

$$\int_A |x|\mu(dx) < \infty \;\; \textit{and } A - m_A \textit{ is non-degenerate,}$$

where $m_A = \frac{1}{\mu(A)} \int_A x\mu(dx)$.

Proof: a2) $\Rightarrow$ d2): Obvious.
d2) $\Rightarrow$ e2): By definition of the set $D(\mu)$, we have $D(\mu) \supset$ supp $[\mu] - a$ for all $a \in$ supp $[\mu]$. Thus, d2) implies that the set $D(\mu)$ is non-degenerate, and hence so is supp $[\mu_2]$ by (1.14).
e2) $\Rightarrow$ a2): We prove the contrapositive. Suppose that there exists an $a \in$ supp $[\mu]$ such that the set supp $[\mu] - a$ is degenerate. Then, $D(\mu)$ is degenerate, since

$$D(\mu) = (\text{supp}\,[\mu] - a) - (\text{supp}\,[\mu] - a).$$

Thus, supp $[\mu_2]$ is degenerate by (1.14).
e2) $\Rightarrow$ f2): We already have conditions a2) and d2) at our disposal. Let $a \in$ supp $[\mu]$ be as in

condition d2). We take $r > 0$ large enough so that $(\overline{B(a,r)} \cap \mathrm{supp}\,[\mu]) - a$ is nondegenerate and set $A = \overline{B(a,r)} \cap \mathrm{supp}\,[\mu]$. Then, A is closed, $\mu(A) = \mu(\overline{B(a,r)}) > 0$ (since $a \in \mathrm{supp}\,[\mu]$), and $A - a$ is non-degenerate. We then set $\mu_A(dx) = \mu(A \cap dx)/\mu(A)$. Then, $a \in A = \mathrm{supp}\,[\mu_A]$ and $A - a$ is non-degenerate. Thus, the measure μ_A satisfies condition d2), which is equivalent to a2). Therefore, $A - m$ is non-degenerate for all $m \in \mathbb{R}^d$, and hence in particular for $m = m_A$.
f2) $\Rightarrow$ e2): Suppose that $\mathrm{supp}\,[\mu_2]$ is degenerate and that $A \subset \mathrm{supp}\,[\mu]$ be any closed set with $\mu(A) > 0$ such that $\int_A |x|\mu(dx) < \infty$. Since $\mathrm{supp}\,[\mu_2]$ is degenerate, so is $\mathrm{supp}\,[(\mu_A)_2]$. Therefore by a2), there exists an $a \in \mathbb{R}^d$ a linear subspace $L \subset \mathbb{R}^d$ with positive codimension such that $A = \mathrm{supp}\,[\mu_A] \subset a + L$. This implies that $m_A \in a + L$, and hence

$$A - m_A \subset A - a + (a - m_A) \subset L.$$

Therefore, $A - m_A$ is degenerate. □

Proof of Proposition 1.2.3: a2) is equivalent to the condition e2) of Lemma 2.3.1. Thus, applying Proposition 1.2.1 to the measure μ_2, we see the equivalence stated in the proposition. □

Proof of Proposition 1.2.4: By Lemma 2.3.1, the condition a2') is equivalent to the degeneracy of the set $\mathrm{supp}\,[\mu_2]$. Thus, applying Proposition 1.2.2 to the measure μ_2, we see the equivalence stated in the proposition. □

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