Localization for Branching Random Walks in Random Environment

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Abstract

We consider branching random walks in d-dimensional integer lattice with time-space i.i.d. offspring distributions. This model is known to exhibit a phase transition: If $d \geq 3$ and the environment is "not too random", then, the total population grows as fast as its expectation with strictly positive probability. If, on the other hand, $d \leq 2$, or the environment is "random enough", then the total population grows strictly slower than its expectation almost surely. We show the equivalence between the slow population growth and a natural localization property in terms of "replica overlap". We also prove a certain stronger localization property, whenever the total population grows strictly slower than its expectation almost surely.

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1 Introduction

1.1 Branching random walks in random environment (BRWRE)

We begin by introducing the model. We write $\mathbb{N} = \{0, 1, 2, ...\}$, $\mathbb{N}^* = \{1, 2, ...\}$ and $\mathbb{Z} = \{\pm x \ ; \ x \in \mathbb{N}\}$. For $x \in \mathbb{Z}^d$, $|x| = (|x_1|^2 + .. + |x_d|^2)^{1/2}$. The following formulation is an analogue of [10, section 4.2], where non-random offspring distributions are considered. See also [3, section 5] for the random offspring case.
Let $X = \{X^\nu_{t,x}\}_{(t,x,\nu)\in\mathbb{N}\times\mathbb{Z}^d\times\mathbb{N}^*}$ be $\mathbb{Z}^d$-valued independent random variables defined on a probability space $(\Omega_X, \mathcal{F}_X, P_X)$ such that

$$P_X(X^\nu_{t,x} = y) = p(x, y) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2d} & \text{if } |x-y| = 1, \\ 0 & \text{if } |x-y| \neq 1. \end{cases} \tag{1.1}$$

For each $(t, x) \in \mathbb{N} \times \mathbb{Z}^d$, let

$$q_{t,x} = (q_{t,x}(k))_{k\in\mathbb{N}} \in [0, 1]^{\mathbb{N}}, \quad \sum_{k\in\mathbb{N}} q_{t,x}(k) = 1$$

be a probability measure on $\mathbb{N}$, which we refer to as the *offspring distribution.* We consider a measurable space $(\Omega_K, \mathcal{F}_K)$ and, for each fixed $q = (q_{t,x})_{(t,x)\in\mathbb{N}\times\mathbb{Z}^d}$, a probability measure P^q_K such that $\mathbb{N}$-valued random variables $K = \{K^\nu_{t,x}\}_{(t,x,\nu)\in\mathbb{N}\times\mathbb{Z}^d\times\mathbb{N}^*}$ defined on $(\Omega_K, \mathcal{F}_K, P^q_K)$ are independent with the laws:

$$P^q_K(K^\nu_{t,x} = k) = q_{t,x}(k), \quad k \in \mathbb{N}. \tag{1.2}$$

For each fixed $q = (q_{t,x})_{(t,x)\in\mathbb{N}\times\mathbb{Z}^d}$, we realize the families X and K of random variables simultaneously on the probability space

$$(\Omega_X \times \Omega_K, \mathcal{F}_X \otimes \mathcal{F}_K, P^q) \ \text{ where } \ P^q = P_X \otimes P^q_K. \tag{1.3}$$

Then, the branching random walk (BRW) with the fixed offspring distributions $q = (q_{t,x})_{(t,x)\in\mathbb{N}\times\mathbb{Z}^d}$ is described as the following dynamics:

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- At time $t = 0$, there is one particle at the origin $x = 0$.
- Suppose that there are $N_{t,x}$ particles at each site $x \in \mathbb{Z}^d$ at time t. At time $t+1$, the ν-th particle at a site x ($\nu = 1, .., N_{t,x}$) jumps to a site $X_{t,x}^{\nu}$. At arrival, it dies, leaving $K_{t,x}^{\nu}$ new particles there.

We now go on to define the *branching random walk in random environment* (BRWRE). We set $\Omega_q = \mathcal{P}(\mathbb{N})^{\mathbb{N}\times\mathbb{Z}^d}$, where $\mathcal{P}(\mathbb{N})$ denotes the set of probability measures on $\mathbb{N}$. Thus, each $q \in \Omega_q$ is a function $(t, x) \mapsto q_{t,x} = (q_{t,x}(k))_{k\in\mathbb{N}}$ from $\mathbb{N} \times \mathbb{Z}^d$ to $\mathcal{P}(\mathbb{N})$. The set $\mathcal{P}(\mathbb{N})$ is equipped with the natural Borel σ-field induced from that of $[0,1]^{\mathbb{N}}$. We denote by $\mathcal{F}_q$ the product σ-field on Ω_q.We fix a probability measure Q on $(\Omega_q, \mathcal{F}_q)$, under which $\{q_{t,x}\}_{(t,x)\in\mathbb{N}\times\mathbb{Z}^d}$ are i.i.d. offspring distributions. Finally, we define the probability space $(\Omega, \mathcal{F}, P)$ by

$$\begin{aligned} &\Omega = \Omega_X \times \Omega_K \times \Omega_q, \quad \mathcal{F} = \mathcal{F}_X \otimes \mathcal{F}_K \otimes \mathcal{F}_q, \\ &P(A) = \int_A Q(dq) P^q(d\omega), \quad A \in \mathcal{F}. \end{aligned} \tag{1.4}$$

In this setup, we consider the dynamics explained as before. In particular, we look at the population $N_{t,x}$ at time-space location $(t, x) \in \mathbb{N} \times \mathbb{Z}^d$, which is defined inductively by

$$N_{0,x} = \delta_{0,x}, \quad N_{t,x} = \sum_{y\in\mathbb{Z}^d} \sum_{\nu=1}^{N_{t-1,y}} \delta_x(X_{t-1,y}^{\nu}) K_{t-1,y}^{\nu}, \quad t \geq 1. \tag{1.5}$$

We consider the filtration:

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_t = \sigma(X_{s,\cdot}^{\cdot}, K_{s,\cdot}^{\cdot}, q_{s,\cdot}\ ;\ s \leq t-1) \quad t \geq 1, \tag{1.6}$$

which the process $t \mapsto (N_{t,x})_{x\in\mathbb{Z}^d}$ is adapted to. The total population at time t is then given by

$$N_t = \sum_{x\in\mathbb{Z}^d} N_{t,x} = \sum_{y\in\mathbb{Z}^d} \sum_{\nu=1}^{N_{t-1,y}} K_{t-1,y}^{\nu}. \tag{1.7}$$

We remark that the total population is exactly the classical Galton-Watson process if $q_{t,x} \equiv q$, where $q \in \mathcal{P}(\mathbb{N})$ is non-random. On the other hand, if $\mathbb{Z}^d$ is replaced by a singleton, then N_t is the population of the Smith-Wilkinson model [11].

For $p > 0$, we write

$$m^{(p)} = Q[m_{t,x}^{(p)}] \ \text{ with } \ m_{t,x}^{(p)} = \sum_{k\in\mathbb{N}} k^p q_{t,x}(k), \tag{1.8}$$

$$m = m^{(1)}. \tag{1.9}$$

Note that for $p \geq 1$,

$$m^p \leq Q[m_{t,x}^p] \leq m^{(p)}$$

by Hölder's inequality. We set

$$\overline{N}_{t,x} = N_{t,x}/m^t \ \text{ and } \ \overline{N}_t = N_t/m^t. \tag{1.10}$$

$\overline{N}_t = N_t/m^t$ is a martingale, and therefore the following limit always exists:

$$\overline{N}_\infty = \lim_t \overline{N}_t, \quad P\text{-a.s.} \tag{1.11}$$

We denote the density of the population by:

$$\rho_{t,x} = \frac{N_{t,x}}{N_t} = \frac{\overline{N}_{t,x}}{\overline{N}_t}, \quad t \in \mathbb{N}, x \in \mathbb{Z}^d \tag{1.12}$$

Interesting objects related to the density would be

$$\rho_t^* = \max_{x \in \mathbb{Z}^d} \rho_{t,x}, \quad \text{and} \quad \mathcal{R}_t = \sum_{x \in \mathbb{Z}^d} \rho_{t,x}^2. \tag{1.13}$$

ρ_t^* is the density at the most populated site, while $\mathcal{R}_t$ is the probability that two particles picked up randomly from the total population at time t are at the same site. We call $\mathcal{R}_t$ the *replica overlap*, in analogy with the spin glass theory. Clearly, $(\rho_t^*)^2 \leq \mathcal{R}_t \leq \rho_t^*$. These quantities convey information on localization/delocalization of the particles. Roughly speaking, large values of ρ_t^* or $\mathcal{R}_t$ indicate that the most of the particles are concentrated on small numbers of "favorite sites" (*localization*), whereas small values of them imply that the particles are spread out over large number of sites (*delocalization*).

Remark: The BRWRE we discuss in this paper is closely related to the directed polymers in random environment (DPRE) [4, 6, 7]. In fact, it is easy to see that

$$P^q[N_{t,x}] = Z_{t,x} \stackrel{\text{def}}{=} \sum_{x_1,\ldots x_{t-1} \in \mathbb{Z}^d} \prod_{u=1}^{t} m_{u-1,x_{u-1}} p(x_{u-1}, x_u), \quad \text{with } x_0 = 0 \text{ and } x_t = x, \tag{1.14}$$

which is exactly the partition function of the DPRE with the end point x. Roughly speaking, the study of DPRE consists in analyzing the large time behavior of $Z_{t,\cdot}$. In this respect, our main object $N_{t,\cdot}$ in this paper is "more random", since it is inside P^q-expectation in (1.14).

1.2 The phase transition in terms of the population growth

Due to the random environment, the population N_t has much more fluctuation as compared with the non-random environment case, e.g.,[10, section 4.2]. This fluctuation results from "disastrous locations" in time-space, where the offspring distribution $q_{t,x}(k)$ happens to assign extremely high probability to small k's. Thanks to the random walk, on the other hand, some of the particles are lucky enough to avoid those disastrous locations. Therefore, the spatial motion component of the model has the effect of moderate the fluctuation, while the random environment intensifies it. These competing factors in the model give rise to a phase transition as we will discuss below.

We first look at the case where the randomness of the offspring distribution is well moderated by that of the random walk.

Let (S_t) be a simple symmetric random walk on $\mathbb{Z}^d$, starting from 0. We denote by π_d the probability of the event $\cup_{t \geq 1}\{S_t = 0\}$. As is well known $\pi_d < 1$ if and only if $d \geq 3$. π_d equals the probability of the event $\cup_{t \geq 1}\{S_t^{(1)} = S_t^{(2)}\}$, where $(S_t^{(1)})$ and $(S_t^{(2)})$ are two independent simple symmetric random walks on $\mathbb{Z}^d$, both starting from 0. To see this, note that $S_t^{(1)} - S_t^{(2)}$ and S_{2t} have the same distribution for each t and that $1/(1-\pi_d) = \sum_{t \geq 0} P(S_{2t} = 0)$.

Proposition 1.2.1 **(a)** *There exists a constant $\alpha^* > \frac{1}{\pi_d}$ such that, if*

$$m > 1, \;\; m^{(2)} < \infty, \;\; d \geq 3, \;\; \text{and} \;\; \alpha \stackrel{\text{def.}}{=} \frac{Q[m_{t,x}^2]}{m^2} < \alpha^*, \tag{1.15}$$

then $P(\overline{N}_\infty > 0) > 0$.

(b) *If one assumes the stronger assumption*

$$m > 1, \ \ m^{(2)} < \infty, \ \ d \geq 3, \ \ \textit{and} \ \ \alpha < \frac{1}{\pi_d}, \tag{1.16}$$

then

$$\mathcal{R}_T = O(T^{-d/2}) \quad \textit{in } P\left(\cdot | \overline{N}_\infty > 0\right)\textit{-probability,}$$

i.e., the laws $P\left(T^{d/2}\mathcal{R}_T \in \cdot | \overline{N}_\infty > 0\right)$, $T \geq 1$ *are tight.*

Conditions (1.15) and (1.16) control the randomness of the environment in terms of the random walk. Proposition 1.2.1(a) says that, under (1.15), the total population grows as fast as its expectation with strictly positive probability. This was obtained in [3, Theorem 4]. Proposition 1.2.1(b) is a quantitative statement for delocalization under (1.16) in terms of the replica overlap [12, Proposition 1.2.3].
Next, we turn to the case where the randomness of the environment dominates:

Proposition 1.2.2 *Suppose one of the following conditions:*

(a1) $d = 1, \ Q(m_{t,x} = m) \neq 1.$

(a2) $d = 2, \ Q(m_{t,x} = m) \neq 1.$

(a3) $d \geq 3, \ Q\left[\frac{m_{t,x}}{m} \ln \frac{m_{t,x}}{m}\right] > \ln(2d).$

Then, $P(\overline{N}_\infty = 0) = 1$*. Moreover, in cases (a1) and (a3), there exists a non-random number* $c > 0$ *such that*

$$\overline{\lim_t} \frac{\ln \overline{N}_t}{t} < -c, \quad a.s. \tag{1.17}$$

Proposition 1.2.2 says that the total population grows strictly slower than its expectation almost surely, in low dimensions or in "random enough" environment. The result is in contrast with the non-random environment case, where $P(\overline{N}_\infty = 0) = 1$ only for offspring distributions with very heavy tails, more precisely, if and only if $P[K^\nu_{t,x} \ln K^\nu_{t,x}] = \infty$ [1, page 24, Theorem 1]. Here, we can have $P(\overline{N}_\infty = 0) = 1$ even when $K^\nu_{t,x}$ is bounded. Also, (1.17) is in sharp contrast with the non-random environment case, where it is well known –see e.g., [1, page 30, Theorem 3] –that

$$\{N_\infty > 0\} \stackrel{\text{a.s.}}{=} \{\lim_t \frac{\ln \overline{N}_t}{t} = 0\} \quad \text{whenever } m > 1.$$

Proposition 1.2.2 was obtained in [3, Theorem 4] without (1.17), and in [12, Corollary 3.3.2] with (1.17).

1.3 The results: the localization/delocalization transition

In this paper, we aim at the localization problem for the branching random walk in random environment. We shall prove that for $d = 1, 2$ and for "random enough environment" in $d \geq 3$, almost surely, there exists a sequence of time t's such that both the maximal density ρ^*_t and overlap $\mathcal{R}_t$ are bigger than some positive constant.

We first characterize the event $\{\overline{N}_\infty = 0\}$ in terms of the replica overlap. Thanks to this characterization, we can rigorously identify the phase transition in terms of population growth as discussed in section 1.2 with the localization/delocalization transition in terms of the replica overlap.

Theorem 1.3.1 *Suppose that*

$$m^{(3)} < \infty, \quad Q(m_{t,x} = m) \neq 1, \quad Q(q_{t,x}(0) = 0) = 1. \tag{1.18}$$

Then,

$$P(\overline{N}_\infty = 0) = P(\sum_{s=0}^{\infty} \mathcal{R}_s = \infty) \in \{0, 1\}, \tag{1.19}$$

where $(\mathcal{R}_t)_{t \geq 0}$ is defined by (1.13). Moreover, if the probabilities in (1.19) are one, then there exist constants $c_1, c_2 \in (0, \infty)$ such that,

$$-c_1 \ln \overline{N}_t \leq \sum_{s=0}^{t-1} \mathcal{R}_s \leq -c_2 \ln \overline{N}_t \quad \text{for large enough } t\text{'s, a.s.} \tag{1.20}$$

We will prove Theorem 1.3.1 in section 2.

As we referred to before, the large values of the replica overlap, or the maximal density, indicate the localization of the particles to a small number of sites. We have the following lower bound for the replica overlap and the maximal density:

Theorem 1.3.2 *Suppose (1.18) and that $P(\overline{N}_\infty = 0) = 1$. Then, there exists a non-random number $c \in (0, 1)$ such that*

$$\overline{\lim_{t \nearrow \infty}} \rho_t^* \geq \overline{\lim_{t \nearrow \infty}} \mathcal{R}_t \geq c, \quad a.s., \tag{1.21}$$

where $(\rho_t^*)_{t \geq 0}$ and $(\mathcal{R}_t)_{t \geq 0}$ are defined by (1.13). In particular, (1.21) holds true if we assume any one of (a1) – (a3) in Proposition 1.2.2.

(1.21) says that the replica overlap persists, in contrast with Proposition 1.2.1(b), where the replica overlap $\mathcal{R}_T$ decays like $O(T^{-d/2})$. The proof of Theorem 1.3.2 will be presented in section 3. Some more remarks on Theorem 1.3.2 are in order:

1) In cases (a1) and (a3) in Proposition 1.2.2, (1.21) follows easily from (1.17) and (1.20). However, the proof we present does not rely on (1.17), so that we can cover two-dimensional case (a2) as well.

2) We prove (1.21) by way of the following stronger estimate:

$$\underline{\lim_{t \nearrow \infty}} \frac{\sum_{s=0}^{t} \mathcal{R}_s^{3/2}}{\sum_{s=0}^{t} \mathcal{R}_s} \geq c_1, \quad a.s. \tag{1.22}$$

for some non-random number $c_1 > 0$. This in particular implies the following quantative lower bound on the number of times, at which the replica overlap is larger than a certain positive number:

$$\underline{\lim_{t \nearrow \infty}} \frac{\sum_{s=0}^{t} 1_{\{\mathcal{R}_s \geq c_2\}}}{\sum_{s=0}^{t} \mathcal{R}_s} \geq c_3, \quad a.s.$$

where c_2 and c_3 are non-random positive numbers.

3) For both Theorem 1.3.1 and Theorem 1.3.2, similar results are known for the directed polymers in random environment (DPRE) [4, 6, 7]. In fact, we have used ideas and techniques from the DPRE case. However, the results for DPRE do not seem to directly imply our results.

2 Proof of Theorem 1.3.1

2.1 Lemmas

For sequences $(a_t)_{t\in\mathbb{N}}$ and $(b_t)_{t\in\mathbb{N}}$ (random or non-random), we write $a_t \preceq b_t$ if there exists a non-random constant $c \in (0,\infty)$ such that $a_t \le cb_t$ for all $t \in \mathbb{N}$. We write $a_t \asymp b_t$ if $a_t \preceq b_t$ and $b_t \preceq a_t$.

Lemma 2.1.1 **(a)** *If $m^{(2)} < \infty$ and $Q(m_{t,x} = m) \neq 1$, then,* $P[(N_t - mN_{t-1})^2|\mathcal{F}_{t-1}] \asymp \sum_{x\in\mathbb{Z}^d} N_{t-1,x}^2$.

(b) *If $m^{(3)} < \infty$, then* $\left|P[(N_t - mN_{t-1})^3|\mathcal{F}_{t-1}]\right| \preceq \sum_{x\in\mathbb{Z}^d} N_{t-1,x}^3$.

Proof: (a): Since

$$N_t - mN_{t-1} = \sum_x \sum_{\nu=1}^{N_{t-1,x}} (K_{t-1,x}^{\nu} - m),$$

we have $(N_t - mN_{t-1})^2 = \sum_{x_1,x_2} F_{x_1,x_2}$, where

$$F_{x_1,x_2} = \sum_{\nu_1=1}^{N_{t-1,x_1}} \sum_{\nu_2=1}^{N_{t-1,x_2}} (K_{t-1,x_1}^{\nu_1} - m)(K_{t-1,x_2}^{\nu_2} - m).$$

If $x_1 \neq x_2$, then $K_{t-1,x_1}^{\nu_1}$ and $K_{t-1,x_2}^{\nu_2}$ are mean m independent random variables under $P(\cdot|\mathcal{F}_{t-1})$, and hence

$$P[F_{x_1,x_2}|\mathcal{F}_{t-1}] = 0.$$

We may therefore focus on the expectation of F_{x_1,x_2} with $x_1 = x_2 = x$. In this case, $\{K_{t-1,x}^{\nu}\}_{\nu=1}^{N_{t-1,x}}$ are independent under $P(\cdot|\widetilde{\mathcal{F}}_{t-1})$, where

$$\widetilde{\mathcal{F}}_{t-1} = \sigma(\mathcal{F}_{t-1}, (q_{t-1,x})_{x\in\mathbb{Z}^d}).$$

Thus,

$$P[F_{x,x}|\widetilde{\mathcal{F}}_{t-1}] = N_{t-1,x}(N_{t-1,x} - 1)(m_{t-1,x} - m)^2 + N_{t-1,x}P^q[(K_{t-1,x}^{\nu} - m)^2].$$

The first and second terms on the right-hand-side come respectively from off-diagonal and diagonal terms in $F_{x,x}$. We now set $\alpha \stackrel{\text{def.}}{=} Q[m_{t,x}^2]/m^2$. Then, $\alpha > 1$ (since $Q(m_{t,x} = m) \neq 1$) and

$$\begin{aligned} P[F_{x,x}|\mathcal{F}_{t-1}] &= (\alpha - 1)m^2 N_{t-1,x}(N_{t-1,x} - 1) + (m^{(2)} - m^2)N_{t-1,x} \\ &= (\alpha - 1)m^2 N_{t-1,x}^2 + (m^{(2)} - \alpha m^2)N_{t-1,x}. \end{aligned}$$

Therefore,

$$P[(N_t - mN_{t-1})^2|\mathcal{F}_{t-1}] = (\alpha - 1)m^2 \sum_x N_{t-1,x}^2 + (m^{(2)} - \alpha m^2)N_{t-1},$$

which implies the desired bound.
(b): We have $(N_t - mN_{t-1})^3 = \sum_{x_1,x_2,x_3} F_{x_1,x_2,x_3}$, where

$$F_{x_1,x_2,x_3} = \sum_{\nu_1=1}^{N_{t-1,x_1}} \sum_{\nu_2=1}^{N_{t-1,x_2}} \sum_{\nu_3=1}^{N_{t-1,x_3}} (K_{t-1,x_1}^{\nu_1} - m)(K_{t-1,x_2}^{\nu_2} - m)(K_{t-1,x_3}^{\nu_3} - m).$$

If, for example, $x_1 \notin \{x_2, x_3\}$, then $K^{\nu_1}_{t-1,x_1}$ is independent of $\{K^{\nu_2}_{t-1,x_2}, K^{\nu_3}_{t-1,x_3}\}$ under $P(\cdot|\mathcal{F}_{t-1})$, and hence $P[F_{x_1,x_2,x_3}|\mathcal{F}_{t-1}] = 0$. This implies that

$$P[(N_t - mN_{t-1})^3|\mathcal{F}_{t-1}] = \sum_x P[F_{x,x,x}|\mathcal{F}_{t-1}].$$

On the other hand, we have that

$$\begin{aligned} P[F_{x,x,x}|\widetilde{\mathcal{F}}_{t-1}] &= N_{t-1,x}P^q[(K^\nu_{t-1,x} - m)^3] \\ &\quad +3N_{t-1,x}(N_{t-1,x}-1)P^q[(K^\nu_{t-1,x} - m)^2]P^q[K^\nu_{t-1,x} - m] \\ &\quad +N_{t-1,x}(N_{t-1,x}-1)(N_{t-1,x}-2)P^q[K^\nu_{t-1,x} - m]^3, \end{aligned}$$

and therefore that

$$\left|P[F_{x,x,x}|\widetilde{\mathcal{F}}_{t-1}]\right| \le N^3_{t-1,x}P^q[|K^\nu_{t-1,x} - m|^3].$$

Putting things together, we obtain

$$\left|P[(N_t - mN_{t-1})^3|\mathcal{F}_{t-1}]\right| \le c\sum_x N^3_{t-1,x}, \quad \text{with } c = Q[|K^\nu_{t-1,x} - m|^3].$$

□

Let us now recall Doob's decomposition in our settings. An $(\mathcal{F}_t)$-adapted process $X = (X_t)_{t\ge 0} \subset \mathbb{L}^1(P)$ can be decomposed in a unique way as

$$X_t = M_t(X) + A_t(X), \quad t \ge 1,$$

where $M(X)$ is an $(\mathcal{F}_t)$-martingale and

$$A_0 = 0, \quad \Delta A_t = P[\Delta X_t|\mathcal{F}_{t-1}], \quad t \ge 1.$$

Here, and in what follows, we write $\Delta a_t = a_t - a_{t-1}$ $(t \ge 1)$ for a sequence $(a_t)_{t\in\mathbb{N}}$ (random or non-random). $M_t(X)$ and $A_t(X)$ are called respectively, the martingale part and compensator of the process X. If X is a square-integrable martingale, then the compensator $A_t(X^2)$ of the process $X^2 = (X_t^2)_{t\ge 0} \subset \mathbb{L}^1(P)$ is denoted by $\langle X \rangle_t$ and is given by the following formula:

$$\Delta\langle X \rangle_t = P[(\Delta X_t)^2|\mathcal{F}_{t-1}].$$

Now, we turn to the Doob's decomposition of $X_t = -\ln \overline{N}_t$, whose martingale part and the compensator will be henceforth denoted M_t and A_t respectively;

$$-\ln \overline{N}_t = M_t + A_t, \quad \Delta A_t = -P[\Delta \ln \overline{N}_t|\mathcal{F}_{t-1}] \tag{2.1}$$

Lemma 2.1.2 *Suppose (1.18). Then,* $\Delta\langle M \rangle_t \preceq \mathcal{R}_{t-1} \asymp \Delta A_t$.

Proof: We set $U_t = \frac{\Delta \overline{N}_t}{\overline{N}_{t-1}}$ to simplify the notation. We first note the following:

(1) $U_t \ge \dfrac{1}{m} - 1 > -1$.

(2) $|\Delta \ln \overline{N}_t| \le m\,|U_t|$.

(3) $P\left[U_t^2|\mathcal{F}_{t-1}\right] \asymp P\left[\varphi\left(U_t\right)|\mathcal{F}_{t-1}\right] \asymp \mathcal{R}_{t-1}$, where $\varphi(x) = x - \ln(1+x)$.

In fact, $N_{t-1} \le N_t$ by (1.18), and hence $(1/m)\overline{N}_{t-1} \le \overline{N}_t$. These imply (1). (2) follows directly from (1) since

$$|\ln x - \ln y| \le \frac{m|x-y|}{y} \quad \text{if } x, y > 0 \text{ and } x/y \ge 1/m.$$

As for (3), we have by Lemma 2.1.1(a) that

$$P\left[U_t^2|\mathcal{F}_{t-1}\right] = \frac{P\left[|\Delta\overline{N}_t|^2|\mathcal{F}_{t-1}\right]}{\overline{N}_{t-1}^2} \asymp \mathcal{R}_{t-1}.$$

We now note that there exists $c \in (0,\infty)$, which depends only on m such that

$$\frac{x^2}{4(2+x)} \le \varphi(x) \le cx^2 \quad \text{for all } x \ge \tfrac{1}{m} - 1.$$

This, together with (1) implies that

$$P\left[\varphi\left(U_t\right)|\mathcal{F}_{t-1}\right] \le cP\left[U_t^2|\mathcal{F}_{t-1}\right] \asymp \mathcal{R}_{t-1}.$$

On the other hand, we have by Lemma 2.1.1(b) that

$$\left|P\left[U_t^3|\mathcal{F}_{t-1}\right]\right| = \frac{1}{N_{t-1}^3}\left|P[(N_t - mN_{t-1})^3|\mathcal{F}_{t-1}]\right| \preceq \frac{1}{N_{t-1}^3}\sum_{x\in\mathbb{Z}^d} N_{t-1,x}^3 \le \mathcal{R}_{t-1}.$$

Thus,

$$\begin{aligned}
\mathcal{R}_{t-1} &\asymp P\left[U_t^2|\mathcal{F}_{t-1}\right] = P\left[\frac{U_t}{\sqrt{2+U_t}}U_t\sqrt{2+U_t}|\mathcal{F}_{t-1}\right] \\
&\le P\left[\frac{U_t^2}{2+U_t}|\mathcal{F}_{t-1}\right]^{1/2} P\left[2U_t^2 + U_t^3|\mathcal{F}_{t-1}\right]^{1/2} \preceq P\left[\varphi(U_t)|\mathcal{F}_{t-1}\right]^{1/2}\mathcal{R}_{t-1}^{1/2},
\end{aligned}$$

and hence $\mathcal{R}_{t-1} \preceq P\left[\varphi(U_t)|\mathcal{F}_{t-1}\right]$.

The rest of the proof is easy. We have by (3) that

$$\Delta A_t = -P[\Delta\ln\overline{N}_t|\mathcal{F}_{t-1}] = -P\left[\ln\left(1+U_t\right)|\mathcal{F}_{t-1}\right] = P\left[\varphi\left(U_t\right)|\mathcal{F}_{t-1}\right] \asymp \mathcal{R}_{t-1}.$$

Similarly, by (2) and Lemma 2.1.1,

$$P[(\Delta\ln\overline{N}_t)^2|\mathcal{F}_{t-1}] \preceq P\left[U_t^2|\mathcal{F}_{t-1}\right] \asymp \mathcal{R}_{t-1}.$$

This, together with $\Delta A_t \asymp \mathcal{R}_{t-1}$ implies that

$$\Delta\langle\, M\,\rangle_t = P[(\Delta M_t)^2|\mathcal{F}_{t-1}] \le 2P[(\Delta\ln\overline{N}_t)^2|\mathcal{F}_{t-1}] + 2(\Delta A_t)^2 \preceq \mathcal{R}_{t-1}.$$

□

Lemma 2.1.3 *Suppose* $Q(q_{t,x}(0) = 0) = 1$. *Then,* $P(\overline{N}_\infty > 0) \in \{0,1\}$.

The proof we present is due to Yuval Peres (private communication). We prepare some notation. For $(s,y) \in \mathbb{N}\times\mathbb{Z}^d$, we define the (s,y)-*branch* $(N_{t,x}^{s,y})_{x\in\mathbb{Z}^d}$, $t \in \mathbb{N}$ of the branching random walk as follows:

$$N_{0,x}^{s,y} = \delta_{x,y}, \quad N_{t,x}^{s,y} = \sum_{z\in\mathbb{Z}^d}\sum_{\nu=1}^{N_{t-1,z}^{s,y}} \delta_x(X_{s+t-1,z}^\nu)K_{s+t-1,z}^\nu, \quad t \in \mathbb{N}^*. \tag{2.2}$$

$N_{t,x}^{s,y}$ can be thought of as the number of particles at time-space $(s+t,x) \in \mathbb{N} \times \mathbb{Z}^d$ which descend from a single particle at time-space $(s,y) \in \mathbb{N} \times \mathbb{Z}^d$. In particular, $N_{t,x} = N_{t,x}^{0,0}$ in this notation. We define the set of occupied sites $\mathcal{O}_t^{s,y}$ of the (s,y)-branch by

$$\mathcal{O}_t^{s,y} = \{x \in \mathbb{Z}^d \; ; \; N_{t,x}^{s,y} \geq 1\}, \quad \mathcal{O}_t = \mathcal{O}_t^{0,0}.$$

Note that $\mathcal{O}_t^{s,y} \subset \mathbb{Z}_{\mathrm{e}}^d$, if $y \in \mathbb{Z}_{\mathrm{e}}^d$ and $t-s \in 2\mathbb{N}$, where

$$\mathbb{Z}_{\mathrm{e}}^d = \{x \in \mathbb{Z}^d \; ; \; x_1 + ... + x_d \in 2\mathbb{Z}\}.$$

Proof of Lemma 2.1.3: We fix $\ell \in \mathbb{N}$ and define an event $E_t^{s,y}$ $(s,t \in 2\mathbb{N},\ y \in \mathbb{Z}_{\mathrm{e}}^d)$ by:

$$E_t^{s,y} = \bigcup_{v \in \mathbb{Z}_{\mathrm{e}}^d} \left\{([-\ell,\ell]^d + v) \cap \mathbb{Z}_{\mathrm{e}}^d \subset \mathcal{O}_t^{s,y}\right\}, \quad E_t = E_t^{0,0},$$

that is, the set of occupied sites of the (s,y)-branch at time t contains a cube with the side-length 2ℓ. We first note that for any $\ell \in \mathbb{N}$,

(1) $\qquad T \stackrel{\text{def}}{=} \inf\{t \in 2\mathbb{N} \; ; \; E_t \text{ occurs}\} < \infty$, a.s.

To prove (1), we take $s \in 2\mathbb{N}$ large enough so that

$$P(E_s) > 0.$$

We then define a sequence $0 = y_0, y_1, ... \in \mathbb{Z}_{\mathrm{e}}^d$ inductively by

$$y_k = \max \mathcal{O}_{ks}^{(k-1)s, y_{k-1}}, \quad k \in \mathbb{N}^*,$$

where the maximum is relative to the lexicographical order of $\mathbb{Z}^d$. By the assumption, $\mathcal{O}_{ks}^{(k-1)s,y_{k-1}} \neq \emptyset$ for all $k \in \mathbb{N}^*$. Now,

$$\begin{aligned}
P(T > ks) &\leq P\left((E_s)^{\mathsf{c}}, (E_s^{s,y_1})^{\mathsf{c}}, ..., (E_s^{(k-1)s,y_{k-1}})^{\mathsf{c}}\right) \\
&= P\left((E_s)^{\mathsf{c}}, (E_s^{s,y_1})^{\mathsf{c}}, ..., (E_s^{(k-2)s,y_{k-2}})^{\mathsf{c}}\right)(1 - P(E_s)) \\
&= (1 - P(E_s))^k.
\end{aligned}$$

This implies that $P[\exp(cT)] < \infty$ for some $c > 0$ and hence (1).
Let T be defined by (1) and

$$v_T = \max\{v \in \mathbb{Z}_{\mathrm{e}}^d \; ; \; ([-\ell,\ell]^d + v) \cap \mathbb{Z}_{\mathrm{e}}^d \subset \mathcal{O}_T^{0,0}\}.$$

We then have that

$$\begin{aligned}
P(\overline{N}_\infty > 0 | \mathcal{F}_T) &\geq P\left(\bigcup_{y \in \mathcal{O}_T} \{\overline{N}_\infty^{T,y} > 0\} \middle| \mathcal{F}_T\right) \\
&\geq P\left(\bigcup_{y \in ([-\ell,\ell]^d + v_T) \cap \mathbb{Z}_{\mathrm{e}}^d} \{\overline{N}_\infty^{T,y} > 0\} \middle| \mathcal{F}_T\right) \\
&= P\left(\bigcup_{y \in [-\ell,\ell]^d \cap \mathbb{Z}_{\mathrm{e}}^d} \{\overline{N}_\infty^{0,y} > 0\}\right),
\end{aligned}$$

where we have used the invariance under the time-space shift on the last line. Hence, by taking expectations and letting $\ell \to \infty$,

$$P(\overline{N}_\infty > 0) \geq P\left(\bigcup_{y \in \mathbb{Z}^d_{\mathrm{e}}} \{\overline{N}^{0,y}_\infty > 0\} \right).$$

The above inequality is in fact an equality, since $\overline{N}^{0,0}_\infty = \overline{N}_\infty$. Moreover, the event on the right-hand-side is invariant under the shift by $x \in \mathbb{Z}^d_{\mathrm{e}}$, which is ergodic with respect to P. It is thus, zero or one depending on whether $P(\overline{N}_\infty > 0)$ is zero or positive. □

2.2 Proof of Theorem 1.3.1

The proof is based on the decomposition (2.1). By Lemma 2.1.3, it is enough to prove the following:

(1) $\{\overline{N}_\infty = 0\} \overset{\text{a.s.}}{\subset} \{A_\infty = \infty\} = \{\sum_{s=0}^\infty \mathcal{R}_s = \infty\}$.

(2) $\{\sum_{s=0}^\infty \mathcal{R}_s = \infty\} \overset{\text{a.s.}}{\subset} \{ -c_1 \ln \overline{N}_t \leq \sum_{s=0}^{t-1} \mathcal{R}_s \leq -c_2 \ln \overline{N}_t \quad \text{for large enough } t\text{'s.}\}$.

To prove these, we recall the following general facts on square-integrable martingales–see for example [9, page 252, (4.9) and page 253, (4.10)]:

(3) $\{\langle M \rangle_\infty < \infty\} \overset{\text{a.s.}}{\subset} \{\lim_t M_t \text{ converges.}\}$.

(4) $\{\langle M \rangle_\infty = \infty\} \overset{\text{a.s.}}{\subset} \{\lim_t \frac{M_t}{\langle M \rangle_t} = 0\}$.

By (3) and Lemma 2.1.2, we get (1) as follows:

$$\begin{aligned}\left\{ \sum_{s=0}^{t-1} \mathcal{R}_s < \infty \right\} &= \{A_\infty < \infty\} = \{A_\infty < \infty, \ \langle M \rangle_\infty < \infty\} \\ &\overset{\text{a.s.}}{\subset} \{A_\infty < \infty, \ \lim_t M_t \text{ converges.}\} \subset \{\overline{N}_\infty > 0\}.\end{aligned}$$

We now turn to (2). Since $\{A_\infty = \infty\} = \{\sum_{s=0}^\infty \mathcal{R}_s = \infty\}$ and

$$-\frac{\ln \overline{N}_t}{\sum_{s=0}^{t-1} \mathcal{R}_s} \asymp -\frac{\ln \overline{N}_t}{A_t} = \frac{M_t}{A_t} + 1,$$

by Lemma 2.1.2, (2) is a consequence of:

(5) $\{A_\infty = \infty\} \overset{\text{a.s.}}{\subset} \left\{ \frac{M_t}{A_t} \longrightarrow 0 \right\}$.

Let us suppose that $A_\infty = \infty$. If $\langle M \rangle_\infty < \infty$, then again by (3), $\lim_t M_t$ converges and therefore (5) holds. If, on the contrary, $\langle M \rangle_\infty = \infty$, then by (4) and Lemma 2.1.2,

$$\frac{M_t}{A_t} = \frac{M_t}{\langle M \rangle_t} \frac{\langle M \rangle_t}{A_t} \longrightarrow 0 \quad \text{a.s.}$$

Thus, (5) is true in this case as well. □

3 Proof of Theorem 1.3.2

We shall prove Theorem 1.3.2 in the same spirit as that of [4]. In the following subsection, we give some preliminary estimates and the final proof is given in the last subsection.

3.1 Lemmas

A technical result at first:

Lemma 3.1.1 *Let η_i, $1 \le i \le n$ $(n \ge 2)$ be positive independent random variables on a probability space with the probability measure $\mathbb{P}$, such that $\mathbb{P}[\eta_i^3] < \infty$ for $i = 1, .., n$. Then,*

$$\begin{aligned} \mathbb{P}\left[\frac{\eta_1\eta_2}{(\sum_{i=1}^n \eta_i)^2}\right] &\ge \frac{m_1 m_2}{M^2} - 2\frac{m_2 var(\eta_1) + m_1 var(\eta_2)}{M^3}, & (3.1)\\ \mathbb{P}\left[\frac{\eta_1^2}{(\sum_{i=1}^n \eta_i)^2}\right] &\ge \frac{\mathbb{P}[\eta_1^2]}{M^2}\left(1 + \frac{2m_1}{M}\right) - 2\frac{\mathbb{P}[\eta_1^3]}{M^3}, & (3.2)\end{aligned}$$

where $m_i = \mathbb{P}[\eta_i]$ and $M = \sum_{i=1}^n m_i$.

Proof: We set

$$U = \sum_{i=1}^n (\eta_i - m_i) = \sum_{i=1}^n \eta_i - M > -M.$$

Note that $(u+M)^{-2} \ge M^{-2}(1 - \frac{2u}{M})$ for $u \in (-M, \infty)$. Thus, we have that

$$\begin{aligned} \mathbb{P}\left[\frac{\eta_1\eta_2}{(\sum_{i=1}^n \eta_i)^2}\right] &= \mathbb{P}\left[\frac{\eta_1\eta_2}{(U+M)^2}\right] \ge M^{-2}\left(m_1 m_2 - \frac{2}{M}\mathbb{P}\left[\eta_1\eta_2 U\right]\right)\\ \mathbb{P}\left[\eta_1\eta_2 U\right] &= \mathbb{P}\left[\eta_1\eta_2(\eta_1 - m_1)\right] + \mathbb{P}\left[\eta_1\eta_2(\eta_2 - m_2)\right] = m_2 \mathrm{var}(\eta_1) + m_1 \mathrm{var}(\eta_2).\end{aligned}$$

These prove (3.1). Similarly,

$$\begin{aligned} \mathbb{P}\left[\frac{\eta_1^2}{(\sum_{i=1}^n \eta_i)^2}\right] &= \mathbb{P}\left[\frac{\eta_1^2}{(U+M)^2}\right] \ge M^{-2}\left(\mathbb{P}\left[\eta_1^2\right] - \frac{2}{M}\mathbb{P}\left[\eta_1^2 U\right]\right),\\ \mathbb{P}\left[\eta_1^2 U\right] &= \mathbb{P}\left[\eta_1^2(\eta_1 - m_1)\right] = \mathbb{P}\left[\eta_1^3\right] - m_1\mathbb{P}\left[\eta_1^2\right].\end{aligned}$$

These prove (3.2). □

As an immediate consequence, we have (by applying Lemma 3.1.1 to $\alpha_i\eta_i$ instead of η_i):

Corollary 3.1.2 *Let η_i, $1 \le i \le n$ $(n \ge 2)$ be as in Lemma 3.1.1. Then, for any $\alpha_i \ge 0$ satisfying $\sum_{i=1}^n \alpha_i = 1$, we have*

$$\begin{aligned} \mathbb{P}\Big(\frac{\eta_1\eta_2}{\big(\sum_{i=1}^n \alpha_i\eta_i\big)^2}\Big) &\ge 1 - (\mathbb{P}(\widetilde{\eta}_1^2) - 1)(\alpha_1 + \alpha_2), & (3.3)\\ \mathbb{P}\Big(\frac{(\eta_1)^2}{\big(\sum_{i=1}^n \alpha_i\eta_i\big)^2}\Big) &\ge (1 + 2\alpha_1)\mathbb{P}(\widetilde{\eta}_1^2) - 2\alpha_1\mathbb{P}(\widetilde{\eta}_1^3), & (3.4)\end{aligned}$$

where $\widetilde{\eta}_1 := \eta_1/\mathbb{P}[\eta_1]$.

Lemma 3.1.3 *Assume $Q(q_{t,x}(0)=0)=1$ and $Q(q_{t,x}(1)=1)<1$. Then,*

$$\varliminf_{t\to\infty}\frac{1}{t}\ln N_t \ge c_0, \qquad a.s. \tag{3.5}$$

where $c_0 = -\ln Q[\sum_{k\ge 1} k^{-1} q_{t,x}] > 0$.

Proof: For any (t,y,ν), $K^{\nu}_{t,y}$ is independent of $\mathcal{F}_t$, hence

$$P\Big((K^{\nu}_{t,y})^{-1}\,|\,\mathcal{F}_t\Big) = P\Big((K^{\nu}_{t,y})^{-1}\Big) = e^{-c_0}.$$

It follows by Jensen's inequality that

$$\begin{aligned}
P\left(\frac{1}{N_t}\,|\,\mathcal{F}_{t-1}\right) &= P\left(\left[\sum_y \sum_{\nu=1}^{N_{t-1,y}} K^{\nu}_{t-1,y}\right]^{-1} |\,\mathcal{F}_{t-1}\right)\\
&= \frac{1}{N_{t-1}} P\left(\left[\frac{1}{N_{t-1}}\sum_y \sum_{\nu=1}^{N_{t-1,y}} K^{\nu}_{t-1,y}\right]^{-1} |\,\mathcal{F}_{t-1}\right)\\
&\le \frac{1}{N_{t-1}} P\left(\frac{1}{N_{t-1}}\sum_y \sum_{\nu=1}^{N_{t-1,y}} \left(K^{\nu}_{t-1,y}\right)^{-1} |\,\mathcal{F}_{t-1}\right)\\
&= \frac{e^{-c_0}}{N_{t-1}}.
\end{aligned}$$

Hence $P\left(\frac{1}{N_t}\right) \le e^{-c_0 t}$, and (3.5) follows from the Borel-Cantelli lemma. □

We denote by $(\mathcal{P}_t, t=0,1,...)$ the semigroup of a simple symmetric random walk on $\mathbb{Z}^d$, namely, $\mathcal{P}_t f(x) := \sum_y \mathcal{P}_t(x,y) f(y)$ where $\mathcal{P}_t(x,y)$ is the probability that the random walk starting from x resides at y on the t-th step. Plainly, $\mathcal{P}_1(x,y) = p(x,y)$. We write $\mathcal{P} = \mathcal{P}_1$. Let for any $z \in \mathbb{Z}^d$,

$$r_t := \mathcal{P}_{2t}(z,z) = \mathcal{P}_{2t}(0,0) \;\sim\; c\,t^{-d/2}, \qquad t\to\infty.$$

For the sake of notational convenience, we write $\rho_t(x) \equiv \rho_{t,x}$, so that ρ_t stands for a function on $\mathbb{Z}^d$.

Lemma 3.1.4 *Assume (1.18). For any (y_1,ν_1) and (y_2,ν_2), $t\ge 1$, we have*

$$P\Big(\frac{K^{\nu_1}_{t,y_1} K^{\nu_2}_{t,y_2}}{N^2_{t+1}}\,|\,\mathcal{F}_t\Big) \;\ge\; \frac{1}{N_t^2}\Big[(\alpha-1)1_{(y_1=y_2)} - c_1\rho_t(y_1) - c_1\rho_t(y_2) - \frac{c_2}{N_t}\Big], \tag{3.6}$$

on the event $\{N_{t,y_1}\wedge N_{t,y_2} \ge 1\}$, where $\alpha = \frac{Q[m^2_{t,x}]}{m^2}$ and c_1 and c_2 are some constants. Consequently,

$$\begin{aligned}
P\Big(\rho_{t+1}(y_1)\rho_{t+1}(y_2)|\,\mathcal{F}_t\Big) \;\ge\;& (1-\frac{c_2}{N_t})\mathcal{P}\rho_t(y_1)\,\mathcal{P}\rho_t(y_2) + (\alpha-1)\sum_z \rho_t^2(z)p(z,y_1)p(z,y_2)\\
&-c_1\Big[\mathcal{P}\rho_t(y_1)\,\mathcal{P}(\rho_t^2)(y_2) + \mathcal{P}\rho_t(y_2)\,\mathcal{P}(\rho_t^2)(y_1)\Big]\\
&-\frac{\alpha}{N_t}\sum_z p(z,y_1)p(z,y_2)\rho_t(z).
\end{aligned} \tag{3.7}$$

Proof: Firstly, we consider (3.6) in the case $(y_1, \nu_1) \neq (y_2, \nu_2)$. Let $A \in \mathcal{F}_t$ and $A \subset \{N_{t,y_1} \wedge N_{t,y_2} \geq 1\}$. Under P^q, $\{K^\nu_{t,y}\}_{y,\nu}$ are independent (but not identically distributed) and independent of $\mathcal{F}_t$. Since $N_{t+1} = \sum_y \sum_{\nu=1}^{N_{t,y}} K^\nu_{t,y}$, we have $M_t \stackrel{\text{def.}}{=} P^q[N_{t+1}|\mathcal{F}_t] = \sum_y m_{t,y} N_{t,y}$. Thus, by applying (3.1) to $\eta_1 = K^{\nu_1}_{t,y_1}$ and $\eta_2 = K^{\nu_2}_{t,y_2}$, we get

$$P^q\Big(1_A \frac{K^{\nu_1}_{t,y_1} K^{\nu_2}_{t,y_2}}{N^2_{t+1}}\Big) \geq P^q\left(1_A \frac{m_{t,y_1} m_{t,y_2}}{M^2_t}\right) - 2P^q\left(1_A \frac{m_{t,y_2} m^{(2)}_{t,y_1} + m_{t,y_1} m^{(2)}_{t,y_2}}{N^3_t}\right),$$

since $M_t \geq N_t$. Therefore, by taking Q-expectation,

$$P\Big(1_A \frac{K^{\nu_1}_{t,y_1} K^{\nu_2}_{t,y_2}}{N^2_{t+1}}\Big) \geq P\left(1_A \frac{m_{t,y_1} m_{t,y_2}}{M^2_t}\right) - 2P\left(1_A \frac{m_{t,y_2} m^{(2)}_{t,y_1} + m_{t,y_1} m^{(2)}_{t,y_2}}{N^3_t}\right).$$

Observe that under P, $m_{t,\cdot}$ are i.i.d. and independent of $\mathcal{F}_t$. It turns out from (3.3) and (3.4), with $\alpha.$ and $\eta.$ playing the roles of $\rho_t(\cdot)$ and $m_{t,\cdot}$ respectively, that

$$\begin{aligned} P\left(\frac{m_{t,y_1} m_{t,y_2}}{M^2_t} | \mathcal{F}_t\right) &= \frac{1}{N^2_t} P\left(\frac{m_{t,y_1} m_{t,y_2}}{(\sum_y \rho_t(y) m_{t,y})^2} | \mathcal{F}_t\right) \\ &\geq \frac{1}{N^2_t}\left[1 + (\alpha - 1) 1_{(y_1 = y_2)} - c_1 \rho_t(y_1) - c_1 \rho_t(y_2)\right]. \end{aligned}$$

On the other hand, we have

$$P\left(m_{t,y_2} m^{(2)}_{t,y_1} + m_{t,y_1} m^{(2)}_{t,y_2} | \mathcal{F}_t\right) \leq 2m^{(3)} < \infty$$

by our integrability assumption. Hence, with $c_2 = 4m^{(3)}$,

$$P\Big(1_A \frac{K^{\nu_1}_{t,y_1} K^{\nu_2}_{t,y_2}}{N^2_{t+1}}\Big) \geq P\left(\frac{1_A}{N^2_t}\left[1 + (\alpha - 1) 1_{(y_1 = y_2)} - c_1 \rho_t(y_1) - c_1 \rho_t(y_2) - \frac{c_2}{N_t}\right]\right),$$

yielding (3.6) in the case $(y_1, \nu_1) \neq (y_2, \nu_2)$. The case $(y_1, \nu_1) = (y_2, \nu_2)$ is obtained in the same way by applying (3.2) instead of (3.1) and by eventually modifying the constants.
To obtain (3.7), we have that

$$\begin{aligned} P\Big(\rho_{t+1}(y_1)\rho_{t+1}(y_2) | \mathcal{F}_t\Big) &= \sum_{z_1, z_2} \sum_{\nu_1=1}^{N_{t,z_1}} \sum_{\nu_2=1}^{N_{t,z_2}} P\left(\frac{\delta_{y_1}(X^{\nu_1}_{t,z_1}) \delta_{y_2}(X^{\nu_2}_{t,z_2}) K^{\nu_1}_{t,z_1} K^{\nu_2}_{t,z_2}}{N^2_{t+1}} | \mathcal{F}_t\right) \\ &= \sum_{z_1, z_2} \sum_{\nu_1=1}^{N_{t,z_1}} \sum_{\nu_2=1}^{N_{t,z_2}} h_{1,2}\, P\left(\frac{K^{\nu_1}_{t,z_1} K^{\nu_2}_{t,z_2}}{N^2_{t+1}} | \mathcal{F}_t\right) \end{aligned} \tag{3.8}$$

by means of the independence between $(X^{\nu_1}_{t,z_1}, X^{\nu_2}_{t,z_2})$ and $(\mathcal{F}_t, N_{t+1}, K^{\nu_1}_{t,z_1} K^{\nu_2}_{t,z_2})$, and the function $h_{1,2}$ is defined as follows:

$$\begin{aligned} h_{1,2} &:= P\Big(\delta_{y_1}(X^{\nu_1}_{t,z_1}) \delta_{y_2}(X^{\nu_2}_{t,z_2})\Big) \\ &= 1_{((z_1,\nu_1)=(z_2,\nu_2))} p(z_1, y_1) 1_{(y_1=y_2)} + 1_{((z_1,\nu_1)\neq(z_2,\nu_2))} p(z_1, y_1) p(z_2, y_2) \\ &\geq 1_{((z_1,\nu_1)\neq(z_2,\nu_2))} p(z_1, y_1) p(z_2, y_2). \end{aligned} \tag{3.9}$$

Applying (3.6) we get

$$
\begin{aligned}
P\Big(\rho_{t+1}(y_1)\rho_{t+1}(y_2)|\,\mathcal{F}_t\Big) &\geq \sum_{z_1,z_2}\sum_{\nu_1,\nu_2} h_{1,2}\,\frac{1}{N_t^2}\left[1+(\alpha-1)1_{(z_1=z_2)}-c_1\rho_t(z_1)-c_1\rho_t(z_2)-\frac{c_2}{N_t}\right] \\
&\geq \sum_{(z_1,\nu_1)\neq(z_2,\nu_2)} p(z_1,y_1)p(z_2,y_2)\,\frac{1}{N_t^2}\Big[g_t(z_1,z_2)+(\alpha-1)1_{(z_1=z_2)}\Big],
\end{aligned}
$$

with $g_t(z_1,z_2) = 1 - c_1\rho_t(z_1) - c_2\rho_t(z_2) - \frac{c_2}{N_t}$. Let us compute explicitly the above sum $\sum_{(z_1,\nu_1)\neq(z_2,\nu_2)}\cdots$:

$$
\begin{aligned}
\sum_{(z_1,\nu_1)\neq(z_2,\nu_2)} &= \sum_{z_1\neq z_2} N_{t,z_1}N_{t,z_2}p(z_1,y_1)p(z_2,y_2)\,\frac{1}{N_t^2}\,g_t(z_1,z_2) \\
&\quad +\sum_z (N_{t,z}^2-N_{t,z})p(z,y_1)p(z,y_2)\,\frac{1}{N_t^2}\,[g_t(z,z)+\alpha-1],
\end{aligned}
$$

by removing the diagonal terms. Using the definition of $\rho_t(z) = N_{t,z}/N_t$, we get

$$
\begin{aligned}
\sum_{(z_1,\nu_1)\neq(z_2,\nu_2)} &= \sum_{z_1,z_2}\rho_t(z_1)\rho_t(z_2)p(z_1,y_1)p(z_2,y_2)\,g_t(z_1,z_2) \\
&\quad +(\alpha-1)\sum_z \rho_t(z)^2 p(z,y_1)p(z,y_2)\;-\sum_z p(z,y_1)p(z,y_2)\,\frac{\rho_t(z)}{N_t}\,[g_t(z,z)+\alpha-1] \\
&\geq (1-\frac{c_2}{N_t})\mathcal{P}\rho_t(y_1)\,\mathcal{P}\rho_t(y_2) + (\alpha-1)\sum_z \rho_t(z)^2 p(z,y_1)p(z,y_2) \\
&\quad -c_1\Big[\mathcal{P}\rho_t(y_1)\,\mathcal{P}(\rho_t^2)(y_2)+\mathcal{P}\rho_t(y_2)\,\mathcal{P}(\rho_t^2)(y_1)\Big] - \frac{\alpha}{N_t}\sum_z p(z,y_1)p(z,y_2)\rho_t(z),
\end{aligned}
$$

as desired. □.

Recall that $\mathcal{R}_t = \sum_x \rho_t^2(x)$. Let $t\geq 2$. The following lemma shows the role played by the semigroup in analyzing $\mathcal{R}_t$.

Lemma 3.1.5 *Assume (1.18). There exists a constant $c_3>0$ such that for all $1\leq s\leq t-1$,*

$$
P\left(\sum_x (\mathcal{P}_{t-(s+1)}\,\rho_{s+1}(x))^2\,|\,\mathcal{F}_s\right) \;\;\geq\;\; \sum_x (\mathcal{P}_{t-s}\rho_s(x))^2 + (\alpha-1)\,r_{t-s}\mathcal{R}_s - 2c_1\mathcal{R}_s^{3/2} - \frac{c_3}{N_s}.
$$

Proof: Observe that

$$
\sum_x (\mathcal{P}_{t-(s+1)}\,\rho_{s+1}(x))^2 = \sum_x\sum_{y_1,y_2}\mathcal{P}_{t-(s+1)}(x,y_1)\mathcal{P}_{t-(s+1)}(x,y_2)\rho_{s+1}(y_1)\rho_{s+1}(y_2).
$$

Applying (3.7) gives

$$
P\left(\sum_x (\mathcal{P}_{t-(s+1)}\,\rho_{s+1}(x))^2\,|\,\mathcal{F}_s\right) \geq (1-\frac{c_2}{N_s})I_2 + (\alpha-1)I_3 - c_1I_4 - \frac{\alpha}{N_s}I_5,
$$

with

$$
I_2 \;:=\; \sum_x\sum_{y_1,y_2}\mathcal{P}_{t-(s+1)}(x,y_1)\mathcal{P}_{t-(s+1)}(x,y_2)\mathcal{P}\rho_s(y_1)\,\mathcal{P}\rho_s(y_2),
$$

$$
\begin{aligned}
I_3 &:= \sum_x \sum_{y_1,y_2} \mathcal{P}_{t-(s+1)}(x,y_1)\mathcal{P}_{t-(s+1)}(x,y_2) \sum_z \rho_s^2(z)p(z,y_1)p(z,y_2),\\
I_4 &:= \sum_x \sum_{y_1,y_2} \mathcal{P}_{t-(s+1)}(x,y_1)\mathcal{P}_{t-(s+1)}(x,y_2) \Big[\mathcal{P}\rho_s(y_1)\,\mathcal{P}(\rho_s^2)(y_2) + \mathcal{P}\rho_s(y_2)\,\mathcal{P}(\rho_s^2)(y_1)\Big],\\
I_5 &:= \sum_x \sum_{y_1,y_2} \mathcal{P}_{t-(s+1)}(x,y_1)\mathcal{P}_{t-(s+1)}(x,y_2) \sum_z p(z,y_1)p(z,y_2)\rho_s(z).
\end{aligned}
$$

Using the semigroup property and noting that $\sum_x (\mathcal{P}_{t-s}(x,z))^2 = \mathcal{P}_{2t-2s}(z,z) = r_{t-s}$ for any z, we obtain

$$
\begin{aligned}
I_2 &= \sum_x (\mathcal{P}_{t-s}\rho_s(x))^2,\\
I_3 &= \sum_x \sum_z (\mathcal{P}_{t-s}(x,z))^2 \rho_s^2(z) = \sum_z \sum_x (\mathcal{P}_{t-s}(x,z))^2 \rho_s^2(z) = r_{t-s} \sum_z \rho_s^2(z),\\
I_4 &= 2\sum_x \mathcal{P}_{t-s}\rho_s(x)\,\mathcal{P}_{t-s}(\rho_s^2)(x),\\
I_5 &= \sum_x \sum_z (\mathcal{P}_{t-s}(x,z))^2 \rho_s(z) = \sum_z \rho_s(z) r_{t-s} = r_{t-s}.
\end{aligned}
$$

By the translation invariance and the Cauchy-Schwarz inequality, we see that

$$
\mathcal{R}_s = \sum_x \mathcal{P}_{t-s}(\rho_s^2)(x) \geq \sum_x (\mathcal{P}_{t-s}(\rho_s)(x))^2 \geq \max_x \mathcal{P}_{t-s}(\rho_s)(x)^2,
$$

and hence that $I_4 \leq 2\,\mathcal{R}_s^{3/2}$. This implies the lemma with $c_3 = \alpha + c_2$. □

Define

$$
V_t = \sum_{s=1}^t \mathcal{R}_s, \qquad t = 1, 2, ...
$$

Lemma 3.1.6 *Assume (1.18). Fix $j \geq 0$. The martingale $Z_j(\cdot)$ defined by*

$$
Z_j(t) := \sum_{s=1}^t \left(\sum_x (\mathcal{P}_j\rho_s(x))^2 - P\Big(\sum_x (\mathcal{P}_j\rho_s(x))^2 \,|\, \mathcal{F}_{s-1} \Big) \right), \qquad t \geq 1.
$$

satisfies the following law of large numbers:

$$
\{V_\infty = \infty\} \overset{a.s.}{\subset} \{\frac{Z_j(t)}{V_t} \to 0, \;\; t \to \infty, \}.
$$

Proof: Let us compute the increasing process $\langle Z_j \rangle.$ associated to Z_j. By the Cauchy-Schwarz inequality, $(\sum_x \mathcal{P}_j\rho_s(x))^2 \leq \sum_x \mathcal{P}_j\rho_s^2(x) = \mathcal{R}_s \leq 1$. It follows that

$$
\begin{aligned}
(Z_j(s) - Z_j(s-1))^2 &\leq 2\left(\sum_x (\mathcal{P}_j\rho_s(x))^2 \right)^2 + 2\left(P\Big(\sum_x (\mathcal{P}_j\rho_s(x))^2 \,|\, \mathcal{F}_{s-1} \Big) \right)^2\\
&\leq 2\mathcal{R}_s^2 + 2P\,(\mathcal{R}_s \,|\, \mathcal{F}_{s-1})^2\\
&\leq 2\mathcal{R}_s + 2P\Big(\mathcal{R}_s \,|\, \mathcal{F}_{s-1}\Big).
\end{aligned}
$$

Hence,

$$
\langle Z_j \rangle_s - \langle Z_j \rangle_{s-1} = P\left((Z_j(s) - Z_j(s-1))^2 \,|\, \mathcal{F}_{s-1} \right) \leq 4P\Big(\mathcal{R}_s \,|\, \mathcal{F}_{s-1}\Big).
$$

We will prove that

$$P\Big(\mathcal{R}_s \,|\, \mathcal{F}_{s-1}\Big) \le 2m^{(2)}\mathcal{R}_{s-1}. \tag{3.10}$$

Then, $\langle\, Z_j \,\rangle_t \le 8m^{(2)}V_{t-1}$, and the lemma follows from the standard law of large numbers for a square-integrable martingale, cf. section 2.2,(4).
It remains to show (3.10). Using (3.8) and (3.9) to $y_1 = y_2 = y$, we have

$$\begin{aligned}
P\Big(\mathcal{R}_s \,|\, \mathcal{F}_{s-1}\Big) &= \sum_y P\Big(\rho_s^2(y) \,|\, \mathcal{F}_{s-1}\Big) \\
&= \sum_y \sum_{z_1,z_2} \sum_{\nu_1=1}^{N_{s-1,z_1}} \sum_{\nu_2=1}^{N_{s-1,z_2}} h_{1,2}\, P\left(\frac{K^{\nu_1}_{s-1,z_1}K^{\nu_2}_{s-1,z_2}}{N_s^2} \,|\, \mathcal{F}_{s-1}\right) \\
&\le \sum_y \sum_{z_1,z_2} \sum_{\nu_1=1}^{N_{s-1,z_1}} \sum_{\nu_2=1}^{N_{s-1,z_2}} h_{1,2}\, \frac{m^{(2)}}{N_{s-1}^2}.
\end{aligned}$$

To obtain the last inequality, we used $N_s \ge N_{s-1}$ and the independence between $K^{\cdot}_{s-1,\cdot}$ and $\mathcal{F}_{s-1}$. We divide the last summation into the summation over $(z_1, \nu_1) = (z_2, \nu_2)$ and that over $(z_1, \nu_1) \neq (z_2, \nu_2)$, to see that

$$\sum_y \sum_{z_1,z_2} \sum_{\nu_1=1}^{N_{s-1,z_1}} \sum_{\nu_2=1}^{N_{s-1,z_2}} \frac{h_{1,2}}{N_{s-1}^2} \le \frac{1}{N_{s-1}} + \sum_x (\mathcal{P}\rho_{s-1}(x))2 \le \frac{1}{N_{s-1}} + \mathcal{R}_{s-1} \le 2\mathcal{R}_{s-1}.$$

Here, we used $\mathcal{R}_{s-1} = \sum_x N^2_{s-1,x}/N^2_{s-1} \ge 1/N_{s-1}$ to see the last inequality. Putting things together, we have (3.10) and the proof of the lemma is now complete. □

3.2 Proof of Theorem 1.3.2:

We first note that there are $\epsilon > 0$ and $t_0 \in \mathbb{N}$ such that

$$\sum_{s=1}^{t_0} r_s \ge \frac{1+\epsilon}{\alpha - 1}, \tag{3.11}$$

where the constant α is from Proposition 1.2.1. For $d = 1, 2$, we take $\epsilon = 1$. Then, (3.11) holds for t_0 large enough, since $\sum_{s=1}^{\infty} r_s = \infty$. For $d \ge 3$, our assumption $P(\overline{N}_\infty = 0) = 1$ implies $\alpha \ge \alpha^* > 1/\pi_d$ by Proposition 1.2.1. Since $\sum_{s=1}^{\infty} r_s = \frac{\pi_d}{1-\pi_d}$, (3.11) holds for small enough $\epsilon > 0$ and large enough t_0.
Let $t > t_0$. Applying Lemma 3.1.5 to $s = t-1, t-2, ..., t-t_0$ and taking the sum on s, we get

$$\begin{aligned}
&\sum_{s=t-t_0}^{t-1} (2c_1\mathcal{R}_s^{3/2} + \frac{c_3}{N_s}) \\
\ge\; &\sum_{s=t-t_0}^{t-1} \left(\sum_x (\mathcal{P}_{t-s}\rho_s(x))^2 - P\Big(\sum_x (\mathcal{P}_{t-(s+1)}\, \rho_{s+1}(x))^2 \,|\, \mathcal{F}_s\Big)\right) + (\alpha - 1) \sum_{s=t-t_0}^{t-1} r_{t-s}\mathcal{R}_s \\
=\; &\sum_{s=t-t_0}^{t-1} \left(\sum_x (\mathcal{P}_{t-(s+1)}\, \rho_{s+1}(x))^2 - P\Big(\sum_x (\mathcal{P}_{t-(s+1)}\, \rho_{s+1}(x))^2 \,|\, \mathcal{F}_s\Big)\right) + \\
&\quad \sum_{s=t-t_0}^{t-1} \left(\sum_x (\mathcal{P}_{t-s}\rho_s(x))^2 - \sum_x (\mathcal{P}_{t-(s+1)}\, \rho_{s+1}(x))^2 \right) + (\alpha - 1) \sum_{s=t-t_0}^{t-1} r_{t-s}\mathcal{R}_s
\end{aligned}$$

$$= \sum_{s=t-t_0}^{t-1} \Big[Z_{t-(s+1)}(s+1) - Z_{t-(s+1)}(s) \Big] + \sum_x \left(\mathcal{P}_{t_0}\, \rho_{t-t_0}(x) \right)^2 - \mathcal{R}_t + (\alpha-1) \sum_{s=t-t_0}^{t-1} r_{t-s} \mathcal{R}_s,$$

where we recall that the martingale $Z_j(\cdot)$ are defined in Lemma 3.1.6. By change of variable $s = t - j$, we have proven that

$$\sum_{j=1}^{t_0} (2c_1 \mathcal{R}_{t-j}^{3/2} + \frac{c_3}{N_{t-j}}) \geq \sum_{j=1}^{t_0} \Big[Z_{j-1}(t-j+1) - Z_{j-1}(t-j) \Big] - \mathcal{R}_t + (\alpha-1) \sum_{j=1}^{t_0} r_j \mathcal{R}_{t-j}.$$

Taking the sum of these inequalities for $t = t_0 + 1, ..., T$, we obtain that

$$\begin{aligned} \sum_{t=t_0+1}^{T} \sum_{j=1}^{t_0} (2c_1 \mathcal{R}_{t-j}^{3/2} + \frac{c_3}{N_{t-j}}) \quad &\geq \quad \sum_{j=1}^{t_0} \Big[Z_{j-1}(T-j+1) - Z_{j-1}(t_0-j+1) \Big] - (V_T - V_{t_0}) \\ &\quad + (\alpha-1) \sum_{j=1}^{t_0} r_j (V_{T-j} - V_{t_0-j}). \end{aligned}$$

Since $\mathcal{R}_s \leq 1$,

$$\begin{aligned} V_{T-j} - V_{t_0-j} \quad &\geq \quad V_T - j - (t_0 - j) = V_T - t_0, \\ (\alpha-1) \sum_{j=1}^{t_0} r_j (V_{T-j} - V_{t_0-j}) \quad &\geq \quad (\alpha-1) \sum_{j=1}^{t_0} r_j V_T - c_9 \geq (1+\epsilon) V_T - c_9, \end{aligned}$$

with constant $c_9 = (\alpha-1) t_0 \sum_{j=1}^{t_0} r_j$. Hence,

$$\sum_{t=t_0+1}^{T} \sum_{j=1}^{t_0} (2c_1 \mathcal{R}_{t-j}^{3/2} + \frac{c_3}{N_{t-j}}) \geq \sum_{j=1}^{t_0} \Big[Z_{j-1}(T-j+1) - Z_{j-1}(t_0-j+1) \Big] + \epsilon V_T - c_9. \quad (3.12)$$

Recall from Lemma 3.1.3 that $\sum_{t=1}^{\infty} \frac{1}{N_t} < \infty, a.s.$, which combined with Lemma 3.1.6 implies that the two sums involving respectively $\frac{c_3}{N_{t-j}}$ and $Z_{j-1}(T-j+1)$ in (3.12) are negligible, relative to V_T. It follows that

$$\liminf_{T\to\infty} \frac{1}{V_T} \sum_{t=t_0+1}^{T} \sum_{j=1}^{t_0} \mathcal{R}_{t-j}^{3/2} \geq \frac{\epsilon}{2c_1}, \quad a.s.$$

Consequently,

$$\liminf_{T\to\infty} \frac{1}{V_T} \sum_{t=1}^{T} \mathcal{R}_t^{3/2} \geq \frac{\epsilon}{2c_1\, t_0}, \quad a.s.,$$

which implies that

$$\limsup_{t\to\infty} \mathcal{R}_t \geq (\frac{\epsilon}{2c_1\, t_0})^2, \qquad a.s.$$

This completes the proof of theorem. □

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