Stochastic Power Law Fluids: Construction of a Weak Solution

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We will report on a work in progress [TeYo09] with Yutaka Terasawa (Tohoku Univ) concerning the existence and the uniqueness of the solution to a certain SPDE.

We consider viscous, incompressible fluid subject to a random perturbation. The container of the fluid is supposed to be the torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \cong [0,1]^d$ as a part of idealization. For a differentiable vector field $v : \mathbb{T}^d \to \mathbb{R}^d$, which is interpreted as the velocity field of the fluid, we will denote the *symmetrized velocity gradient* by:

$$e(v) = \left(\frac{\partial_i v_j + \partial_j v_i}{2} \right) : \mathbb{T}^d \to \mathbb{R}^d \otimes \mathbb{R}^d. \tag{0.1}$$

We assume that the extra stress tensor:

$$\tau(v) : \mathbb{T}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$$

depends on the deformation rate tensor via the following power law: for $\nu > 0$ (the kinematic viscosity) and $p > 1$,

$$\tau(v) = 2\nu(1 + |e(v)|^2)^{\frac{p-2}{2}} e(v). \tag{0.2}$$

The linearly dependent case $p = 2$ is the *Newtonian fluid*, which is described by the Navier-Stokes equation, the special case of (0.3)–(0.4) below. On the other hand, both the *shear thinning* ($p < 2$) and the *shear thickening* ($p > 2$) cases are considered in many fields in science and engineering. For example, shear thinning fluids are used for automobile engine oil and pipeline for crude oil transportation, while applications of shear thinckening fluids can be found in modeling of body armors and automobile four wheels driving systems [Wi09].

Given an initial velocity $u_0 : \mathbb{T}^d \to \mathbb{R}^d$, the dynamics of the fluid is described by the following SPDE:

$$\operatorname{div} u = 0, \tag{0.3}$$

$$\partial_t u + (u \cdot \nabla)u = -\nabla \Pi + \operatorname{div} \tau(u) + \partial_t W, \tag{0.4}$$

where

$$u \cdot \nabla = \sum_{j=1}^d u_j \partial_j \ \text{ and } \ \operatorname{div} \tau(u) = \left(\sum_{j=1}^d \partial_j \tau_{ij}(u) \right)_{i=1}^d. \tag{0.5}$$

The unknown process in the SPDE are the velocity field $u = u(t,x) = (u_i(t,x))_{i=1}^d$ and the pressure $\Pi = \Pi(t,x)$. The Brownian motion $W = W(t,x) = (W_i(t,x))_{i=1}^d$ with values in $L^2(\mathbb{T}^d \to \mathbb{R}^d)$ is added as the random perturbation. Physical interpretation of (0.3) and (0.4) are the conservation laws of the mass and the motion equation, repectively. We note that the SPDE (0.3)–(0.4) for the case $p = 2$ is the stochastic Navier-Stokes equation [Fl08].

We first state the existence result:

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Theorem 1 For certain ranges of the power p (e.g.,$p \in (3/2, \infty)$ for $d = 2$, $p \in (9/5, \infty)$ for $d = 3$...), there exists a weak solution to the SPDE (0.3)–(0.4) globally in time.

For $d = 2, 3$, this generalizes the known result for the stochastic Navier-Stokes equation ([Fl08] and references therein). Also, by considering the degenerate noise, our result recovers the PDE result [MNRR96].

Let us briefly sketch the outline of the proof of Theorem 1:

Step 1: Set up a finite dimensional subspace of a smooth, divergence-free vector fields, say $\mathcal{V}^n$, and an approximating equation to the SPDE (0.3)–(0.4) in $\mathcal{V}^n$. A good news here is that the approximating equation is a well posed SDE, admitting a unique strong solution $u^n \in \mathcal{V}^n$.

Step 2: Establish some a priori bounds for the solution $u^n \in \mathcal{V}^n$ of the approximating SDE. The point here is that the bounds should be *uniform in* n for them to be useful. Martingale inequalities (e.g., the Burkholder-Davies-Gundy inequality) are effectively used here, working in combination with the Sobolev imbedding theorem.

Step 3: Show that the solutions $u^n \in \mathcal{V}^n$ to the approximating SDE are tight as $n \to \infty$. This is where the a priori bounds in Step 2 play their roles as the moment estimates to ensure that the tails of the solutions are thin enough in certain Sobolev norms.

Step 4: By Step 3, u^n ($n \to \infty$) converges in law along a subsequence to a limit. Verify that the limit is a weak solution to the SPDE (0.3)–(0.4).

Here are some comments concerning the technical difference between the Navier-Stokes equation ($p = 2$) and the power law fluids. For the Navier-Stokes equations (both stochastic and deterministic), it is reasonable to discuss solutions in the L_2-space [Fl08, Te79]. On the other hand, for the power law fluids given by (0.2), it is the L_p-space and its dual space that become relevant. Also, due to the extra non-linearlity introduced by (0.2), some of the arguments (e.g. Step 2 above) for $p \neq 2$ become considerably more involved than the case of $p = 2$, especially for $p < 2$. We will overcome this difficulty by carrying the ideas in [MNRR96] over to the framework of Itô's calclus.

We also have the following uniqueness result:

Theorem 2 For $p \geq 1 + \frac{d}{2}$, the weak solution to the SPDE (0.3)–(0.4) is pathwise unique.

This is also in consistent with the uniqueness result known for deterministic PDE [MNRR96].

References

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