Level-Rank Duality in Fusion RSOS Models

by

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ABSTRACT. Let n and ℓ be integers not less than 2 and let Y be a Young diagram with the depth at most $n-1$, and the width at most $\ell-1$. With a data $(A^{(1)}_{n-1}, Y, \ell)$ there is associated a solvable restricted solid-on-solid (RSOS) model in two-dimensional statistical mechanics. We propose a duality between the models corresponding to the datas $(A^{(1)}_{n-1}, Y, \ell)$ and $(A^{(1)}_{\ell-1}, {}^tY, n)$, where ${}^tY$ denotes the transposition of Y along the NW-SE diagonal. The duality is due to the equivalence of the fusion rule in $A^{(1)}_{n-1}$ level ℓ Wess-Zumino-Witten model with the ℓ and the rank n interchanged. Based on this we explain curious phenomena concerning the RSOS 1-point functions.

For proceedings of *International Colloquium on Modern Quantum Field Theory* , January 8-14, 1990, TIFR, Bombay, India.

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1. INTRODUCTION

Integrable systems in two-dimensional statistical mechanics and quantum field theories have been a subject of active research[1]. Considerable amount of such systems have been found in both fields as well as their interrelations. Among these accumulations, restricted solid-on-solid (RSOS) models form an important class in the realm of solvable lattice models. They possess remarkable structures related to quantum groups at q a root of unity, representation theory of affine Lie algebras and rational conformal field theories, etc.

The $A^{(1)}_{n-1}$ face models[2] are the RSOS models of such sort basicallly characterized by a triad $(A^{(1)}_{n-1}, Y, \ell)$. Here n and ℓ are integers not less than 2. We call n the rank and ℓ the level of the model. The Y is a Young diagram with the depth $< n$ and the width $< \ell$. It specifies the irreducible representation of the classical part $sl(n, \mathrm{C})$ of $A^{(1)}_{n-1}$ in the usual way on the finite dimensional space which we write as $\mathcal{V}_Y$. The model is an (generalized) interaction round face model in the sense of Baxter[3] having the following features.

(i) The site variables range over the level ℓ dominant integral weights of $A^{(1)}_{n-1}$.

(ii) A domiant integral weight b is allowed to occupy the right or lower neighbor site of another one a only if

$$\begin{aligned}
&N^Y_{ab} \geq 1, \\
&N^Y_{ab} = \min{}_{\sigma}[\mathcal{V}(\sigma(a)) \otimes \mathcal{V}_Y : \mathcal{V}(\sigma(b))], \qquad (1.1) \\
&\sigma : \text{ cyclic Dynkin diagram automorphism of } A^{(1)}_{n-1}.
\end{aligned}$$

Here $\mathcal{V}(\sigma(a))$, for example, stands for the irreducible $sl(n, \mathrm{C})-$ module whose highest weight is the classical part $\overline{\sigma(a)}$ of $\sigma(a)$. The symbol $[\ :\ ]$ denotes the multiplicity of $\mathcal{V}(\sigma(b))$ in the irreducible decomposition of $\mathcal{V}(\sigma(a)) \otimes \mathcal{V}_Y$. In the case the decomposition is not multiplicity free, the corresponding degree of freedom is taken into account as the *edge* variable. Under these conditions a solution of the Yang-Baxter equation has been obtained[2]. The construction uses the *fusion procedure*[4] piling the simplest case $Y = (1)$ (a Young diagram with a single node) up to the Y in question. The resulting solution provides elliptic parametrization of the Boltzmann weights as the function of the *spectral parameter* u. At the critical point the parametrization reduces to the trigonometric one originating from the representation theory of the quantized universal enveloping algebra $U_q(sl(n, \mathbf{C}))$ at $q = e^{\frac{2\pi i}{n+\ell}}$.

In this paper, we propose a duality between the pair of the models corresponding to $(A^{(1)}_{n-1}, Y, \ell)$ and $(A^{(1)}_{\ell-1}, {}^tY, n)$. Here ${}^tY$ denotes the transposition of the Y along the NW-SE diagonal. We claim that the two models are equivalent to each other with the spectral parameter u replaced by $-u$. Under some assumptions each configuration in one model can be mapped to that in the duality counterpart having the equal statistical weight. A key to the configuration correspondence is the following property of N^Y_{ab}.

$$N^Y_{ab} = N^{{}^tY}_{{}^ta\,{}^tb}. \tag{1.2}$$

The left and the right sides are referring to the datas $(A^{(1)}_{n-1}, Y, \ell)$ and $(A^{(1)}_{\ell-1}, {}^tY, n)$, respectively. See section 3 for the precise description of the level n dominant integral weights ${}^ta$ and ${}^tb$ of $A^{(1)}_{\ell-1}$. We note that the quantity N^Y_{ab} (1.1) also appears as the *fusion rule* in $A^{(1)}_{n-1}$ level ℓ Wess-Zumino-Witten model. Therefore (1.2) implies an equivalence of the fusion rule to the one with the level ℓ and the rank n interchanged. Our proof of (1.2) actually relies on Verlinde's formula[5,6]) expressing N^Y_{ab} in terms of modular transformation matrices.

By the construction, the duality naturally extends to the case of non-unitary models where the relevant q becomes $q = e^{\frac{2\pi i t}{n+\ell}}$ with t coprime to $n + \ell$. Based on these facts, we explain curious phenomena concerning the 1-point functions observed so far in various cases[2,11,12]). These are physical consequences of the complementary role of the level and the rank in the representation theory of affine Lie algebras (cf. ref.7).

The paper is organized as follows. In the next section we recall the construction of the $A^{(1)}_{n-1}$ face models following ref. 2. The duality is formulated in section 3 between the models for the datas $(A^{(1)}_{n-1}, Y, \ell)$ and $(A^{(1)}_{\ell-1}, {}^tY, n)$. Their equivalence follows from two assertions, i.e., the configuration correspondence and the proportionality of the statistical weights. The latter is yet to be verified except for Y representing the $N-$symmetric ($Y = (N)$) and the $N-$antisymmetric ($Y = (1^N)$) tensors. Section 4 demonstrates the examples of the duality for some small values of n and ℓ. We depict the correspondence of the configurations between the duality pairs. In section 5 we discuss some consequences of the duality on the RSOS 1-point functions. Appendix is devoted to the proof of (1.2). We note that in the simplest case $Y = (1)$, our duality reduces to that found in ref. 8, which was a motivation of our study.

2. THE $A_{n-1}^{(1)}$ FACE MODELS

Let us recall the construction of the RSOS model corresponding to $(A_{n-1}^{(1)}Y, \ell)$[2]. Let $P_+(n,\ell)$ be the set of level ℓ dominant integral weights of the affine Lie algebra $A_{n-1}^{(1)}$. We shall call an element of $P_+(n,\ell)$ a *state*. We set $L = n+\ell$. A state $a \in P_+(n,\ell)$ is written as

$$a = (L + a_{n-1} - a_0 - 1)\Lambda_0 + (a_0 - a_1 - 1)\Lambda_1 + \cdots + (a_{n-2} - a_{n-1} - 1)\Lambda_{n-1}, \quad (2.1)$$

with a_μ's obeying the conditions $a_{\mu\nu} \stackrel{\text{def}}{=} a_\mu - a_\nu \in \mathbf{Z}$ for all $0 \leq \mu, \nu \leq n-1$ and $L + a_{n-1} > a_0 > a_1 > \cdots > a_{n-1}$. We extend the index of the fundamental weights by setting $\Lambda_\mu = \Lambda_{\mu+n}, \forall \mu$ and denote by σ the cyclic Dynkin diagram automorphim: $\sigma(\Lambda_\mu) = \Lambda_{\mu+1}$. Let Y be a Young diagram with the depth less than n and the width less than ℓ. Our aim is to build, in one to one correspondence with Y, a vector space V_Y and an operator $W_Y(u) \in \mathrm{End}(V_Y \otimes V_Y), u \in \mathbf{C}$ such that;

(1) The Yang-Baxter equation is fulfilled in $\mathrm{End}(V_Y^{\otimes 3})$:

$$\begin{aligned}&\big(1 \otimes W_Y(u)\big)\big(W_Y(u+u') \otimes 1\big)\big(1 \otimes W_Y(u')\big)\\ &= \big(W_Y(u') \otimes 1\big)\big(1 \otimes W_Y(u+u')\big)\big(W_Y(u) \otimes 1\big).\end{aligned}$$

(2) The space V_Y has the direct sum decomposition $V_Y = \oplus_{a,b \in P_+(n,\ell)}(V_Y)_{ab}$ and the composition

$$(V_Y)_{ab} \otimes (V_Y)_{b'c'} \xrightarrow{i} V_Y \otimes V_Y \xrightarrow{W_Y(u)} V_Y \otimes V_Y \xrightarrow{\pi} (V_Y)_{a'd'} \otimes (V_Y)_{dc}$$

is zero unless $a = a', b = b', c = c'$ and $d = d'$.

Hereafter we shall write $W_Y\left(\begin{matrix} ab \\ dc \end{matrix}\middle| u\right) = \pi \circ W_Y(u) \circ i \in \mathrm{Hom}_{\mathbf{C}}\big((V_Y)_{ab} \otimes (V_Y)_{bc},$

$(V_Y)_{ad} \otimes (V_Y)_{dc}\big)$,i.e., $W_Y(u) = \oplus_{a,b,c,d \in P_+(n,\ell)} W_Y\left(\begin{matrix} ab \\ dc \end{matrix}\middle| u\right)$. Note that $W_Y(u)$ here corresponds to a less general case $Y = Y'$ of $W_{YY'}(u)$ treated in ref. 2. The operator $W_Y\left(\begin{matrix} ab \\ dc \end{matrix}\middle| u\right)$ gives rise to a solvable statistical mechanical model, which we shall now explain.

Consider a two-dimensional square lattice $\mathcal{L}$ of size, say, $(M+1)$ by $(M'+1)$. To each site we attach a coordinate $(i,j)(0 \leq i \leq M, 0 \leq j \leq M', i,j \in \mathbf{Z})$

so that the points $(0,0),(0,M'),(M,0)$ and (M,M') signify the NW, NE, SW and SE corner of $\mathcal{L}$, respectively. We also assign a coordinate $(i-\frac{1}{2},j)$ (resp. $(i,j-\frac{1}{2})$) to the vertical (resp. horizontal) edge between the sites $(i-1,j)$ and (i,j) (resp. $(i,j-1)$ and (i,j)). We introduce two kinds of physical degrees of freedom, one sitting on the sites and the other on the edges of $\mathcal{L}$. The site variables $\{a^{(ij)}\}_{0\le i\le M, 0\le j\le M'}$ take their values in the set $P_+(n,\ell)$. On the other hand the variables $\{\alpha^{(i-\frac{1}{2}j)}\}_{1\le i\le M, 0\le j\le M'}$ (resp. $\{\alpha^{(ij-\frac{1}{2})}\}_{0\le i\le M, 1\le j\le M'}$) on the vertical (resp. horizontal) edges range over the base vectors of $(V_Y)_{a^{(i-1j)}a^{(ij)}}$ (resp. $(V_Y)_{a^{(ij-1)}a^{(ij)}}$). Thus $b\in P_+(n,\ell)$ is allowed to occupy the lower or right neighbor site of another state a if and only if $\dim(V_Y)_{ab}\ge 1$. By a configuration on $\mathcal{L}$, we mean an assignment of all the variables $a^{(ij)}, \alpha^{(i-\frac{1}{2}j)}$ and $\alpha^{(ij-\frac{1}{2})}$ on $\mathcal{L}$ satisfying the above constraint on every edge. The elementary interaction takes place among the eight round the face (Fig.1). Here α is a base vector of $(V_Y)_{ab}$, etc. To this configuration round the face we assign the $(\alpha\otimes\beta,\delta\otimes\gamma)$ matrix element $W_Y\left(\begin{matrix} ab \\ dc \end{matrix}\middle| u\right)^{\alpha\beta}_{\delta\gamma}$ of the operator $W_Y\left(\begin{matrix} ab \\ dc \end{matrix}\middle| u\right)$ as the statistical weight. The partition function of the system is then given by

$$Z=\sum_{\text{configurations}}\prod_{\substack{0\le i\le M\\ 0\le j\le M'}} W_Y\left(\begin{matrix} a^{(ij)} & a^{(ij+1)} \\ a^{(i+1j)} & a^{(i+1j+1)} \end{matrix}\middle| u\right)^{\alpha^{(ij+\frac{1}{2})}\alpha^{(i+\frac{1}{2}j+1)}}_{\alpha^{(i+\frac{1}{2}j)}\alpha^{(i+1j+\frac{1}{2})}}. \tag{2.2}$$

Here the factors involving $a^{(iM'+1)}$ or $a^{(M+1j)}$ are to be understood as 1. This defines an (generalized) interaction round face model[3)] whose Boltzmann weights obey the Yang-Baxter equation.

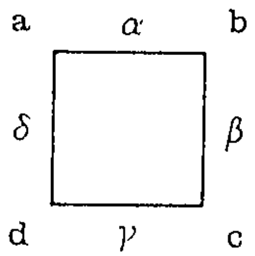


Fig.1 A configuration round a face. Its Boltzmann weight is denoted by $W_Y\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right)^{\alpha\beta}_{\delta\gamma}$.

Before going to the description of V_Y and $W_Y(u)$, we need to prepare further notations. Let $(\rho_a, \mathcal{V}(a))$ be the finite dimensional irreducible representation of $sl(n, \mathbf{C})$ with the highest weight $\overline{a}$, the classical part of a. Thus in particular $\mathcal{V}(\Lambda_1)$ becomes the space of the vector representation of $sl(n, \mathbf{C})$, which will be denoted as $\mathcal{A} \stackrel{\text{def}}{=} \oplus_{\lambda=0}^{n-1}\mathbf{C}e_\lambda$, $e_\lambda \stackrel{\text{def}}{=} (\underbrace{0,\ldots,0}_{\lambda},1,\underbrace{0,\ldots,0}_{n-1-\lambda})$. We will also regard the dominant integral weight (2.1) as $(a_0, \ldots, a_{n-1})$ belonging to $\mathcal{A}$. A sequence $p = (a^{(j)})_{j=0}^{N}$ of the states $a^{(j)} \in P_+(n,\ell)$ is called an $N-$admissible path from a to b if $a^{(0)} = a, a^{(N)} = b$ and $a^{(j)} - a^{(j-1)} = e_{\lambda_j}$ for some $\lambda_j (j = 1, \ldots, N)$. We put $e_p^{(N)} = e_{\lambda_1} \otimes \cdots \otimes e_{\lambda_N} \in \mathcal{A}^{\otimes N}$. In particular, we write e_{ab} to mean the 1-admissible path from a to b. Set $(\tilde{V}^N)_{ab} = \oplus_p \mathbf{C} e_p^{(N)}$, where the direct sum extends over all the $N-$admissible paths p from a to b. Notice that $(\tilde{V}^N)_{ab} \subset \mathcal{A}^{\otimes N}$.

We invoke the fusion method to construct V_Y and $W_Y(u)$ out of the simplest case $Y = (1)$. In this latter case, the dimensionality of the space $(V_{(1)})_{ab}$ does not exceed 1 and we have

$$(V_{(1)})_{ab} = \begin{cases} \mathbf{C}e_{ab} & \text{if } b - a \in \mathcal{A}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

Thus $W_{(1)}\left(\begin{matrix} ab \\ dc \end{matrix}\middle| u\right)$ becomes a scalar operator:

$$W_{(1)}\left(\begin{matrix} ab \\ dc \end{matrix}\middle| u\right) e_{ab} \otimes e_{bc} = B\left(\begin{matrix} ab \\ dc \end{matrix}\middle| u\right) e_{ad} \otimes e_{dc},$$

where the Boltzmann weights in the r.h.s. are given as follows[8] $(\mu \neq \nu)$.

$$\begin{aligned}
B\left(\begin{matrix} a & a+e_\mu \\ a+e_\mu & a+2e_\mu \end{matrix}\middle| u\right) &= \frac{[1+u]}{[1]}, \\
B\left(\begin{matrix} a & a+e_\nu \\ a+e_\mu & a+e_\mu+e_\nu \end{matrix}\middle| u\right) &= \epsilon \frac{[a_{\mu\nu}+1][u]}{[a_{\mu\nu}][1]}, \quad \epsilon = \pm 1, \\
B\left(\begin{matrix} a & a+e_\mu \\ a+e_\mu & a+e_\mu+\nu \end{matrix}\middle| u\right) &= \frac{[a_{\mu\nu}-u]}{[a_{\mu\nu}]}.
\end{aligned} \tag{2.4}$$

Either sign $\epsilon = \pm 1$ is allowed so long as the choice does not vary depending on a, μ, ν. The symbol $[u]$ denotes the elliptic theta function with nome p $(-1 <$

$p < 1$):

$$[u] = 2p^{1/8}\sin\frac{\pi u}{L}\prod_{k=1}^{\infty}(1 - 2p^k\cos\frac{2\pi u}{L} + p^{2k})(1 - p^k). \tag{2.5}$$

Using the coefficients (2.4) we introduce the elementary operators $W_i(u) \in \mathrm{End}(\tilde{V}^M)\,(i = 1, \ldots, M-1)$ by (M is assumed to be sufficiently large)

$$\begin{aligned} W_i(u)e_{a^{(0)}a^{(1)}} \otimes \cdots \otimes e_{a^{(M-1)}a^{(M)}} &= \sum B\left(\begin{matrix} a^{(i-1)}a^{(i)} \\ a'^{(i)}a^{(i+1)} \end{matrix}\middle| u\right) \\ &\times e_{a^{(0)}a^{(1)}} \otimes \cdots \otimes e_{a^{(i-1)}a'^{(i)}} \otimes e_{a'^{(i)}a^{(i+1)}} \otimes \cdots \otimes e_{a^{(M-1)}a^{(M)}}, \end{aligned} \tag{2.6}$$

where the sum is taken over $a'^{(i)} \in P_+(n,\ell)$ with the condition $\dim(V_{(1)})_{a^{(i-1)}a'^{(i)}} = \dim(V_{(1)})_{a'^{(i)}a^{(i+1)}} = 1$. They satisfy the Yang-Baxter equation:

$$W_i(u)W_{i+1}(u+u')W_i(u') = W_{i+1}(u')W_i(u+u')W_{i+1}(u).$$

Let $Y = (y_1, y_2, \ldots, y_{n-1})$ be the Young diagram that constitutes the basic data $(A^{(1)}_{n-1}, Y, \ell)$ of the model. Here y_i denotes the length of the i-th row of Y and is assumed to obey

$$\begin{aligned} &\ell \geq y_1 \geq \cdots \geq y_m > 0 \quad \text{for some} \quad 1 \leq m \leq n-1, \\ &y_j = 0 \quad \text{for } m < j \leq n-1. \end{aligned} \tag{2.7}$$

We set $N = y_1 + \cdots + y_{n-1}$, which is the number of the nodes contained in Y. Now we are going to define an operator $F \in \mathrm{End}(\tilde{V}^N)$ which is analogous to the Young symmetrizer for a standard tableau on Y (cf. appendix A of ref.2). We introduce an auxiliary parameter z and assign $v_i(z)(i = 1, \ldots, N)$ to each node i of Y as follows.

$$\begin{aligned} v_1(z), \ldots, v_N(z) = 0, 1, \ldots, y_1 - 1, z-1, z, \ldots, z + y_2 - 2, \ldots, \\ (m-1)z - m + 1, (m-1)z - m + 2, \ldots, (m-1)z - m + y_m. \end{aligned}$$

Using the elementary operator $W_i(u)$ (2.6) and $v_{ij} \stackrel{\text{def}}{=} v_i(z) - v_j(z)$ we set

$$F(z) = W_1(v_{21})\big(W_2(v_{31})W_1(v_{32})\big)\cdots\big(W_{N-1}(v_{N1})\cdots W_1(v_{NN-1})\big). \tag{2.8}$$

It turns out that $F(z)$ is vanishing as $z^{\kappa}O(1)$ in the limit $z \to 0$, where κ is the number of pairs of the nodes on the NW-SE diagonals of Y. Removing this factor we define $F \in \operatorname{End}(\tilde{V}^N)$ as follows.

$$
\begin{aligned}
F &= \lim_{z\to 0} F(z) \prod_{1\le i<j\le m} \beta_{ij}(z)^{-1}, \\
\beta_{ij}(z) &= \prod_{r=2}^{y_j} \prod_{s=1}^{y_i} [(j-i)(z-1)+r-s].
\end{aligned} \tag{2.9}
$$

One can check that $\prod_{1\le i<j\le m}\beta_{ij}(z)$ has precisely κ zeros at $z=0$. In the limit $L\to\infty, |\, a_{\mu\nu}\,| \to \infty$, the F tends to $c_T \times$(invertible element), where c_T is a Young symmetrizer for a standard tableau T on Y (cf. ref. 2). Our V_Y is defined to be the image of $\tilde{V}^N$ under F.

$$
(V_Y)_{ab} = F((\tilde{V}^N)_{ab}). \tag{2.10}
$$

The operator $W_Y(u) \in \operatorname{End}(\tilde{V}^{2N})$ is also made from the following composition of the elementary operators.

$$
\begin{aligned}
W_Y(u) &= w_N w_{N-1}\cdots w_1, \\
w_i &= W_i(u_1-u_i)W_{i+1}(u_2-u_i)\cdots W_{i+N-1}(u_N-u_i), \\
u_i &= u + v_i(0).
\end{aligned} \tag{2.11}
$$

By the construction, the $W_Y(u)$ is shown to satisfy the Yang-Baxter equation in $\operatorname{End}\big((\tilde{V}^N)^{\otimes 3}\big)$. It can be also verified that

$$
W_Y\left(\begin{matrix} ab \\ dc \end{matrix}\middle| u\right)\big((V_Y)_{ab}\otimes(V_Y)_{bc}\big) \subset \big((V_Y)_{ad}\otimes(V_Y)_{dc}\big).
$$

Thus $W_Y(u)$ indeed belongs to $\operatorname{End}(V_Y\otimes V_Y)$. In general one has to fix base vectors α of $(V_Y)_{ab}$, etc to write down the Boltzmann weights $W_Y\left(\begin{matrix} ab \\ dc \end{matrix}\middle| u\right)^{\alpha\beta}_{\delta\gamma}$. This has been done for some non-trivial cases in ref. 2. Although, the general case is still to be worked out.

Finally, let us include a remark on the dimensionality of the space $(V_Y)_{ab}$. Based on the invariance of the Boltzmann weights under any cyclic Dynkin diagram automorphism σ, it has been conjectured as[2] (see (1.1))

$$\dim(V_Y)_{ab} = N_{ab}^{Y}, \tag{2.12a}$$

$$N_{ab}^{Y} \stackrel{\text{def}}{=} \min_\sigma \overline{N}_{\sigma(a)\sigma(b)}^{Y}, \tag{2.12b}$$

$$\overline{N}_{ab}^{Y} \stackrel{\text{def}}{=} [\mathcal{V}(a) \otimes \mathcal{V}_Y : \mathcal{V}(b)]. \tag{2.12c}$$

Here $\mathcal{V}_Y$ is the irreducible $sl(n, \mathbf{C})-$module whose highest weight is the classical part of

$$(\ell - y_1)\Lambda_0 + (y_1 - y_2)\Lambda_1 + \cdots + (y_{n-2} - y_{n-1})\Lambda_{n-2} + y_{n-1}\Lambda_{n-1} \in P_+(n, \ell). \tag{2.13}$$

The N_{ab}^{Y} is known as the fusion rule[5,6] in $A_{n-1}^{(1)}$ level ℓ Wess-Zumino-Witten model. Fixing $a, b \in P_+(n, \ell)$ and Y, one can show that $\dim(V_Y)_{ab}$ is equal to $\overline{N}_{ab}^{Y}$ for L sufficiently large.

$$\lim_{L \to \infty} \dim\, (V_Y)_{ab} = \overline{N}_{ab}^{Y}. \tag{2.14}$$

The quantities $\overline{N}_{ab}^{Y}$ and N_{ab}^{Y} will be described in detail in the next section.

3. DUALITY

Having described the fusion RSOS models, we now turn to the duality among the pair $(A^{(1)}_{n-1}, Y, \ell)$ and $(A^{(1)}_{\ell-1}, {}^tY, n)$. We claim that the corresponding models are equivalent to each other with the spectral parameter u replaced by $-u$. In order to establish this one has to verify the following which we will discuss separately in the sequel.

(i) Correspondence between the configurations.

(ii) Proportionality of the Boltzmann weights for the corresponding configurations.

3.1 Configuration Correspondence.

Firstly, let us explain a way going from $P_+(n,\ell)$ to $P_+(\ell,n)$. For this purpose we introduce Young diagram representation of the dominant integral weights. Let

$$\mathcal{Y}^{(n\ell)} = \{(g_1,\ldots,g_n) \mid g_i \in \mathbf{Z}, g_1 \geq \cdots \geq g_n \geq 0, g_1 - g_n \leq \ell\}, \qquad (3.1a)$$

$$\mathcal{Y}^{(\infty\ell)} = \{(g_1,g_2,\ldots) \mid g_i \in \mathbf{Z}, \ell \geq g_1,\ g_i \geq g_{i+1}\ \forall i,\ g_i = 0 \text{ for } i \gg 1\} \quad (3.1b)$$

be the sets consisting of infinitely many Young diagrams. For $X \in \mathcal{Y}^{(n\ell)}$ or $\mathcal{Y}^{(\infty\ell)}$, we will write $|X|$ to mean the number of the nodes contained in X. We define the maps $D_{n\ell} : \mathcal{Y}^{(n\ell)} \to P_+(n,\ell)$ and $\hat{D}_{n\ell} : \mathcal{Y}^{(\infty\ell)} \to P_+(n,\ell)$ as follows.

$$\begin{aligned} &D_{n\ell}((g_1,\ldots,g_n)) \\ &\quad = (\ell - g_1 + g_n)\Lambda_0 + (g_1 - g_2)\Lambda_1 + \cdots + (g_{n-1} - g_n)\Lambda_{n-1}, \end{aligned} \qquad (3.2a)$$

$$\hat{D}_{n\ell}((g_1,g_2,\ldots)) = (\ell - g_1)\Lambda_0 + \sum_{i=1}^{\infty}(g_i - g_{i+1})\Lambda_i. \qquad (3.2b)$$

Note that the sum in (3.2b) is actually finite and the convention $\Lambda_{i+n} = \Lambda_i$ has been implied. By the definition, the set $\mathcal{Y}^{(n\ell)}$ is the disjoint union of the inverse image $\mathcal{Y}^{(n\ell)}_a$ of $a \in P_+(n,\ell)$ (2.1) under $D_{n\ell}$.

$$\mathcal{Y}^{(n\ell)} = \cup_{a\in P_+(n,\ell)} \mathcal{Y}^{(n\ell)}_a,$$

$$\mathcal{Y}^{(n\ell)}_a = \{(a_0 - n + 1 + k, a_1 - n + 2 + k, \ldots, a_{n-1} + k) \mid k + a_{n-1} \in \mathbf{Z}_{\geq 0}\}. \qquad (3.3)$$

In view of (3.2a) we naturally extend the Dynkin diagram automorphism $\sigma(\Lambda_\mu) = \Lambda_{\mu+1}$ to the operation $\sigma : \mathcal{Y}^{(n\ell)} \to \mathcal{Y}^{(n\ell)}$ as follows.

$$\sigma((g_1,\ldots,g_n)) = (g_n + \ell, g_1, \ldots, g_{n-1}). \qquad (3.4)$$

Suppose that h_i appears k_i times $(1 \leq i \leq s \leq n)$ in the sequence $g_1, \ldots, g_n$ and $0 = h_0 < h_1 < \cdots < h_s$. We then define the map $T_{n\ell} : \mathcal{Y}^{(n\ell)} \to \mathcal{Y}^{(\infty n)}$ by

$$
\begin{aligned}
T_{n\ell}((g_1, \ldots, g_n)) &= (g_1', \ldots, g_s', 0, 0, \ldots), \\
g_i' &= k_j + \cdots + k_s \quad \text{for} \quad h_{j-1} < i \leq h_j.
\end{aligned}
\tag{3.5}
$$

This is the transposition of the Young diagram along the NW-SE diagonal. Now consider the following scheme of these maps.

$$
\begin{array}{ccc}
\mathcal{Y}_a^{(n\ell)} & \xrightarrow{T_{n\ell}} & \mathcal{Y}^{(\infty n)} \\
\big\downarrow D_{n\ell} & & \big\downarrow \hat{D}_{\ell n} \\
a \in P_+(n, \ell) & & P_+(\ell, n)
\end{array}
\tag{3.6}
$$

By this one can associate the subset $\hat{D}_{\ell n}(T_{n\ell}(\mathcal{Y}_a^{(n\ell)})) \subset P_+(\ell, n)$ to each $a \in P_+(n, \ell)$.

Secondly, we explain an effective tool to evaluate the multiplicity $\overline{N}_{ab}^{Y}$ in (2.12c). This is known as the *Littlewood-Richardson rule*[9]. In our context (2.12), it describes the configurations $(a^{(ij)}, \alpha^{(i+\frac{1}{2}j)}, a^{(i+1j)})$ and $(a^{(ij)}, \alpha^{(ij+\frac{1}{2})}, a^{(ij+1)})$ in terms of the Young diagrams. Below we assume that $Y = (y_1, \ldots, y_{n-1})$, $N = y_1 + \cdots + y_{n-1}$ is as given in (2.7) and $a, b \in P_+(n, \ell)$. The procedure goes as follows.

Step 1.

Fix an element $X = (x_1, \ldots, x_n) \in \mathcal{Y}_a^{(n\ell)}$. Then $\overline{N}_{ab}^{Y} = 0$ unless there exists $Z = (z_1, \ldots, z_n) \in \mathcal{Y}_b^{(n\ell)}$ such that $d_i \stackrel{\text{def}}{=} z_i - x_i \geq 0$ for $1 \leq i \leq n$ and $\sum_{i=1}^{n} d_i = N$ (i.e. $|Z| = |X| + |Y|$). If such Z exists, it is unique.

Step 2.

Take the Z as above. Let $r_1, \ldots, r_N$ be an array of integers involving $i\,(1 \leq i < n)$ for y_i times. We inscribe $r_{d_1+\cdots+d_{i-1}+j}$ $(1 \leq i \leq n,\ 1 \leq j \leq d_i)$ on the node of Z located at the $i-$th row and the $(z_i - j + 1)-$th column. The following conditions must be satisfied.

(1)

$$
r_{d_1+\cdots+d_i} \leq \cdots \leq r_{d_1+\cdots+d_{i-1}+1} \quad \text{for } 1 \leq i \leq n.
$$

(2) In every column of Z, the appearing r_k's strictly increase downwards.

(3)

$$c_1^k \geq c_2^k \geq \cdots \geq c_m^k \quad \text{for } 1 \leq k \leq N,$$

$$c_j^k = \sharp\{s \mid 1 \leq s \leq k,\ r_s = j\}.$$

Step 3.

$$\overline{N}_{ab}^{Y} = \text{ the number of the possible ways to do } \textit{Step 2}.$$

This determines the multiplicity $\overline{N}_{ab}^{Y}$. Combining it with (2.12b) and (3.4) one gets N_{ab}^{Y}.

Example. $\overline{N}_{ab}^{Y}$ for $A_3^{(1)}, Y = (2,1,0), \ell = 5$ and $a = 2\Lambda_0 + 2\Lambda_1 + \Lambda_2$.
Take $X = (3,1,0,0) \in \mathcal{Y}_a^{(4\,5)}$. The possible ways to do *Step 2* above are

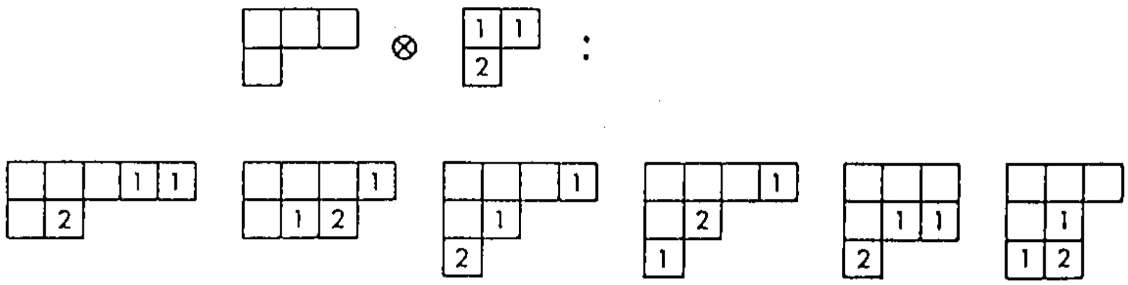


Fig.2 The possible ways of doing Step 2 for $n = 4, \ell = 5, X = (3,1)$ and $Y = (2,1)$.

According to *Step 3* we find that $\overline{N}_{ab}^{Y} = 1$ for $b = 3\Lambda_1 + 2\Lambda_2, \Lambda_0 + \Lambda_1 + 3\Lambda_2, 2\Lambda_0 + 2\Lambda_2 + \Lambda_3, 2\Lambda_0 + \Lambda_1 + 2\Lambda_3$, $\overline{N}_{ab}^{Y} = 2$ for $b = \Lambda_0 + 2\Lambda_1 + \Lambda_2 + \Lambda_3$ and $\overline{N}_{ab}^{Y} = 0$ otherwise.

Thirdly, we state a duality property of the fusion rule N_{ab}^{Y}. For $a \in P_+(n,\ell)$, pick up a Young diagram $X \in \mathcal{Y}_a^{(n\ell)}$. Assume that $N_{ab}^{Y} > 0$, hence $\overline{N}_{ab}^{Y} > 0$. Then there exists a unique $Z \in \mathcal{Y}_b^{(n\ell)}$ as described in *Step1* above. Denote $\hat{D}_{\ell n}(T_{n\ell}(X))$ and $\hat{D}_{\ell n}(T_{n\ell}(Z)) \in P_+(\ell, n)$ by ${}^t a$ and ${}^t b$, respectively. The crucial point we find in the configuration correspondence is the following property.

$$N_{ab}^{Y} = N_{{}^t a {}^t b}^{{}^t Y}. \tag{3.7}$$

The fusion rules in the left and the right sides are referring to the datas $(A_{n-1}^{(1)}, Y, \ell)$ and $(A_{\ell-1}^{(1)}, {}^tY, n)$, respectively. The proof has been done in Appendix by an extensive use of Verlinde's formula[5]. The duality property (3.7) together with (2.12a) lead to

$$\dim\,(V_Y)_{ab} = \dim\,(V_{{}^tY})_{{}^ta\,{}^tb}. \tag{3.8}$$

In the sequel we explain the configuration correspondence between the models for $(A^{(1)}_{n-1}, Y, \ell)$ and $(A^{(1)}_{\ell-1}, {}^tY, n)$ admitting (3.8). Consider a configuration on $\mathcal{L}$ in the model $(A^{(1)}_{n-1}, Y, \ell)$. As presented in section 2, the relevant physical degrees of freedom are the site variables $\{a^{(ij)}\}_{0\le i\le M, 0\le j\le M'}$, the edge variables $\{\alpha^{(i-\frac{1}{2}\,j)}\}_{1\le i\le M, 0\le j\le M'}$ and $\{\alpha^{(i\,j-\frac{1}{2})}\}_{0\le i\le M, 1\le j\le M'}$. The former belongs to $P_+(n,\ell)$ whereas the latter two range over the base vectors of $(V_Y)_{a^{(i-1j)}a^{(ij)}}$ and $(V_Y)_{a^{(ij-1)}a^{(ij)}}$, respectively. By the definition, the site variables have to be chosen so as to satisfy $\dim\,(V_Y)_{a^{(i-1j)}a^{(ij)}} \ge 1$ and $\dim\,(V_Y)_{a^{(ij-1)}a^{(ij)}} \ge 1$ on every edge on $\mathcal{L}$. Upon fixing a Young diagram representation $X^{(00)} \in \mathcal{Y}^{(n\,\ell)}_{a^{(00)}}$, we have consequently a unique assignment $X^{(ij)} \in \mathcal{Y}^{(n\,\ell)}_{a^{(ij)}}$ that obeys $|X^{(i\,j+1)}| = |X^{(i+1\,j)}| = |X^{(ij)}| + N$ (cf. *Step 1*). Now consider the model $(A^{(1)}_{\ell-1}, {}^tY, n)$ and the relevant physical variables which we write as ${}^ta^{(ij)}, {}^t\alpha^{(i-\frac{1}{2}\,j)}$ and ${}^t\alpha^{(i\,j-\frac{1}{2})}$. We put them on a lattice ${}^t\mathcal{L}$, the transposition of $\mathcal{L}$ with respect to the NW-SE diagonal. The corresponding configuration is then built by setting ${}^ta^{(ij)} = \hat{D}_{\ell\,n}(T_{n\,\ell}(X^{(ji)})) \in P_+(\ell, n)$. Notice that (3.8) guarantees that such a state configuration is indeed possible. It also assures that there are as many possible assignments for the edge variables ${}^t\alpha^{(i-\frac{1}{2}\,j)}$ and ${}^t\alpha^{(i\,j-\frac{1}{2})}$ as $\alpha^{(j\,i-\frac{1}{2})}$ and $\alpha^{(j-\frac{1}{2}\,i)}$ in the model $(A^{(1)}_{n-1}, Y, \ell)$, respectively. To summarize, given $a^{(ij)} \in P_+(n,\ell)$ in the model $(A^{(1)}_{n-1}, Y, \ell)$ there exist a Young diagram representation $X^{(ij)} \in \mathcal{Y}^{(n\,\ell)}_{a^{(ij)}}$ that is uniquely determined by the choice $X^{(00)}$. The state configuration is mapped to a possible assignment in the model $(A^{(1)}_{\ell-1}, {}^tY, n)$ on ${}^t\mathcal{L}$ through ${}^ta^{(ij)} = \hat{D}_{\ell\,n}(T_{n\,\ell}(X^{(ji)})) \in P_+(\ell, n)$. The corresponding edge variables therein are allowed to assume the same number of choices in both models. In the next section we will exhibit several examples of the configuation correspondence.

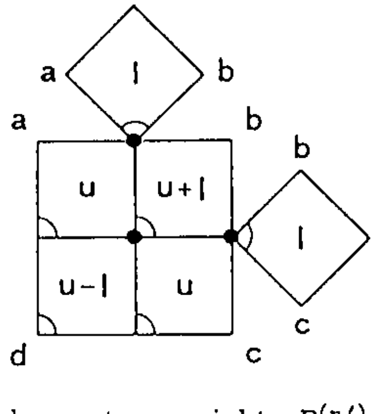
3.2 Statistical Weights.

Here we shall only consider those Y representing the symmetric tensors ($Y = (N)$) and the anti-symmetric tensors ($Y = (1^N)$). For these cases we prove that the statistical weights, if the spectral parameter u is negated, are proportional for the corresponding configurations described in 3.1. Firstly, consider the simplest case $Y = (1)$ (the Young diagram with one node). Let $a, b, c, d \in P_+(n, \ell)$ be the four states such that $N^{(1)}_{ab} = N^{(1)}_{bc} = N^{(1)}_{ad} = N^{(1)}_{dc} = 1$. The Boltzmann weight $B\left(\begin{matrix} ab \\ dc \end{matrix}\middle| u\right)$ of the state configuration round a face has been given in (2.4). As explained in 3.1, one has accordingly the states ${}^t a, {}^t b, {}^t c, {}^t d \in P_+(\ell, n)$ so that $N^{(1)}_{{}^t a {}^t b} = N^{(1)}_{{}^t b {}^t c} = N^{(1)}_{{}^t a {}^t d} = N^{(1)}_{{}^t d {}^t c} = 1$. (Note that ${}^t(1) = (1)$.) Under this correspondence the following relation is valid[8].

$$B^{(n\ell)}\left(\begin{matrix} ab \\ dc \end{matrix}\middle| u\right) = (-)^{1-\delta_{bd}} B^{(\ell n)}\left(\begin{matrix} {}^t a\, {}^t d \\ {}^t b\, {}^t c \end{matrix}\middle| -u\right), \tag{3.9}$$

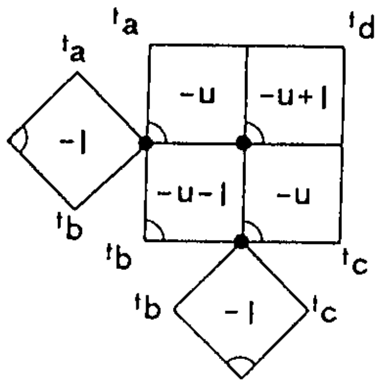
where we have exhibited the rank and the level dependences. The sign factor $(-)^{1-\delta_{bd}}$ here can be absorbed into ϵ in (2.4) and actually irrelevant. This is because the two choices $\epsilon = \pm 1$ lead to the same probabilities for the configurations on $\mathcal{L}$. In this way (3.9) assures the proportionality of the statistical weights for $Y = (1)$.

Let us proceed to the case $Y = (N)$, ${}^t Y = (1^N)$. We compare the statistical weights in the models $(A^{(1)}_{n-1}, (N), \ell)$ and $(A^{(1)}_{\ell-1}, (1^N), n)$. A simplifying feature here is that we still have $\dim(V_Y)_{ab} = \dim(V_{{}^t Y})_{{}^t a {}^t b} \le 1$, hence are left with the unique choice for the edge variables. (See (4.1).) We illustrate the idea of proving the proportionality using the example $Y = (2)$, ${}^t Y = (1^2)$. By the construction in section 2, the Boltzmann weight $W_{(2)}\left(\begin{matrix} ab \\ dc \end{matrix}\middle| u\right)$ for the face configuration $\left(\begin{matrix} ab \\ dc \end{matrix}\right)$ in the model $(A^{(1)}_{n-1}, (2), \ell)$ is proportional to



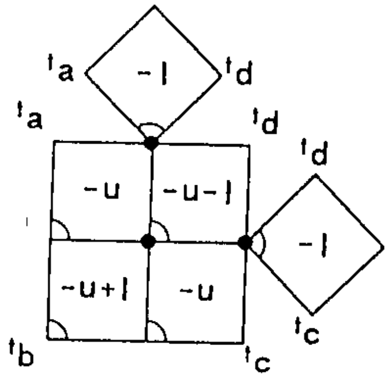
(3.10)

Here each square signifies the elementary weights $B^{(n\ell)}$ (2.4) with the spectral parameter taking the value specified inside. The figure represents the quantity obtained by making the product of these weights and taking the summation afterwards with respect to the site variables marked by the solid circles. By means of (3.9), one can also express (3.10) in terms of $B^{(\ell n)}$'s as



(3.11)

The squares now stand for $B^{(\ell n)}$'s in place of $B^{(n\ell)}$'s. With the aid of the Yang-Baxter equation this can be rearranged into the form



(3.12)

which is proportional to the Boltzmann weight $W_{(1^2)}\left(\begin{matrix} {}^t a & {}^t d \\ {}^t b & {}^t c \end{matrix} \middle| -u\right)$ in the model $(A^{(1)}_{\ell-1}, (1^2), n)$. The case $Y = (N)$ for $N > 2$ can be dealt with in the same manner.

For general Y other than $Y = (N)$ or (1^N), the proportionality of the statistical weights are left as a conjecture.

4. EXAMPLES

Consider a state configuration $\{a^{(ij)}\}_{0\le i\le M, 0\le j\le M'}$, $a^{(ij)} \in P_+(n,\ell)$ on the $M \times M'$ lattice $\mathcal{L}$ in the model $(A^{(1)}_{n-1}, Y, \ell)$. It may be regarded as a collection of the paths of the states $(a^{(j)})_{j=1}^{M+M'}$ going from the NW corner $a^{(1)} = a^{(00)}$ right or downwards to the SE corner $a^{(M+M')} = a^{(MM')}$. In each step one has to satisfy the condition $N^{Y}_{a^{(j)}a^{(j+1)}} \ge 1$. In view of this we shall depict the configuration correspondence in terms of such paths. The examples will include the Young diagrams of signature $(2,1)$ (a hook diagram) as well as $(1),(2)$ and (1^2).

When $Y = (N)$ or (1^N), the fusion rule N^{Y}_{ab} (2.12b) is determined to be 0 or 1 via the following rule.

N^{Y}_{ab} is zero unless $b - a = e_{\lambda_1} + \cdots + e_{\lambda_N}$ for some $0 \le \lambda_1, \ldots, \lambda_N \le n-1$.

If $b - a$ is of this form, we have

$$N^{(N)}_{ab} = \begin{cases} 1 & \text{if } a + e_{\lambda_{\tau(1)}} + \cdots + e_{\lambda_{\tau(j)}} \in P_+(n,\ell) \text{ for all } 1 \le j \le N \\ & \text{and all the permutations } \tau \text{ of } N \text{ letters } 1,\ldots,N, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1a}$$

$$N^{(1^N)}_{ab} = \begin{cases} 1 & \text{if all the } \lambda_i\text{'s are distinct} \\ 0 & \text{otherwise.} \end{cases} \tag{4.1b}$$

These are easily established by combining the Littlewood-Richardson rule and (2.12b) as explained in section 3.1. Note that (4.1) reduces to (2.3) when $N = 1$.

4.1 $(A^{(1)}_1, Y, 3)$ and $(A^{(1)}_2, {}^tY, 2)$.

We label the the states in $P_+(2,3)$ and $P_+(3,2)$ as in Fig. 3. Here the roman

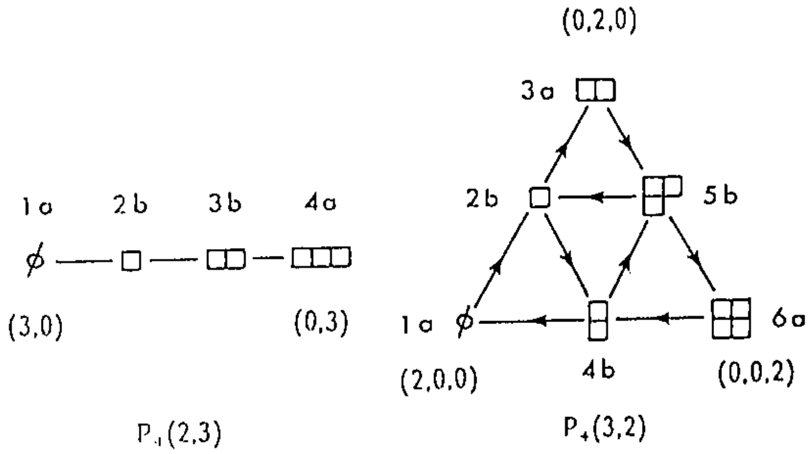
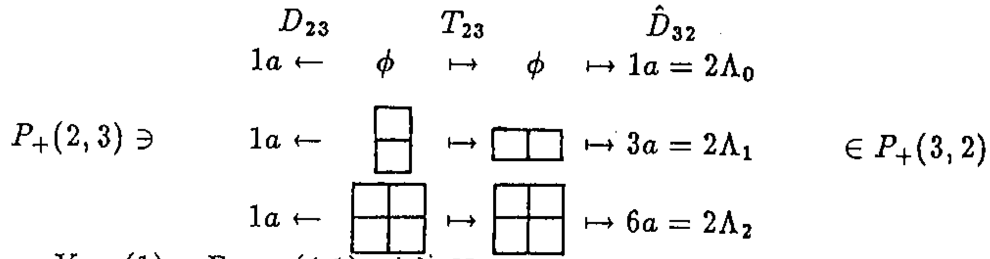
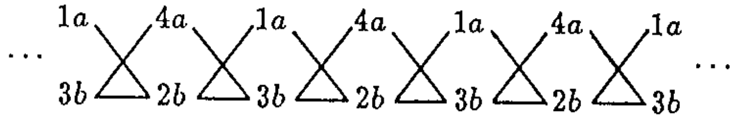


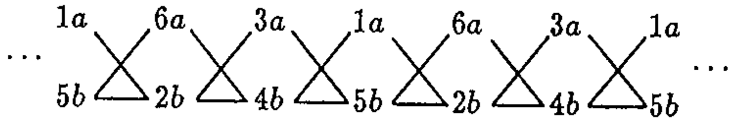
Fig.3 Labelings of the elements in $P_+(2,3)$ and $P_+(3,2)$. In $P_+(2,3)$, the coordinate (3,0) signifies a dominant integral weight $3\Lambda_0 + 0\Lambda_1$, etc. A bond is put between j and $k \in P_+(2,3)$ if and only if $N^{(1)}_{jk} = N^{(1)}_{kj} = 1$. In $P_+(3,2)$, an arrow is put from j to k if and only if $N^{(1)}_{jk} = 1$.

characters a, b, etc signify the $\mathbf{Z}_2-$ (resp. $\mathbf{Z}_3-$) equivalence classes of $P_+(2,3)$ (resp. $P_+(3,2)$) under the automorphism σ (cf. Appendix). Consider an element $1a \in P_+(2,3)$, for example. Depending on the choice of the Young diagram representation in $\mathcal{Y}^{(23)}_{1a}$, there are three states $1a, 3a, 6a \in P_+(3,2)$ that correspond to $1a = 3\Lambda_0 \in P_+(2,3)$ under the scheme (3.6).

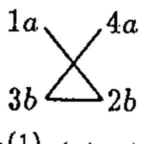


The case $Y = (1)$. From (4.1) with $N = 1$, one has the following paths.

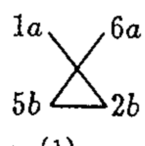
$P_+(2,3)$  (4.2a)

$P_+(3,2)$  (4.2b)

Here the paths grow from left to right following the bonds connecting the states. For example, one has $N^{(1)}_{1a\ 2b} = N^{(1)}_{3b\ 4a} = N^{(1)}_{3b\ 2b} = 1$ but $N^{(1)}_{1a\ 4a} = 0$, etc in (4.2a). Under the scheme (3.6), each path in (4.2a) is mapped to the one in (4.2b) lying in the corresponding position. The paths consisit of the following fundamental patterns.



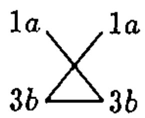
$(A^{(1)}_1, (1), 3)$



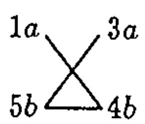
$(A^{(1)}_2, (1), 2)$

The other patterns in (4.2) can be deduced from these by using the symmetry $N^Y_{jk} = N^Y_{\sigma(j)\sigma(k)}$ (see (2.12)). In what follows we shall exclusively exhibit the identical structure of the fundamental patterns as above.

The case $Y = (2)$.



$(A^{(1)}_1, (2), 3)$



$(A^{(1)}_2, (1^2), 2)$

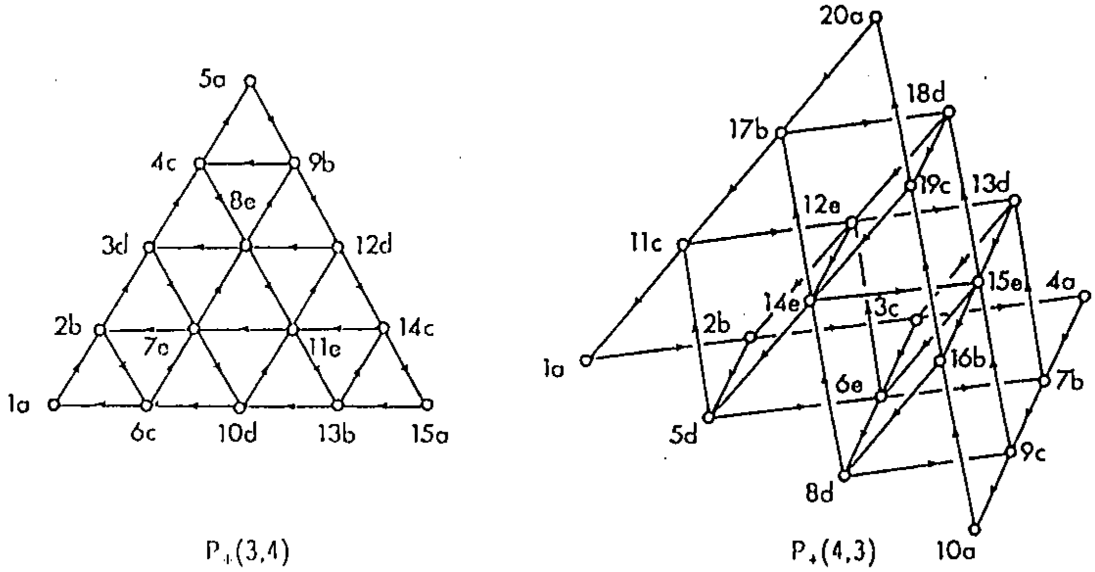
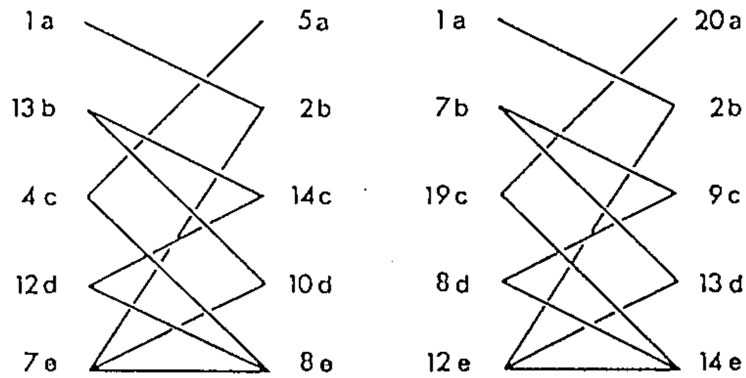


Fig.4 Labelings of the elements in $P_+(3,4)$ and $P_+(4,3)$.

4.2 $(A_2^{(1)}, Y, 4)$ and $(A_3^{(1)}, {}^tY, 3)$.

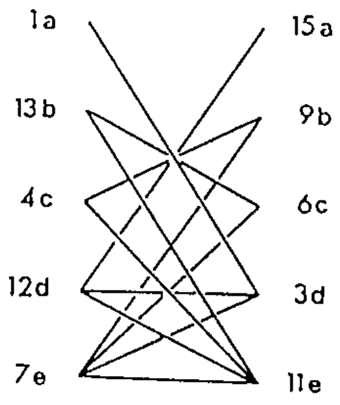
The labeling of the elements in $P(3,4)$ and $P_+(4,3)$ is shown in Fig. 4. Likewise in Fig. 3, the roman characters stand for the $\mathbf{Z}_3-$ and the $\mathbf{Z}_4-$equivalence classes.

The case $Y = (1)$. The fundamental patterns are as follows.

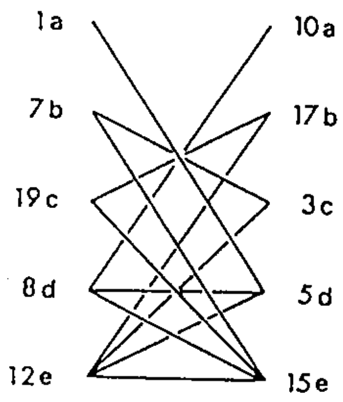


$(A_2^{(1)}, (1), 4)$ $(A_3^{(1)}, (1), 3)$

The case $Y = (2)$.



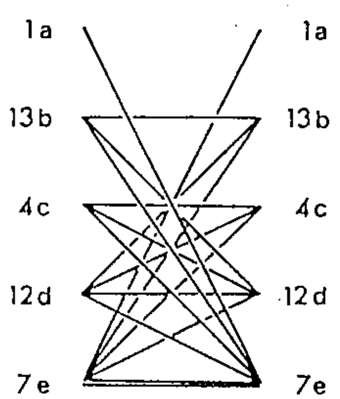
$(A_2^{(1)}, (2), 4)$



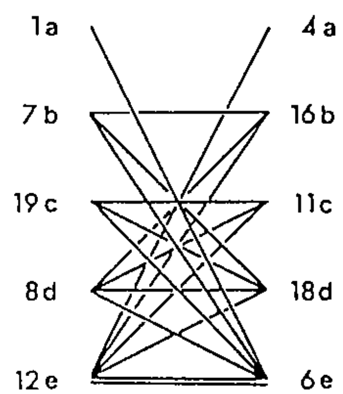
$(A_3^{(1)}, (1^2), 3)$

Finally we present an example including the situation $N_{ab}^{Y} > 1$.

The case $Y = (2, 1)$. The fusion rule is calculated through the method in 3.1. As the result we find the following fundamental patterns.



$(A_2^{(1)}, (2, 1), 4)$



$(A_3^{(1)}, (2, 1), 3)$

In this diagram, double lines represent that the fusion rules in between are equal to 2 (e.g. $N_{7c7c}^{(2,1)} = 2$ for $(n, \ell) = (3, 4)$). Single lines show that they are 1.

5. LOCAL STATE PROBABILITIES

Let us discuss the physical implications of the duality detailed so far. The quantity of our interest here is the 1-point function, which we call the local state probability. By definition, it is the probability of finding a site variable, say $a^{(1)}$, in a preassigned state $a \in P_+(n, \ell)$.

$$P(a) = \lim_{M, M' \to \infty} Z^{-1} \sum_{\text{configurations}} \delta_{a\, a^{(1)}} \prod_{\substack{0 \leq i \leq M \\ 0 \leq j \leq M'}} W_Y \left(\begin{matrix} a^{(ij)} & a^{(i\, j+1)} \\ a^{(i+1\, j)} & a^{(i+1\, j+1)} \end{matrix} \middle| u \right)^{\alpha^{(i\, j+\frac{1}{2})}\, \alpha^{(i+\frac{1}{2}\, j+1)}}_{\alpha^{(i+\frac{1}{2}\, j)}\, \alpha^{(i+1\, j+\frac{1}{2})}}, \tag{5.1}$$

wherein Z is given by (2.2). There are some regimes of distinct physical behaviors depending on the values of the spectral parameter u and the elliptic nome p (see (2.5)). Here we shall exclusively consider

$$\begin{aligned} &\text{Regime II}: && 0 < u < \ell/2, && 0 < p < 1, \\ &\text{Regime III}: && -n/2 < u < 0, && 0 < p < 1. \end{aligned} \tag{5.2}$$

The LSPs are exactly computable by the corner transfer matrix (CTM) method[3] if the Yang-Baxter equation and some additional conditions are satisfied. The results turn out to be the functions of the nome p alone and dependent on u only through the choice of the regimes. A remarkable feature is that the LSPs are neatly described by the coset construction of the Virasoro modules[10] (not necessarily irreducible ones), which we now explain quickly.

Let $\mathcal{G} \supset \mathcal{H}$ be a pair of an affine Lie algebra and its subalgebra such that the highest weight representations of $\mathcal{H}$ are labeled by $a \in P_+(n, \ell)$. We write $\chi^{\mathcal{G}}_{\xi}$, $\chi^{\mathcal{H}}_{a}$ to mean their characters for integrable representations with the specified highest weights. Then the character decomposition describing the $\mathcal{G}/\mathcal{H}$ coset construction has the form

$$\chi^{\mathcal{G}}_{\xi} = \sum_{a} b_{\xi a} \chi^{\mathcal{H}}_{a}, \tag{5.3}$$

where $b_{\xi a}$, the branching coefficient, gives rise to the character of the so-called GKO-Virasoro algebra[10]. Our LSP (5.1) is related to the above construction in that the CTM method leads to the following expression.

$$P(a) = \left.\frac{b_{\xi a}\chi^{\mathcal{H}}_a}{\chi^{\mathcal{G}}_\xi}\right|_{\text{principal specialization}} . \tag{5.4}$$

Note that (5.3) ensures the correct normalization; $\sum_a P(a) = 1$. In (5.4) the argument of the specialized characters is chosen to be a suitable power of the conjugate nome of p. The highest weight ξ of $\mathcal{G}$ is determined by the boundary condition in (5.1). Here we are not going into the details about that matters. So far the following cases have been treated along this line.

models	regime II	regime III	
$(A^{(1)}_1, (N), \ell)$[12]	$\underset{1}{A^{(1)}_{2\ell-1}} \supset \underset{1}{C^{(1)}_\ell}$	$\underset{\ell-N}{A^{(1)}_1} \oplus \underset{N}{A^{(1)}_1} \supset \underset{\ell}{A^{(1)}_1}$	(5.5a)
$(A^{(1)}_{n-1}, (1), \ell)$[8]	$\underset{n-1}{A^{(1)}_{\ell-1}} \oplus \underset{1}{A^{(1)}_{\ell-1}} \supset \underset{n}{A^{(1)}_{\ell-1}}$	$\underset{\ell-1}{A^{(1)}_{n-1}} \oplus \underset{1}{A^{(1)}_{n-1}} \supset \underset{\ell}{A^{(1)}_{n-1}}$	(5.5b)
$(A^{(1)}_{n-1}, (N), \ell)$[2]		$\underset{\ell-N}{A^{(1)}_{n-1}} \oplus \underset{N}{A^{(1)}_{n-1}} \supset \underset{\ell}{A^{(1)}_{n-1}}$	(5.5c)
$(A^{(1)}_{n-1}, (1^N), \ell)$[2]		$\underset{\ell-1}{A^{(1)}_{n-1}} \oplus \underset{1}{A^{(1)}_{n-1}} \supset \underset{\ell}{A^{(1)}_{n-1}}$	(5.5d)

Here we have exhibited the relevant coset pairs $\mathcal{G} \supset \mathcal{H}$ with the levels of their representations. Except for the cases $(A^{(1)}_1, (N), \ell)$ and $(A^{(1)}_{n-1}, (1), \ell)$, the latter two cases (5.5c) and (5.5d) are conjectures emerged from the computer experiments. (Further results concerning the non-unitary cases are also available[11]), where L in (2.5) is replaced by L/t with t coprime to L.)

Now recall the duality of the models $(A^{(1)}_{n-1}, Y, \ell)$ and $(A^{(1)}_{\ell-1}, {}^tY, n)$. They are equivalent to each other if one replaces the spectral parameter u by $-u$. In view of (5.2) this implies the equivalence between regime II (resp. III) of the model $(A^{(1)}_{n-1}, Y, \ell)$ and regime III (resp. II) of $(A^{(1)}_{\ell-1}, {}^tY, n)$. For the simplest case $Y = {}^tY = (1)$, such an equivalence was effectively used to determine the LSPs as in (5.5b)[8].

Notice that in regime II of (5.5a) the coset pair, hence the LSPs are *independent* of the degree N of the fusion. (The dependence only enters in the description of the ground state structures.) This is a curious phenomenon in view of that the calculation with the CTM method becomes highly non-trivial for general N[12]. Our duality for the case $Y = (1)$ provides a simple explanation of this. Namely, it attributes the problem to regime III of the model $(A^{(1)}_{\ell-1}, (1^N), 2)$ (5.5d), where the $N-$independence property is actually enjoyed! In this way the curiousity in regime II of (5.5a) is found to be a simple consequence of (5.5d) through the duality.

Then what is the mechanism responsible for the $N-$independence in (5.5d) ? Here we only point out that the indenpendence property can be restated as a symmetry of the LSPs under σ; the Dynkin diagram automorphism $A^{(1)}_{n-1}$. To explain this, recall that the data $(A^{(1)}_{n-1}, Y, \ell)$ in general refers to a finite dimensional irreducible $sl(n, \mathrm{C})-$module $\mathcal{V}_Y$ (1.1). Denoting Y by $(y_1, \ldots, y_{n-1})$, its highest weight is given by the classical part of $D_{n\ell}(Y)\big|_{\ell=y_1} \in P_+(n, y_1)$ (see (3.2)). In our case of $Y = (1^N)$, this is identical with the $N-$th fundamental weight Λ_N, which is just related to the $N = 1$ case Λ_1 by $\Lambda_N = \sigma^{N-1}(\Lambda_1)$. Thus the invariance of the LSPs under the change of N may be viewed as reflecting the Dynkin diagram symmetry.

Turning to the general case $(A^{(1)}_n, Y, \ell)$, such a feature can also be observed in the following conjecture due to Date, Jimbo, Kuniba, Miwa and Okado[13] for the coset pair describing the LSP in regime III.

$$\underset{k_0}{A^{(1)}_{n-1}} \oplus \underset{k_1}{A^{(1)}_{n-1}} \oplus \cdots \oplus \underset{k_r}{A^{(1)}_{n-1}} \supset \underset{\ell}{A^{(1)}_{n-1}} \tag{5.6}$$

Here the levels $k_0, \ldots, k_r$ $(k_0 + \cdots + k_r = \ell)$ are specified from the diagram Y as follows.

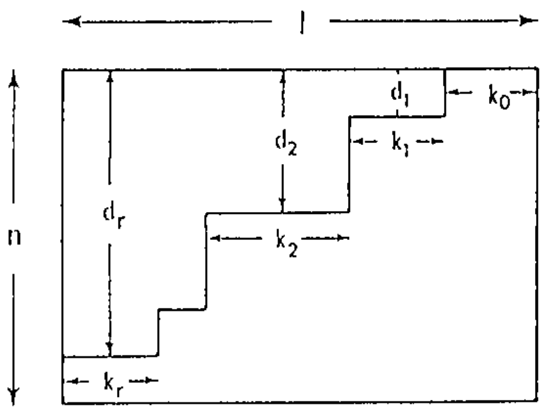


Fig.5 The Young diagram Y of the data $(A^{(1)}_{n-1}, Y, \ell)$.

Note that all the cases of (5.5) for regime III are included in (5.6) as the special case $r = 1$. For some small values of n and ℓ, the conjecture (5.6) has been checked also for the Young diagram $Y = (2,1)$ by the computer experiments[13]. The Dynkin diagram symmetry may be thought to be reflected in the fact that the levels $k_0, \ldots, k_r$ are determined from Y independently of the depths $d_1, \ldots, d_r$. They appear only in labeling the ground states in terms of one dimensional sequence of the states[13]; $a^{(j)} = \sum_{s=0}^{r} \sigma^{j d_s}(\xi_s), \xi_s \in P_+(n, k_s)$. We remark that a similar Dynkin diagram symmetry seems to be valid between the $D_n^{(1)}$ vertex models associated with the vector[14] and the spinor[15] representations.

After completing the paper, the authors were informed of ref.17 where similar concepts to the duality were studied in the Hecke algebra.

APPENDIX PROOF OF (3.7)

Let us prove the duality (3.7) in the fusion rule N_{ab}^{Y}. In order to treat the Young diagram $Y=(y_1,\dots,y_{n-1})$ on the same footing as $a,b\in P_+(n,\ell)$, we hereafter identify it with the element (2.13) in $P_+(n,\ell)$. We exploit Verlinde's formula[5)]

$$N_{ab}^{Y}=\sum_{c\in P_+(n,\ell)}\frac{S_{Yc}S_{ac}S_{bc}^{*}}{S_{\phi c}}, \tag{A.1}$$

where the symbol ϕ denotes the empty Young diagram corresponding to $\ell\Lambda_0$ and $*$ stands for the complex conjugate. See ref. 6 for various aspects of this formula. The $\left(S_{ab}\right)_{a,b\in P_+(n,\ell)}$ is the unitary matrix representing a modular transformation of the $A_{n-1}^{(1)}$ level ℓ characters[16)] $(L=n+\ell)$.

$$\begin{aligned} S_{ab}=S_{ba}&=\frac{i^{n(n-1)/2}}{\sqrt{nL^{n-1}}}\det\left(\zeta^{a_\mu b_\nu}\right)_{0\leq\mu,\nu\leq n-1},\\ &=\frac{i^{n(n-1)/2}}{\sqrt{nL^{n-1}}}\zeta^{-na_{n-1}b_{n-1}}\det\left(\zeta^{a_{\mu\, n-1}b_{\nu\, n-1}}\right)_{0\leq\mu,\nu\leq n-1},\\ \zeta&=e^{-2\pi i/L}. \end{aligned} \tag{A.2}$$

Here a_μ's and b_μ's are the coordinates uniquely specified from a and b by (2.1) and the extra constraints $\sum_{0\leq\mu\leq n-1}a_\mu=\sum_{0\leq\mu\leq n-1}b_\mu=0$. Under the cyclic Dynkin diagram automorphism $\sigma(\Lambda_\mu)=\Lambda_{\mu+1}$, the S_{ab} transforms as follows.

$$S_{\sigma(a)b}=(-)^{n-1}e^{-2\pi i b_0}S_{ab}. \tag{A.3}$$

As a result, the fusion rule N_{ab}^{Y} enjoys the following symmetry.

$$N_{ab}^{Y}=N_{\sigma(a)\sigma(b)}^{Y}=N_{a\sigma(b)}^{\sigma(Y)}. \tag{A.4}$$

We shall call two elements $a,b\in P_+(n,\ell)$ $\mathbf{Z}_n$*-equivalent* if $b=\sigma^k(a)$ for some $0\leq k\leq n-1$. Accordingly, (A.1) is rewritten as follows.

$$N_{ab}^{Y}=\sum_{c\in P_+(n,\ell)/\mathbf{Z}_n}\frac{1}{m_n(c)}\sum_{0\leq k\leq n-1}\frac{S_{Y\sigma^k(c)}S_{a\sigma^k(c)}S_{b\sigma^k(c)}^{*}}{S_{\phi\sigma^k(c)}}, \tag{A.5}$$

where the outer sum extends over $\mathbf{Z}_n$-equivalence classes and $m_n(c)$ is the multiplicity of c occuring in the σ-orbit $c,\sigma(c),\dots,\sigma^{n-1}(c)$. Eliminating σ^k by (A.3) we get an extra phase factor $e^{-2\pi i k(a_0+Y_0-b_0-\phi_0)}$. Under the assumption $N_{ab}^{Y} > 0$, this is equal to 1 since we have $a_0 \in \mathbf{Z}+\phi_0 - \frac{1}{n}|X|, X \in \mathcal{Y}_a^{(n\ell)}, Y_0 \in \mathbf{Z}+\phi_0 - \frac{1}{n}|Y|, b_0 \in \mathbf{Z}+\phi_0 - \frac{1}{n}|Z|, Z \in \mathcal{Y}_a^{(n\ell)}$ and $|Z| = |X|+|Y|$ (cf. *Step 1* and (A.10)). Thus (A.5) reduces to

$$N_{ab}^{Y} = \sum_{c\in P_+(n,\ell)/\mathbf{Z}_n} \frac{n}{m_n(c)} \frac{S_{Yc}S_{ac}S_{bc}^*}{S_{\phi c}}. \tag{A.6}$$

The following fact simplifies the proof of (3.7).

Lemma 1. *There is a one to one correspondence between the $\mathbf{Z}_n-$equivalence classes $P_+(n,\ell)/\mathbf{Z}_n$ and the $\mathbf{Z}_\ell-$equivalence classes $P_+(\ell,n)/\mathbf{Z}_\ell$. Suppose that $c \in P_+(n,\ell)$ and ${}^t c \in P_+(\ell,n)$ belong to the corresponding classes. Then we have $m_n(c) = m_\ell({}^t c)$.*

Proof. To see the former part, recall the scheme (3.6). With each $a \in P_+(n,\ell)$, there is associated a subset $\hat{D}_{\ell n}(T_{n\ell}(\mathcal{Y}_a^{(n\ell)}))$, which forms a $\mathbf{Z}_\ell$-equivalence class in $P_+(\ell,n)$. Varying a, these subsets cover $P_+(\ell,n)$ while $\sigma^k(a)$'s for $0 \le k \le n-1$ are mapped into the same $\mathbf{Z}_\ell-$equivalence class. Hence the assertion holds. To verify the latter part, assume that $m_n(c) = s$ for some positive integer s. Then n is divisible by s and $c = c^0\Lambda_0 + \cdots + c^{n-1}\Lambda_{n-1}$ takes the form $(c^0,\ldots,c^{n-1}) = (\underbrace{\alpha_1,\ldots,\alpha_r}_{1-\mathrm{st}},\ldots,\underbrace{\alpha_1,\ldots,\alpha_r}_{s-\mathrm{th}})$. Here $\alpha_i \in \mathbf{Z}_{\ge 0}, s = n/r$ and r is a divisor of n. Take a Young diagram representation $C \in \mathcal{Y}_c^{(n\ell)}$ with the depth less than n. See Fig. 6. By the transposition we see that $m_\ell({}^t c) = s$.

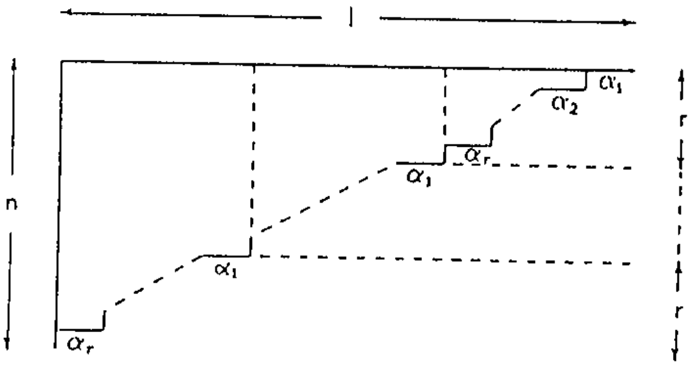


Fig.6 The Young diagram C representing $D_{n\ell}(C) = c^0\Lambda_0 + \cdots + c^{n-1}\Lambda_{n-1} \in P_+(n,\ell)$ with the coefficients having the form $(c^0,\ldots,c^{n-1}) = (\underbrace{\alpha_1,\ldots,\alpha_r}_{1-\mathrm{st}},\ldots,\underbrace{\alpha_1,\ldots,\alpha_r}_{s-\mathrm{th}})$.

In the sequel we shall show

Theorem. *Let c and ${}^tc$ be as in Lemma 1. For a, b such that $N_{ab}^{Y} > 0$ and ${}^ta, {}^tb$ as described in (3.7), we have*

$$n\frac{S_{Yc}^{(n\ell)}S_{ac}^{(n\ell)}S_{bc}^{(n\ell)^*}}{S_{\phi c}^{(n\ell)}} = \ell\frac{S_{{}^tY{}^tc}^{(\ell n)^*}S_{{}^ta{}^tc}^{(\ell n)^*}S_{{}^tb{}^tc}^{(\ell n)}}{S_{{}^t\phi{}^tc}^{(\ell n)^*}}. \tag{A.7}$$

Here we have exhibited the opposite level-rank dependences on the both sides. By virtue of (A.6), Lemma 1 and the fact that N_{ab}^{Y} is real, the duality (3.7) follows from Theorem. To verify (A.7) we need a few Lemmas. The first one is

Lemma 2. *((1.7) of ref. 9.) Let $X = (x_1, x_2, \ldots)$ be a Young diagram, ${}^tX = ({}^tx_1, {}^tx_2, \ldots)$ be its transposition and $x_1 \le \ell$ and ${}^tx_1 \le n$. Then the $n+\ell$ numbers $x_\mu + n - \mu\,(1 \le \mu \le n)$, $n - 1 + \nu - {}^tx_\nu\,(1 \le \nu \le \ell)$ are a permutation of $\{0, 1, \ldots, n+\ell-1\}$.*

Proof. Consider an n by ℓ rectangle containing X. Starting from the bottom left, we assign the numbers $0, 1, \ldots, n+\ell-1$ to each edge on the boundary line separating X and its complement. See Fig. 7. Those attached to the vertical edges are $x_\mu + n - \mu\,(1 \le \mu \le n)$. On the other hand the numbers appearing on the horizontal edges are by transposition

$$(n+\ell-1) - ({}^tx_\nu + \ell - \nu) = n - 1 + \nu - {}^tx_\nu,$$

for $1 \le \nu \le \ell$. Hence follows the Lemma.

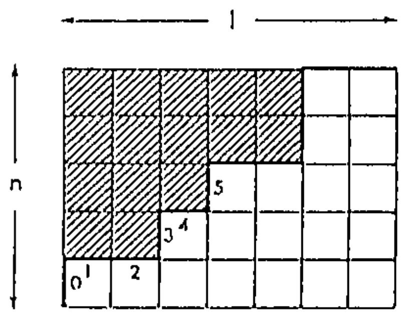


Fig.7 The n by ℓ rectangle. The Young diagram X contained therein is hatched. $n = 5, \ell = 7$ and $X = (5, 5, 3, 2)$.

Lemma 3. *Let ζ be as in (A.2) and define a symmetric L by L matrix Ω by*

$$\Omega = \frac{1}{\sqrt{L}} e^{-\pi i(L-1)(L+2)/4L} (\zeta^{\mu\nu})_{0\leq\mu,\nu\leq L-1}. \tag{A.8}$$

Then we have $\det \Omega = 1$ and $\Omega^{-1} = \Omega^*$.

Proof. The latter assertion is obvious. The determinant is of Vandermonde's type and is rewritten as

$$\det \Omega = \sqrt{\frac{2^{L(L-1)}}{L^L}} \prod_{0\leq\mu<\nu\leq L-1} \sin \frac{\pi(\nu-\mu)}{L}.$$

This is shown to be 1 by considering the involusion property of a modular transformation for the denomenator of the $A^{(1)}_{L-1}$ characters.

There is a simple relation between the matrices $S^{(n,\ell)}$ and $S^{(\ell,n)^*}$.

Lemma 4. *For $a, b \in P_+(n,\ell)$ let $X = (x_1, \ldots, x_n) \in \mathcal{Y}^{(n\ell)}_a$ and $Z = (z_1, \ldots, z_n)$ $\in \mathcal{Y}^{(n\ell)}_b$ be the Young diagram representations with $x_1 \leq \ell$ and $z_1 \leq \ell$. Define ${}^ta, {}^tb \in P_+(\ell, n)$ by ${}^ta = \hat{D}_{\ell n}(T_{n\ell}(X)), {}^tb = \hat{D}_{\ell n}(T_{n\ell}(Z))$. Then we have*

$$\frac{S^{(n\ell)}_{ab}}{S^{(\ell n)^*}_{{}^ta{}^tb}} = \sqrt{\frac{\ell}{n}}\, e^{\frac{2\pi i}{\ell n}|X||Z|}. \tag{A.9}$$

Proof. By the definition, the coordinates a_μ's of a obeying $\sum_{\mu=0}^{n-1} a_\mu = 0$ are given as

$$a_\mu = x_{\mu+1} - \mu - \frac{1}{n}|X| + \frac{n-1}{2}, \quad 0 \leq \mu \leq n-1. \tag{A.10a}$$

See (2.1) and (3.2a). Similarly, denoting the transposition $T_{n\ell}(X)$ by $({}^tx_1, {}^tx_2, \ldots)$, we have

$${}^ta_\nu = {}^tx_{\nu+1} - \nu - \frac{1}{\ell}|X| + \frac{\ell-1}{2}, \quad 0 \leq \nu \leq \ell-1. \tag{A.10b}$$

Define the sets I and $\overline{I}$ of integers by

$$I = \{a_{\mu\, n-1} + x_n \mid 0 \leq \mu \leq n-1\}, \tag{A.11a}$$

$$\overline{I} = \{L - 1 - {}^tx_\ell - {}^ta_{\nu\,\ell-1} \mid 0 \leq \nu \leq \ell-1\}. \tag{A.11b}$$

Using Lemma 2 and (A.10) we see that I and $\overline{I}$ are disjoint and $I \cup \overline{I} = \{0, 1, \ldots, L-1\}$. Replacement of X by Z yields $b_\mu\ (0 \leq \mu \leq n-1), {}^tb_\nu\ (0 \leq \nu \leq \ell-1)$ in the same way as in (A.10) and also the following sets J and $\overline{J}$ satisfying $J \cup \overline{J} = \{0, 1, \ldots, L-1\}$.

$$J = \{b_{\mu\, n-1} + z_n \mid 0 \le \mu \le n-1\}, \tag{A.12a}$$

$$\overline{J} = \{L - 1 - {}^t z_\ell - {}^t b_{\nu\, \ell-1} \mid 0 \le \nu \le \ell - 1\}. \tag{A.12b}$$

Now consider the following identity for the symmetric unimodular matrix Ω in Lemma 3.

$$\det\, (\Omega)_{I,J} = (-)^{|X|+|Z|} \det\, (\Omega^{-1})_{\overline{I},\overline{J}}. \tag{A.13}$$

Here the left hand side, for example, stands for the minor of Ω with the row indices I (A.11a) and the column indices J (A.12a). Combining (A.2),(A.8),(A.11a) and (A.12a) we get

$$\begin{aligned}\det\, (\Omega)_{I,J} &= e^{-\frac{n\pi i}{4L}(L-1)(L+2)} L^{\frac{-n}{2}} \det\, (\zeta^{(a_{\mu n-1}+x_n)(b_{\nu n-1}+z_n)})_{0\le\mu,\nu\le n-1} \\ &= \sqrt{\frac{n}{L}}\, i^{-\frac{n(n-1)}{2}} e^{-\frac{n\pi i}{4L}(L-1)(L+2)} \zeta^{n(\frac{1}{n}|X|+\frac{n-1}{2})(\frac{1}{n}|Z|+\frac{n-1}{2})} S^{(n\ell)}_{ab}.\end{aligned} \tag{A.14a}$$

A similar calculation using (A.11b) and (A.12b) leads to

$$\begin{aligned}&\det\, (\Omega^{-1})_{\overline{I},\overline{J}} \\ &= \sqrt{\frac{\ell}{L}}\, i^{\frac{\ell(\ell-1)}{2}} e^{\frac{\ell\pi i}{4L}(L-1)(L+2)} \zeta^{-\ell(\frac{1}{\ell}|X|+\frac{\ell-1}{2}-L+1)(\frac{1}{\ell}|Z|+\frac{\ell-1}{2}-L+1)} S^{(\ell n)*}_{{}^t a\, {}^t b}.\end{aligned} \tag{A.14b}$$

Substituting (A.14) into (A.13) we find that all the phase factors cancell except $e^{\frac{2\pi i}{\ell n}|X||Z|}$ and thus obtain (A.9).

Proof of Theorem. With no loss of generality we assume that $C \in \mathcal{Y}^{(n\ell)}_c$ is of width $\le \ell$ and set ${}^t c = \hat{D}_{\ell\, n}(T_{n\, \ell}(C))$. Consider $a, b \in P_+(n, \ell)$ and their Young diagram representations $X = (x_1, \ldots, x_n) \in \mathcal{Y}^{(n\ell)}_a$, $Z = (z_1, \ldots, z_n) \in \mathcal{Y}^{(n\ell)}_b$. Due to the assumption $N^Y_{ab} > 0$, Z is uniquely specified by the requirement $|Z| = |X| + |Y|$ (cf. *Step 1*). According to the scheme (3.6) we have then ${}^t a = \hat{D}_{\ell\, n}(T_{n\, \ell}(X)), {}^t b = \hat{D}_{\ell\, n}(T_{n\, \ell}(Z))$. Let us consider the X as consisting of an n by x_n rectangle diagram $(x_n, \ldots, x_n)$ and the rest $X_R = (x_1 - x_n, \ldots, x_{n-1} - x_n, 0) \in \mathcal{Y}^{(n\ell)}_a$. Similarly, we divide the Z into $(z_n, \ldots, z_n)$ and $Z_R = (z_1 - z_n, \ldots, z_{n-1} - z_n, 0) \in \mathcal{Y}^{(n\ell)}_b$. Set

${}^t a_R = \hat{D}_{\ell n}(T_{n\ell}(X_R))$ and ${}^t b_R = \hat{D}_{\ell n}(T_{n\ell}(Z_R))$. Then one can deduce from (3.6) that

$$ {}^t a = \sigma^{x_n}({}^t a_R), \quad {}^t b = \sigma^{z_n}({}^t b_R). \tag{A.15} $$

Since both X_R and Z_R have the width not greater than ℓ, Lemma 4 evaluates $S^{(n\ell)}_{ac}/S^{(\ell n)*}_{{}^t a_R {}^t c}$ and $S^{(n\ell)*}_{bc}/S^{(\ell n)}_{{}^t b_R {}^t c}$. Combining the results with (A.3) and (A.15) we obtain

$$ \begin{aligned} \frac{S^{(n\ell)}_{ac}}{S^{(\ell n)*}_{{}^t a {}^t c}} &= \sqrt{\frac{\ell}{n}}\,(-)^{(\ell-1)x_n} e^{\frac{2\pi i}{\ell n}|X_R||C| - 2\pi i {}^t c_0 x_n}, \\ \frac{S^{(n\ell)*}_{bc}}{S^{(\ell n)}_{{}^t b {}^t c}} &= \sqrt{\frac{\ell}{n}}\,(-)^{(\ell-1)z_n} e^{-\frac{2\pi i}{\ell n}|Z_R||C| + 2\pi i {}^t c_0 z_n}. \end{aligned} \tag{A.16} $$

Using (A.9) and (A.16) we now simplify the ratio of the left and the right hand sides of (A.7) as follows.

$$ (-)^{(\ell-1)(x_n - z_n)} e^{-2\pi i\, {}^t c_0 (x_n - z_n)} e^{\frac{2\pi i}{\ell n}|C|(|X_R|+|Y|-|Z_R|)}. \tag{A.17} $$

Recall that from (A.10) and (A.12) one has ${}^t c_0 \in \mathbf{Z} + {}^t\phi_0 - \frac{1}{\ell}|C|$, ${}^t\phi_0 = \frac{\ell-1}{2}$, $|X| = |X_R| + nx_n$, $|Z| = |Z_R| + nz_n$ and $|Z| = |X| + |Y|$. Thus we conclude that the ratio (A.17) is equal to 1. This completes the proof of (A.7) and therefore (3.7).

Remark. The duality (3.7) has the *classical version* $\overline{N}^{Y}_{ab} = \overline{N}^{{}^t Y}_{{}^t a {}^t b}$ for n and ℓ sufficiently large. It can be verified by using the Jacobi-Trudi identity for the Schur polynomials (cf. ref. 9).

ACKNOWLEDGEMENTS

The authors thank Prof. S. Tanaka for his kind interest and generous helps at Kyushu university. Thanks are also due to Profs. T.Eguchi, T.Hayashi, M. Okado, V. Pasquier, J. Suzuki, I. Terada and A.Tsuchiya for useful discussions. One of the authors (A.K.) thanks Profs. E. Date, M. Jimbo and T. Miwa for their supports in attending the conference. He is also grateful to Profs. S. Das, A. Dhar, S. Mukhi, A. Raina, A. Sen and the members of the theoretical physics group at Tata Institute of Fundamental Research for their hospitality.

REFERENCES

1. *Integrable Systems in Quantum Field Theory and Statistical Mechanics*, M. Jimbo, T. Miwa and A. Tsuchiya eds., Adv. Stud. Pure Math **19**. Kinokuniya, Tokyo, (1989).
2. M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Commun. Math. Phys., **119**, 543, (1988).
3. R. J. Baxter, *Exactly solved models in statistical mechanics*, Academic, London, (1982).
4. P. P. Kulish, N. Yu. Reshetikhin and E. K. Sklyanin, Lett. Math. Phys. **5**, 309, (1981).
5. E. Verlinde, Nucl. Phys., **B300**, 360, (1988).
6. G. Moore and N. Seiberg, Commun. Math. Phys. **123**, 177, (1989).

   L. Alvarez-Gaumé, G. Sierra and C. Gomez in *Physics and Mathematics of Strings*, L. Brink, D. Friedan and A. M. Polyakov eds., World Scientific, Sigapore, (1990).

   M. A. Walton, "Algorithm for WZW fusion rules: A Proof", Laval University, Canada, preprint (1990).
7. I. B. Frenkel, Lecture Notes in Math. **933**, 71 Springer (1982).

   M. Jimbo and T. Miwa, Adv. Stud. Pure Math. **6**, 17, (1985).
8. M. Jimbo, T. Miwa and M. Okado, Lett. Math. Phys., **14**, 123, (1987).
9. I. G. Macdonald, *Symmetric functions and Hall Polynomials*, Oxford University Press (1979).
10. P. Goddard, A. Kent and D. Olive, Phys. Lett. **B152**, 88, (1985).
11. T. Nakanishi, "Non-unitary minimal models and RSOS models", Nucl. Phys. in press.

    A. Kuniba, T. Nakanishi and J. Suzuki, "Ferro- and Antiferro- Magnetizations in RSOS Models", preprint.
12. E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Nucl. Phys. **B290**, 231, (1987), Adv. Stud. Pure Math. **16**, 17, (1988).
13. E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Talk at Taniguchi conference on *Integrable Systems in Quantum Field Theory and Statistical*

Mechanics, Kyoto (1988), unpublished.

14. E. Date, M. Jimbo, A. Kuniba, T. Miwa and M. Okado, Lett. Math. Phys. **17**, 69, (1989).
15. M. Okado, "Quantum R matrices related to the spin representations of B_n and D_n", RIMS-677, Kyoto University preprint (1990).
16. V. G. Kac and D. H. Peterson, Adv. Math., **53**, 125, (1984).
17. F.M.Goodman and H.Wenzl, "Littlewood Richardson Coefficients for Hecke algebras at Roots of Unity", preprint.