Cluster Algebras
and
Scattering Diagrams

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Preface

As the title suggests, the theme of this monograph is the relation between cluster algebras and scattering diagrams. The former was introduced by Fomin and Zelevinsky as an algebraic and combinatorial structure originated in Lie theory, while the latter was introduced by Kontsevich and Soibelman, and also by Gross and Siebert in the study of homological mirror symmetry.

The text consists of three parts.

- Part I is a first step guide to the theory of cluster algebras for readers without any knowledge on cluster algebras. We especially focus on basic notions, techniques, and results concerning seeds, cluster patterns, and cluster algebras.
- Part II is considered as the main part of the monograph, where we focus on the column sign-coherence of C-matrices and the Laurent positivity for cluster patterns, both of which were conjectured by Fomin and Zelevinsky and proved by Gross, Hacking, Keel, and Kontsevich based on the scattering diagram method. We also give a detailed account of the correspondence between the notions of cluster patterns and scattering diagrams.
- Part III is a self-contained exposition of several fundamental properties of cluster scattering diagrams which are admitted and used in Part II without proof. In particular, detailed proofs are presented for the construction, the mutation invariance, and the positivity of theta functions of cluster scattering diagrams with emphasis on the roles of the dilogarithm elements and the pentagon relation. This is regarded as a supplement to Part II, but also it may be read as an introductory text to cluster scattering diagrams.

More detailed introductions will be found in each part.

As a specific feature of this monograph, each part is written without explicitly relying on the other parts. Thus, readers can start reading from any part depending on their interest and knowledge as suggested below.

- If the reader is new to cluster algebras, simply start from Part I.
- If the reader is already familiar with basic notions in cluster algebras, skip Part I and start from Part II or III depending on the reader's interest.

Both cluster algebras and scattering diagrams are still young subjects. Therefore, it is likely that their notions and formulations would be drastically altered in near future. In fact, the proofs of the sign-coherence and the Laurent positivity in Part II suggest that a cluster pattern and a cluster scattering diagram are regarded as one inseparable object, so that they will be eventually integrated. On the other hand, the usefulness of the current formulation of cluster algebras has been already established in various applications; therefore, the current formulation should also remain to be used especially in practical applications to various subjects due to its efficiency and simplicity. It is my hope that this monograph serves as a useful guide in both perspectives.

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The errata and update will be posted on arXiv and the following webpage:
`http://www.math.nagoya-u.ac.jp/~nakanisi/index.html`

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