Exact WKB analysis and cluster algebras

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Based on joint work with Kohei Iwaki (RIMS), arXiv:1401.7074, 98 pages.

The pdf file of this slide (or updated one) will be available at my web site.

exact WKB analysis and cluster algebras

WKB approximation

Wentzel, Kramers, Brilloin (1926)

semiclassical approximation method for Schrödinger equation

exact WKB analysis (80 $\sim$)

$\fallingdotseq$ study of WKB solution of 1d (complex) Schrödinger equation
by Borel resummation

Voros (83)

Aoki-Kawai-Takei (91)

Dellabaere-Dillinger-Pham (DDP) (93)

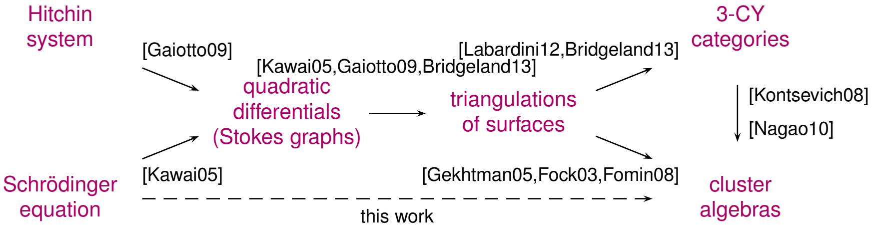
cluster algebras

Fomin-Zelevinsky (00 $\sim$)

combinatorial structure in representation theory in several contexts

appearing in several areas in mathematics

Overview & Keywords



“pentagon relation" [DDP93]

$$\mathfrak{S}_{\gamma_1}\mathfrak{S}_{\gamma_2} = \mathfrak{S}_{\gamma_2}\mathfrak{S}_{\gamma_1+\gamma_2}\mathfrak{S}_{\gamma_1}\,, \qquad \mathfrak{S}_{\gamma}\text{: Stokes automorphism for cycle } \gamma$$

“There is a striking similarity between our [their] wall-crossing formula and identities for the Stokes automorphisms in the theory of WKB asymptotics..." [Kontsevich-Soibelman08]

$$\begin{cases} \text{Voros symbols (2)} \\ \text{Stokes phenomenon} \end{cases} + \begin{cases} \text{Stokes graph (1)} \\ \text{flip} \end{cases} \longrightarrow \begin{cases} \text{seed} \begin{cases} \text{quiver (1)} \\ \text{cluster variables (2)} \end{cases} \\ \text{mutation} \end{cases}$$

WKB solutions (1)

T. Kawai and Y. Takei, *Algebraic analysis of singular perturbation theory*, AMS, 2005.

Schrödinger equation

$$\left(\frac{d^2}{dz^2} - \eta^2 Q(z,\eta)\right)\psi(z,\eta) = 0$$

z: complex (local) coordinate, $\eta = \hbar^{-1}$: large parameter

$$\psi(z,\eta) = \exp\left(\int^z S(z,\eta)dz\right)$$

$$\frac{dS}{dz} + S^2 = \eta^2 Q \quad \text{(Riccati equation)}$$

$$\begin{cases} Q(z,\eta) = Q_0(z) + \eta^{-1}Q_1(z) + \cdots \\ S(z,\eta) = \eta S_{-1}(z) + S_0(z) + \cdots \end{cases}$$

$$S_{-1}^2 = Q_0, \quad \frac{dS_{-1}}{dz} + 2S_{-1}S_0 = Q_1, \quad \ldots$$

$$\begin{aligned} S_{\pm}(z,\eta) &= \pm\eta\sqrt{Q_0(z)} + \cdots \\ &= S_{\text{even}}(z,\eta) \pm S_{\text{odd}}(z,\eta) \end{aligned}$$

$$S_{\text{odd}}(z,\eta) = \eta\sqrt{Q_0(z)} + \cdots$$

$$S_{\text{even}}(z,\eta) = -\frac{1}{2}\frac{d}{dz}\log S_{\text{odd}}(z,\eta)$$

WKB solutions (2)

(in previous page)

$$\psi(z,\eta) = \exp\left(\int^z S(z,\eta)dz\right)$$

$$\begin{aligned} S_{\pm}(z,\eta) &= \pm\eta\sqrt{Q_0(z)} + \cdots \\ &= S_{\text{even}}(z,\eta) \pm S_{\text{odd}}(z,\eta) \end{aligned}$$

$$\begin{aligned} S_{\text{odd}}(z,\eta) &= \eta\sqrt{Q_0(z)} + \cdots \\ S_{\text{even}}(z,\eta) &= -\frac{1}{2}\frac{d}{dz}\log S_{\text{odd}}(z,\eta) \end{aligned}$$

Hence

$$\begin{aligned} \psi_{\pm}(z,\eta) &= \frac{1}{\sqrt{S_{\text{odd}}(z,\eta)}}\exp\left(\pm\int^z S_{\text{odd}}(z,\eta)dz\right) \quad \text{WKB solutions} \\ &= \frac{1}{\sqrt{\eta\sqrt{Q_0(z)}}}\exp\left(\pm\int^z \eta\sqrt{Q_0(z)}dz\right)(1+O(\eta^{-1})) \end{aligned}$$

WKB approximation divergent series!

The exact WKB analysis manages this divergent series by Borel resummation.

Borel resummation

$$f(\eta) = \sum_{n=0}^{\infty} f_n \eta^{-n} \quad \text{(possibly divergent) formal series}$$

$$f_B(y) = \sum_{n=1}^{\infty} \frac{f_n}{(n-1)!} y^{n-1} \quad \text{Borel transform of } f$$

$$\mathcal{S}[f](\eta) = f_0 + \int_0^{\infty} e^{-\eta y} f_B(y) dy \quad \text{Borel sum of } f$$

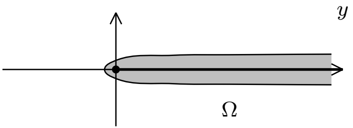
(not necessarily convergent)

Example.
(1) $f(\eta) = \eta^{-n} \quad \Longrightarrow \quad \mathcal{S}[f](\eta) = \eta^{-n}$
(2) $f(\eta)$: a convergent series of $\eta^{-1} \quad \Longrightarrow \quad \mathcal{S}[f](\eta) = f(\eta)$ near $\eta = \infty$.

Borel summability

Definition. A formal series $f(\eta) = \sum_{n=0}^{\infty} f_n \eta^{-n}$ is Borel summable if

- $f_B(y)$ is a convergent series of y.
- $f_B(y)$ is analytically continued in the domain Ω.
- $|f_B(y)| \le c_1 e^{c_2|y|}$ for some $c_1, c_2 > 0$.



Example. The following $f(\eta)$ is divergent, but Borel summable.

$$f(\eta) = \sum_{n=0}^{\infty} (-1)^{n-1}(n-1)!\eta^{-n}$$

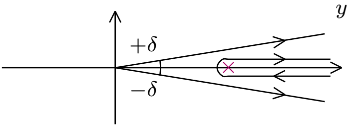
$$f_B(y) = \sum_{n=1}^{\infty} (-1)^{n-1} y^{n-1} \quad \left(= \frac{1}{1+y} \quad \text{for } |y| < 1 \right)$$

Theorem. (e.g. [Costin08]) If $f(\eta)$ is Borel summable, then the Borel sum $\mathcal{S}[f](\eta)$ converges near $\eta = \infty$; moreover, it is asymptotically expanded to $f(\eta)$,

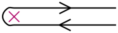
$$\mathcal{S}[f](\eta) \sim f(\eta) \quad (\eta \to \infty).$$

Stokes phenomenon

Formulation of Stokes phenomenon by Borel resummation



$\times$: isolated singularity of $f_B(y)$
($f(\eta)$ is not Borel summable)

analytic function $\mathcal{S}_{+\delta}[f] = \mathcal{S}_{-\delta}[f] +$ 

asymptotic expansion $f \quad f + g$

jump
(Stokes phenomenon)

This method will be applied to the WKB solutions of the Schrödinger equation

$$\left(\frac{d^2}{dz^2} - \eta^2 Q(z,\eta)\right)\psi(z,\eta) = 0,$$

$$\psi_{\pm}(z,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}(z,\eta)}} \exp\left(\pm \int^z S_{\mathrm{odd}}(z,\eta)dz\right).$$

S^1-action on $y \implies S^1$-action on $\eta \implies S^1$-action on z, or $Q(z,\eta)$
(various viewpoints for Stokes phenomenon)

Trajectories and Stokes curves

Schrödinger equation on a compact Riemann surface Σ

$$\left(\frac{d^2}{dz^2} - \eta^2 Q(z,\eta)\right)\psi(z,\eta) = 0$$

In this section, concentrate on the classic situation

$$\Sigma = \text{Riemann sphere}, \quad Q(z,\eta) = Q_0(z) \text{ polynomial in } z.$$

Assume that $Q_0(z)$ has only simple zeros.

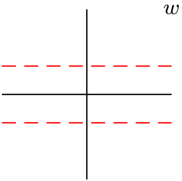
$Q_0(z)$ determines a foliation in Σ:

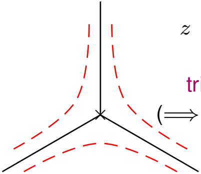
leaf (trajectory) $\quad \mathrm{Im} \int_a^z \sqrt{Q_0(z)}dz = \mathrm{const} \quad$ (a: zero of $Q_0(z)$)

Stokes curve $\quad \mathrm{Im} \int_a^z \sqrt{Q_0(z)}dz = 0 \quad$ (a: zero of $Q_0(z)$)

Around a simple zero $\times$

$$\int_0^z \sqrt{z}dz = \frac{2}{3}z^{3/2} = w$$

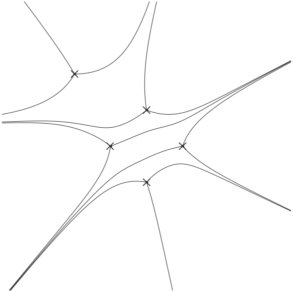




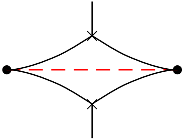
trivalency
($\Longrightarrow$ triangulation)

Stokes graph

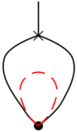
Stokes graph: graph on Σ with vertices = zeros & pole (∞), edges = Stokes curves
Stokes region: domain on Σ surrounded by Stokes curves
Example: $Q_0(z) = i(z-1)(z+1)(z-2i+2)(z-i)(z+i)$



Assume that the Stokes graph is saddle-free (i.e., without saddle trajectory).
Then, Stokes regions fall into the following two classes [Strebel84].



(regular) horizontal strip

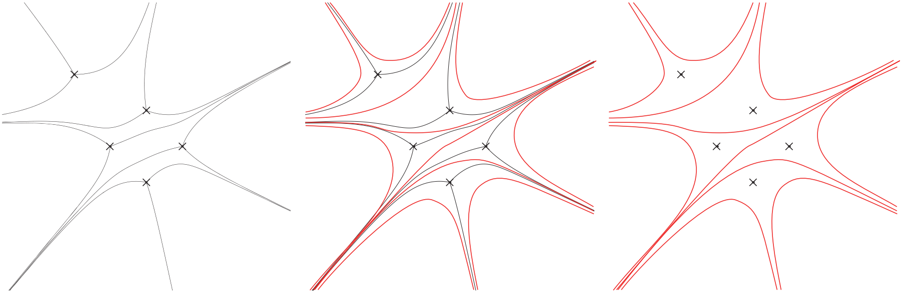


half plane

From Stokes graph to triangulation

To each saddle-free Stokes graph, one can associate a triangulation of a polygon.

Example: $Q_0(z) = i(z-1)(z+1)(z-2i+2)(z-i)(z+i)$



Stokes graph	triangulation
horizontal strip	(internal) arc
half plane	(boundary) edge
simple zero	triangle

Voros' connection formula

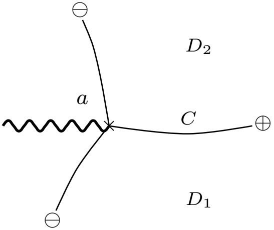
Theorem. [Voros83, Koike-Schäfke (to appear)]
Assume that the Stokes graph is saddle-free.

(1) The WKB solutions $\psi_\pm(z,\eta)$ are Borel summable in each Stokes region D, so that the Borel sums $\mathcal{S}[\psi_\pm](z,\eta)$ for $|\eta| \gg 1$ define analytic functions of z in each D.

(2) The following connection formula holds:

$$\mathcal{S}[\psi_+^{D_1}] = \mathcal{S}[\psi_+^{D_2}] + i\mathcal{S}[\psi_-^{D_2}],$$
$$\mathcal{S}[\psi_-^{D_1}] = \mathcal{S}[\psi_-^{D_2}],$$

where



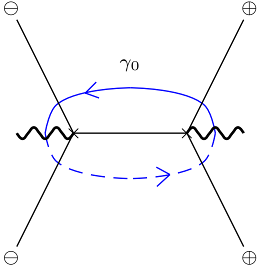
in the double covering $\hat{\Sigma}$ of Σ
s.t. $\sqrt{Q_0(z)}dz$ is single-valued

and we normalize the WKB solutions at a, i.e.,

$$\psi_\pm(z,\eta) = \frac{1}{\sqrt{S_{\mathrm{odd}}(z,\eta)}} \exp\left(\pm\int_a^z S_{\mathrm{odd}}(z,\eta)dz\right).$$

DDP's jump formula

Assume that the Stokes graph has the following unique saddle trajectory:



γ_0: saddle class

For each path $\beta \in H_1(\hat{\Sigma} \setminus \hat{P}_0, \hat{P}_\infty)$, we define

$$W_\beta(\eta) := \int_\beta \left(S_{\mathrm{odd}}(z,\eta) - \eta\sqrt{Q_0(z)} \right) dz \quad \text{Voros coefficient}$$

e^{W_β} Voros symbol

Let $\langle \gamma_0, \beta \rangle$ be the intersection number of γ_0 and β.

Theorem. [Dellabaere-Dillinger-Pham93]
(1) If $\langle \gamma_0, \beta \rangle = 0$, then e^{W_β} is Borel summable.
(2) If $\langle \gamma_0, \beta \rangle \neq 0$, then e^{W_β} is not Borel summable; the following jump formula holds:

$$\mathcal{S}_{-\delta}[e^{W_\beta}] = \mathcal{S}_{+\delta}[e^{W_\beta}](1 + \mathcal{S}_{+\delta}[e^{V_{\gamma_0}}])^{-\langle \gamma_0, \beta \rangle}, \quad V_{\gamma_0}(\eta) := \int_{\gamma_0} S_{\mathrm{odd}}(z,\eta) dz.$$

Quadratic differential

Schrödinger equation on a surface Σ:

$$\left(\frac{d^2}{dz^2} - \eta^2 Q(z,\eta)\right)\psi(z,\eta) = 0,$$

In this section, we consider a more general situation than the classic one:

Σ: compact Riemann surface,

$Q(z,\eta) = Q_0(z) + Q_1(z)\eta^{-1} + \cdots + Q_k(z)\eta^{-k}$, $\quad Q_n(z)$: meromorphic

Under the coordinate transformation, the leading term $Q_0(z)$ transforms as a quadratic differential (e.g. [Kawai-Takei05]),

$$Q_0(z)dz^{\otimes 2}.$$

$\Longrightarrow \quad \sqrt{Q_0(z)}dz$: 1-form single-valued on the double covering $\hat{\Sigma}$ of Σ

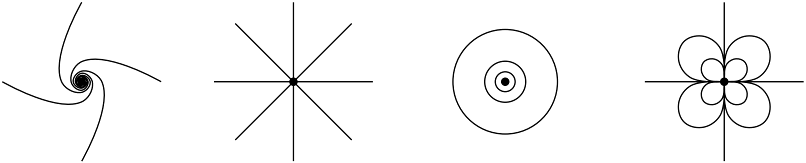
Assume

- Every zero of $Q_0(z)$ is simple.
- Every pole of $Q_0(z)$ is not simple.

We also assume some technical condition on the poles of $Q_n(z)$ $(n \geq 2)$.

Stokes graphs

Patterns of foliations around a pole [Strebel84]



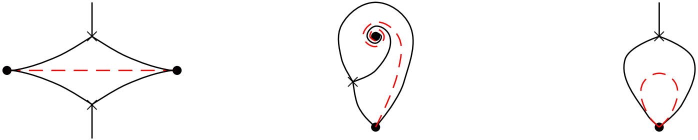
(a) (b) (c) $m-2$ tangent lines

double pole

pole of order $m \geq 3$ $(m=6)$

The Stokes graph and the Stokes regions are defined as before.

Assume that the Stokes graph is saddle-free (i.e., without saddle trajectory).
Then, Stokes regions fall into three classes [Strebel84].



regular horizontal strip degenerate horizontal strip half plane

From Stokes graph to triangulation

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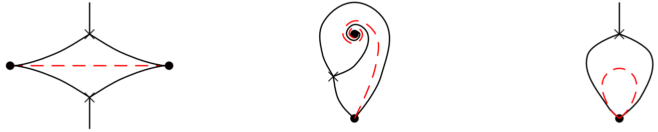
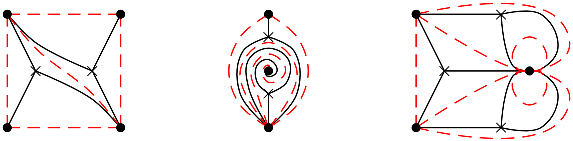


regular horizontal strip degenerate horizontal strip half plane

Assume that the Stokes graph is saddle-free.

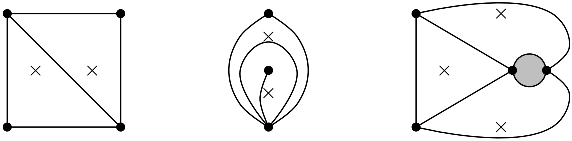
(a part of) Stokes graph



pole of order $m \geq 3$

$(m = 4)$

(a part of) triangulation



hole with

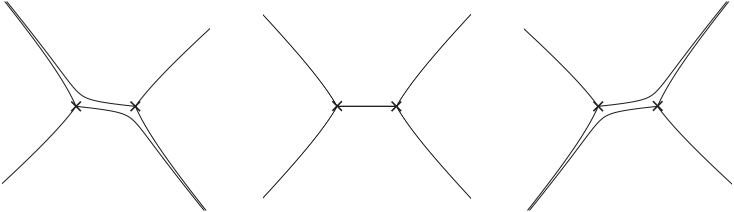
$m - 2$ marked points

These triangulations fit the surface realization of cluster algebras.

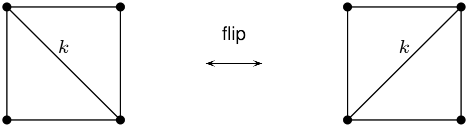
Mutation of Stokes graphs

Under a continuous deformation of the potential $Q(z, \eta)$, the Stokes graph may change its topology. (= mutation of Stokes graphs)

Example.

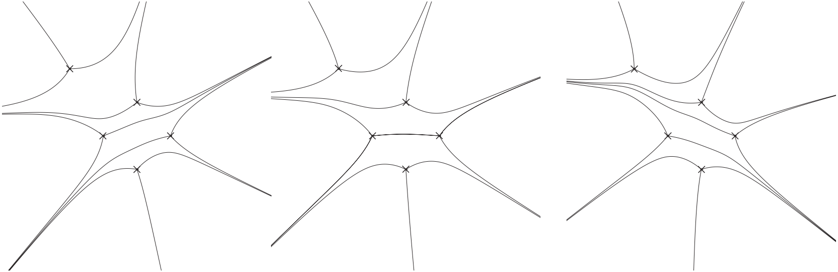


$e^{+\pi i/10}(1 - z^2)$ $\quad$ $1 - z^2$ $\quad$ $e^{-\pi i/10}(1 - z^2)$



Mutation of Stokes graphs

Example. $\Sigma = \mathbb{P}^1$, $Q(z) = e^{\delta \pi i}(z-1)(z+1)(z-2i+2)(z-i)(z+i)$



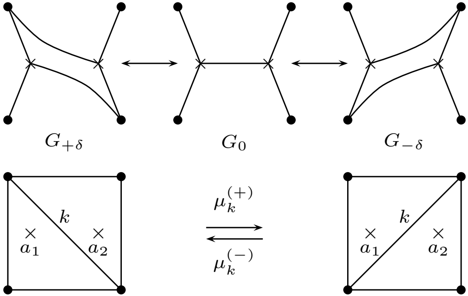
$\delta = 0.5$ $\delta = 0.374$ $\delta = 0.25$

Mutation of Stokes graphs

Mutation of Stokes graphs

- Under a continuous deformation of the potential $Q(z,\eta)$, the Stokes graph may change its topology. (= mutation of Stokes graphs)
- When the mutation occurs, one or more saddle connections appear.
- If only one saddle connection simultaneously appears during the mutation, it locally reduces to two types of elementary mutations called flip and pop [Gaiotto-Neitzke-Moore09], [Bridgeland-Smith13].
- They are refined to signed flip and signed pop.

Singed flip $\mu_k^{(\varepsilon)}$

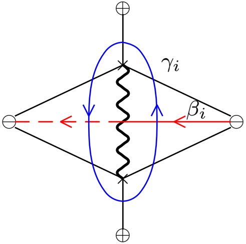


It is induced from the S^1-action on $Q(z,\eta)$ for G_0, $Q^{(\theta)}(z,\eta) := e^{2i\theta}Q(z,e^{i\theta}\eta)$.

Simple paths and simple cycles

For simplicity, assume that the Stokes graph has no degenerate horizontal strip.
($\leftrightarrow$ no self-folding triangle in the triangulation)

To each Stokes region D_i, we assign
simple path $\beta_i \in H_1(\hat{\Sigma} \setminus \hat{P}_0, \hat{P}_\infty)$
simple cycle $\gamma_i \in H_1(\hat{\Sigma} \setminus \hat{P}_0 \sqcup \hat{P}_\infty)$



duality $\langle \gamma_i, \beta_j \rangle = \delta_{ij}$

Under the signed flip $\mu_k^{(\varepsilon)}$, they mutate as

$$\beta_i' = \begin{cases} -\beta_k + \sum_{j=1}^{n} [-\varepsilon b_{jk}]_+ \beta_j & i = k \\ \beta_i & i \neq k \end{cases} \quad (g\text{-vector-like})$$

$$\gamma_i' = \begin{cases} -\gamma_k & i = k \\ \gamma_i + [\varepsilon b_{ki}]_+ \gamma_k & i \neq k \end{cases} \quad (c\text{-vector-like})$$

Main Theorem: Mutation formula of Voros symbols

We set

$$x_i = e^{W_i}, \quad W_i = \int_{\beta_i} \left(S_{\mathrm{odd}}(z,\eta) - \eta\sqrt{Q_0(z)} \right) dz \quad \text{Voros symbol for } \beta_i,$$

$$\hat{y}_i = e^{V_i}, \quad V_i = \int_{\gamma_i} S_{\mathrm{odd}}(z,\eta) dz \quad \text{Voros symbol for } \gamma_i,$$

$$y_i = e^{v_i}, \quad v_i = \int_{\gamma_i} \eta\sqrt{Q_0(z)} dz, \qquad \hat{y}_i = y_i \prod_{j=1}^{n} x_j^{b_{ji}}$$

where we follow Fomin-Zelevinsky's notation for cluster algebras with coefficients.

Theorem. [IN14] Under the S^1-action $Q^{(\theta)}(z,\eta) = e^{2i\theta}Q(z, e^{i\theta}\eta)$ which induces the signed mutation $\mu_k^{(\varepsilon)}$ of Stokes graphs, the Voros symbols "mutate" as

$$y_i' \rightsquigarrow \begin{cases} y_k^{-1} & i = k \\ y_i {y_k}^{[\varepsilon b_{ki}]_+} & i \neq k, \end{cases}$$

$$x_i' \rightsquigarrow \begin{cases} {x_k}^{-1} \left(\prod_{j=1}^{n} {x_j}^{[-\varepsilon b_{jk}]_+} \right) (1 + {\hat{y}_k}^{\varepsilon}) & i = k \\ x_i & i \neq k, \end{cases}$$

$$\hat{y}_i' \rightsquigarrow \begin{cases} {\hat{y}_k}^{-1} & i = k \\ \hat{y}_i {\hat{y}_k}^{[\varepsilon b_{ki}]_+} (1 + {\hat{y}_k}^{\varepsilon})^{-b_{ki}} & i \neq k. \end{cases}$$

Remark. The jump terms come form DDP's jump formula.