Wonder of sine-Gordon Y-systems

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Based on a joint work with Salvatore Stella (Northeastern U.), arXiv:1212.6853.
All beautiful figures in this slide are prepared by him.

The pdf file of this slide is available at my web site.

Summary

- The sine-Gordon (SG) Y-systems are some (very complicated) systems of algebraic functional equations introduced by Roberto Tateo in 90's.
- They originate from the SG equation, which is a famous soliton equation.
- To be more precise, there are two subfamilies:
  SG Y-systems and reduced SG (RSG) Y-systems.

Both are associated with continued fractions.

Main Message

There is a wonderful interplay among

- continued fractions
- SG/RSG Y-systems
- triangulations of polygons
- cluster algebras of types A and D

- Since the SG equation is associated with affine sl_2, this should be a tip of iceberg of some gigantic thing!

Important remark

Don't be discouraged by the horrible appearance of the SG/RSG Y-systems, since they are beautiful in nature.

Outline

Background in integrable models

- 2d sine-Gordon (SG) equation

$$\partial_t^2\phi - \partial_x^2\phi + g\sin(\beta\phi) = 0, \quad (g,\ \beta\text{: parameters})$$

a deformation of Klein-Gordon equation $\partial_t^2\phi - \partial_x^2\phi + m^2\phi = 0$.

- integrable, having solitons, (affine) Toda field theory of $\widehat{sl}_2$
- quantization of ϕ (SG model) $\Longrightarrow$ factorizable S-matrix [Zamolodchikov^2 79]
- Tateo (95) applied the thermodynamic Bethe ansatz (TBA) method (started by Yang^2, and developed by Takahashi, Suzuki, Faddeev, Takhtajan, Bazhanov, Kirillov, Reshetikhin, Al. Zamolodchikov, ...).

SG S-matrix $\Longrightarrow$ SG TBA equation (integral equation)
$\Longrightarrow$ SG Y-system & RSG Y-system (algebraic equation)

- We are interested in the situation when the parameter

$\xi := \frac{\beta^2/2}{1-\beta^2/2}$ is rational, and $0 < \xi < 1$ (the attractive region).

The SG and RSG Y-systems depend on the continued fraction expression of ξ.

- Boldly speaking, the SG equation, the SG S-matrix (SG model), and the SG/RSG Y-systems are like cow, milk, and cheese.

Continued fractions (1)

- For any finite sequence of positive integers $(n_1, \ldots, n_F)$, $n_1 \geq 2$, we associate a continued fraction

$$\xi = [n_F, \ldots, n_1] := \cfrac{1}{n_F + \cfrac{1}{n_{F-1} + \cfrac{1}{\ddots + \cfrac{1}{n_1}}}}.$$

We have the one-to-one correspondence:

$$(n_1, \ldots, n_F) \quad \leftrightarrow \quad \xi \in \mathbb{Q}, \quad 0 < \xi < 1.$$

For each $(n_1, \ldots, n_F)$, we have a family of continued fractions:

$$\xi_a = [n_a, \ldots, n_1], \quad (a = 1, \ldots, F)$$

Define positive integers p_a, q_a $(a = 1, \ldots, F)$ by

$$\xi_a = \frac{p_a}{q_a}, \quad \gcd(p_a, q_a) = 1.$$

Set

$$r_a := p_a + q_a.$$

Continued fractions (2)

- Running example. $F = 3$, $(n_1, n_2, n_3) = (6, 4, 3)$.

$$\xi_1 = \frac{1}{6}, \qquad (p_1, q_1) = (1, 6), \qquad r_1 = 7,$$
$$\xi_2 = \frac{1}{4 + \frac{1}{6}} = \frac{6}{25}, \qquad (p_2, q_2) = (6, 25), \qquad r_2 = 31,$$
$$\xi_3 = \frac{1}{3 + \frac{1}{4 + \frac{1}{6}}} = \frac{25}{81}, \qquad (p_3, q_3) = (25, 81), \qquad r_3 = 106.$$

- We use the numbers $p_1, \ldots, p_F$ to define the RSG/SG Y-systems. They satisfy the recursion relation

$$p_1 = 1, \quad p_2 = n_1,$$
$$p_a = n_{a-1} p_{a-1} + p_{a-2}.$$

e.g., in the above example, $p_1 = 1$, $p_2 = n_1 = 6$, $p_3 = n_2 p_2 + p_1 = 4 \cdot 6 + 1 = 25$. All the nice properties originate from this recursion relation.

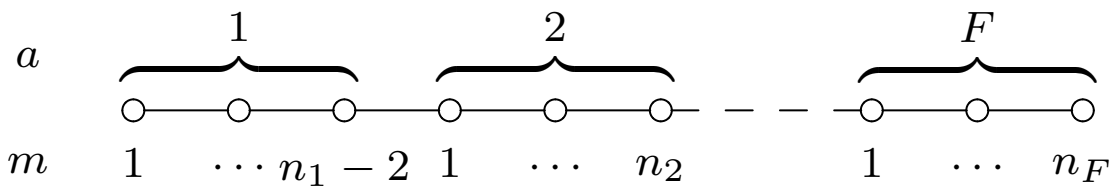
RSG Y-systems (1)

- Input: $(n_1, \ldots, n_F)$. Here, assume $n_1 \geq 3$ (the case $n_1 = 2$ is exceptional). We will define the RSG Y-system $\mathbb{Y}_{\mathrm{RSG}}(n_1, \ldots, n_F)$.
- the Y-variables $Y_m^{(a)}(u)$, where

$$u \in \mathbb{Z} \text{ (spectral parameter)}, \quad a = 1 \ldots, F \text{ (generation)},$$

$$m = \begin{cases} 1, \ldots, n_1 - 2 & \text{if } a = 1 \\ 1, \ldots, n_a & \text{if } a = 2, \ldots, F. \end{cases}$$

- the Dynkin diagram $X_{\mathrm{RSG}}(n_1, \ldots, n_F)$ of type A indexed by (a, m) in the above range



- signs: We set

$$\varepsilon_a := (-1)^{a+1} \quad a = 1, \ldots, F$$

.

- General relations: For a general (a, m) other than $(2, 1)$, $(3, 1)$, $\ldots$, $(F, 1)$,

$$Y_m^{(a)}(u - p_a) Y_m^{(a)}(u + p_a) = \prod_{(b,k) \sim (a,m)} (1 + Y_k^{(b)}(u)^{\varepsilon_b})^{\varepsilon_b},$$

where $(b, k) \sim (a, m)$ means (b, k) is adjacent to (a, m) in $X_{\mathrm{RSG}}(n_1, \ldots, n_F)$.

- For $F = 1$, it is the classic Y-system of type A.

RSG Y-systems (2)

Exceptional relations: For $(a, m) = (2, 1)$,

$$Y_1^{(2)}(u-p_2)Y_1^{(2)}(u+p_2) = (1+Y_2^{(2)}(u)^{-1})^{-1}(1+Y_1^{(1)}(u)) \times \prod_{m=1}^{n_1-2}(1+Y_m^{(1)}(u-1-m)^{-1})^{-1} \times \prod_{m=1}^{n_1-2}(1+Y_m^{(1)}(u+1+m)^{-1})^{-1}.$$

For $(a, m) = (3, 1), \ldots, (F, 1)$,

$$\begin{aligned} &Y_1^{(a)}(u-p_a)Y_1^{(a)}(u+p_a) \\ &= (1+Y_2^{(a)}(u)^{\varepsilon_a})^{\varepsilon_a}(1+Y_{n_{a-2}-2\delta_{a3}}^{(a-2)}(u)^{\varepsilon_a})^{\varepsilon_a} \\ &\quad \times \prod_{m=1}^{n_{a-1}}(1+Y_m^{(a-1)}(u-p_a+(n_{a-1}+1-m)p_{a-1})^{\varepsilon_a})^{\varepsilon_a} \\ &\quad \times \prod_{m=1}^{n_{a-1}}(1+Y_m^{(a-1)}(u+p_a-(n_{a-1}+1-m)p_{a-1})^{\varepsilon_a})^{\varepsilon_a}, \end{aligned}$$

where δ_{a3} is the Kronecker delta.

- In short: (very complicated) generalization of the Y-systems of type A.

Bisection of RSG Y-systems

• Let $\mathcal{Y} = \mathcal{Y}_{\mathrm{RSG}}(n_1, \ldots, n_F)$ be the set of all Y-variables of $\mathbb{Y}_{\mathrm{RSG}}(n_1, \ldots, n_F)$.
Divide $\mathcal{Y} = \mathcal{Y}_+ \sqcup \mathcal{Y}_-$, where

$$\mathcal{Y}_+ := \{Y_m^{(a)}(u) \in \mathcal{Y} \mid \theta_m^{(a)}(u) := u + p_{a+1} - (n_a - m)p_a \text{ is even}\},$$
$$\mathcal{Y}_- := \{Y_m^{(a)}(u) \in \mathcal{Y} \mid \theta_m^{(a)}(u) := u + p_{a+1} - (n_a - m)p_a \text{ is odd.}\}$$

Fact

Each relation of the Y-system involves variables only from $\mathcal{Y}_+$ or only from $\mathcal{Y}_-$.

So, the Y-system $\mathbb{Y}_{\mathrm{RSG}}(n_1, \ldots, n_F)$ is bisected into the one for $\mathcal{Y}_+$ and the one for $\mathcal{Y}_-$.
One can concentrate on the Y-system for $\mathcal{Y}_+$, for example.

SG Y-systems (1)

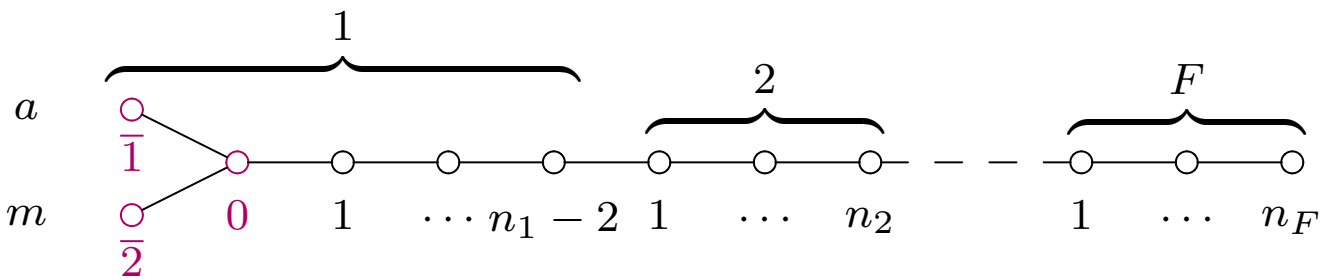
- Input: $(n_1, \ldots, n_F)$

We will define the SG Y-system $\mathbb{Y}_{\mathrm{SG}}(n_1, \ldots, n_F)$.

- the Y-variables $Y_m^{(a)}(u)$, where $u \in \mathbb{Z}$, $a = 1 \ldots, F$, and

$$m = \begin{cases} \overline{1}, \overline{2}, 0, 1, \ldots, n_1 - 2 & \text{if } a = 1 \\ 1, \ldots, n_a & \text{if } a = 2, \ldots, F. \end{cases}$$

- the Dynkin diagram $X_{\mathrm{SG}}(n_1, \ldots, n_F)$ of type D indexed by (a, m) in the above range



- General relations: For a general (a, m) other than $(2, 1)$, $(3, 1)$, $\ldots$, $(F, 1)$,

$$Y_m^{(a)}(u - p_a) Y_m^{(a)}(u + p_a) = \prod_{(b,k) \sim (a,m)} (1 + Y_k^{(b)}(u)^{\varepsilon_b})^{\varepsilon_b},$$

where $(b, k) \sim (a, m)$ means (b, k) is adjacent to (a, m) in $X_{\mathrm{SG}}(n_1, \ldots, n_F)$.

- For $F = 1$, it is the classic Y-system of type D.

SG Y-systems (2)

Exceptional relations: For $(a,m)=(2,1)$,

$$
\begin{aligned}
&Y_1^{(2)}(u-p_2)Y_1^{(2)}(u+p_2)\\
&\quad=(1+Y_2^{(2)}(u)^{-1})^{-1}(1+Y_{\overline{1}}^{(1)}(u)^{-1})^{-1}(1+Y_{\overline{2}}^{(1)}(u)^{-1})^{-1}\\
&\qquad\times\prod_{m=0}^{n_1-2}(1+Y_m^{(1)}(u-1-m)^{-1})^{-1}\\
&\qquad\times\prod_{m=0}^{n_1-2}(1+Y_m^{(1)}(u+1+m))^{-1})^{-1}.
\end{aligned}
$$

For $(a,m)=(a,1)$ with $a=3,\dots,F$,

$$
\begin{aligned}
&Y_1^{(a)}(u-p_a)Y_1^{(a)}(u+p_a)\\
&\quad=(1+Y_2^{(a)}(u)^{\varepsilon_a})^{\varepsilon_a}(1+Y_{n_{a-2}-2\delta_{a3}}^{(a-2)}(u)^{\varepsilon_a})^{\varepsilon_a}\\
&\qquad\times\prod_{m=1}^{n_{a-1}}(1+Y_m^{(a-1)}(u-p_a+(n_{a-1}+1-m)p_{a-1})^{\varepsilon_a})^{\varepsilon_a}\\
&\qquad\times\prod_{m=1}^{n_{a-1}}(1+Y_m^{(a-1)}(u+p_a-(n_{a-1}+1-m)p_{a-1})^{\varepsilon_a})^{\varepsilon_a}.
\end{aligned}
$$

- In short: (very complicated) generalization of the Y-systems of type D.

Reduction of SG Y-systems

- Under the specialization

$$Y_0^{(1)}(u) = 0, \quad Y_{\overline{1}}^{(1)}(u) = Y_{\overline{2}}^{(1)}(u) = -1,$$

$\mathbb{Y}_{\mathrm{SG}}(n_1, \ldots, n_F)$ reduces to $\mathbb{Y}_{\mathrm{RSG}}(n_1, \ldots, n_F)$.

Outline

Y-system and CFT

- General picture:

S-matrix model massive theory	massless limit $\longrightarrow$ $\longleftarrow$ massive deformation	conformal field theory (CFT) massless theory

- In TBA method, the Y-system is a tool to extract some essential information of CFT limit of the S-matrix model.

  period of Y-system $\Longrightarrow$ conformal dimension of the deformation field

  dilogarithm identity of Y-system $\Longrightarrow$ central charge of CFT

- This perspective leads to several conjectures on the periodicities and dilogarithm identities for various Y-systems in 90's.

- In particular, Tateo made conjectures on explicit formulas of period and dilogarithm identities for RSG/SG Y-systems.

Tateo's Conjectures (1): Periodicity

- Recall that

$$\xi_F = [n_F, \ldots, n_1] = \frac{p_F}{q_F}, \quad r_F := p_F + q_F.$$

Conjecture [Tateo 95].
"... the periodicity properties that can be numerically verified with high precision..."

(a). The RSG Y-system $\mathbb{Y}_{\mathrm{RSG}}(n_1, \ldots, n_F)$ has the following periodicity.

$$Y_m^{(a)}(u + 2r_F) = Y_m^{(a)}(u).$$

(b). The SG Y-system $\mathbb{Y}_{\mathrm{SG}}(n_1, \ldots, n_F)$ has the following periodicity.
(i). If r_F is even, we have

$$Y_m^{(a)}(u + 2r_F) = Y_m^{(a)}(u).$$

(ii). If r_F is odd, we have

$$\text{(half periodicity)} \quad Y_m^{(a)}(u + 2r_F) = \begin{cases} Y_{\overline{2}}^{(1)}(u) & (a, m) = (1, \overline{1}) \\ Y_{\overline{1}}^{(1)}(u) & (a, m) = (1, \overline{2}) \\ Y_m^{(a)}(u) & \text{otherwise,} \end{cases}$$

$$\text{(full periodicity)} \quad Y_m^{(a)}(u + 4r_F) = Y_m^{(a)}(u).$$

Tateo's Conjectures (2): Dilogarithm identities

- Let $L(x)$ be the Rogers dilogarithm

$$L(x) = -\frac{1}{2}\int_0^x \left(\frac{\log(1-y)}{y} + \frac{\log y}{1-y}\right) dy.$$

Conjecture [Tateo 95].
"... with an appropriate number of initial conditions we find the following identities ..."

For any real positive solution of the RSG/SG Y-system $\mathbb{Y}_{\mathrm{RSG/SG}}(n_1,\ldots,n_F)$ for $\mathcal{Y}_+$, the following identities hold:

$$\frac{6}{\pi^2}\sum_{\substack{Y_m^{(a)}(u)\in\mathcal{Y}_+\\ 0\le u<2r_F}} L\left(\frac{Y_m^{(a)}(u)}{1+Y_m^{(a)}(u)}\right)$$

$$= \begin{cases} r_F\left(\sum_{a:\mathrm{odd}} n_a - 4 + \sum_{a=1}^{F-1}(-1)^{a+1}\frac{6}{p_a q_a} + (-1)^{F+1}\frac{6}{p_F r_F}\right) & \text{for RSG} \\ r_F\left(\sum_{a:\mathrm{odd}} n_a\right) & \text{for SG.} \end{cases}$$

The most complicated example of the dilog identities arising from the TBA method.

Cross-ratio solution of RSG Y-system by [Gliozzi-Tateo 96]

- cross-ratio:

$$(\alpha,\beta,\gamma,\delta) := \frac{(\alpha-\delta)(\beta-\gamma)}{(\alpha-\beta)(\gamma-\delta)}.$$

Introduce formal variables $z(n)$ ($n \in \mathbb{Z}$).

Theorem [Gliozzi &Tateo 96]. (cross-ratio solution of RSG Y-system)
"... it is straightforward to verify by direct substitution that the general solution of Eqs. (3.34-3.39) is ..."

Assume the periodicity $z(n+r_F) = z(n)$. Them, the cross-ratios

$$Y_m^{(a)}(u) = (z(\alpha_m^{(a)}(u)), z(\beta_m^{(a)}(u)), z(\gamma_m^{(a)}(u)), z(\delta_m^{(a)}(u)))^{\varepsilon_a},$$

give the general solution of the RSG Y-system for $\mathcal{Y}_+$, where

$$\alpha_m^{(1)}(u) = \frac{1}{2}(u+m+2), \quad \beta_m^{(1)}(u) = \frac{1}{2}(u+m),$$
$$\gamma_m^{(1)}(u) = \frac{1}{2}(u-m), \qquad \delta_m^{(1)}(u) = \frac{1}{2}(u-m-2).$$

and, for $a = 2,\dots,F$,

$$\alpha_m^{(a)}(u) = \frac{1}{2}(u+p_{a+1}-(n_a-m)p_a), \qquad \beta_m^{(a)}(u) = \frac{1}{2}(u+p_{a+1}-(n_a+2-m)p_a),$$
$$\gamma_m^{(a)}(u) = \frac{1}{2}(u-p_{a+1}+(n_a+2-m)p_a), \quad \delta_m^{(a)}(u) = \frac{1}{2}(u-p_{a+1}+(n_a-m)p_a).$$

In particular, the periodicity $Y_m^{(a)}(u+2r_F) = Y_m^{(a)}(u)$ is proved.

Known and related results on Tateo's conjectures

- chronological table

1. [Gliozzi &Tateo 96] Periodicities for RSG (for any F).
2. [Frenkel & Szenes 95] Dilog identities for type A Y-system (=RSG for $F = 1$).

- There was no systematic method to prove these conjectures.
  (↑ B.C.)

- '00, Introduction of cluster algebras by Fomin & Zelevinsky
  (↓ A.D.)

3. [Fomin & Zelevinsky 02] Periodicity for type D Y-system (=SG for $F = 1$).
4. [Chapoton 05] Dilog identities for type D Y-system (=SG for $F = 1$).

- Similar conjectures for the RSOS Y-systems were proved by tropicalization-categorification method in cluster algebra [Keller 10], [N. 09], [Inoue-Iyama-Keller-Kuniba-N. 10]

5. [N.& Tateo 10] Periodicities/Dilog identities for RSG/SG for $F = 2$.
   (applying the above tropicalization-categorification method)

- However, the extension of the analysis [N.& Tateo 10] to general F seems not easy, because analysis becomes too complicated.
  Source of problem: We do not see any good structure in RSG/SG Y-systems.
  Again, we are stuck.

Main result

Main Theorem [N. & Stella 12].

Tateo's conjectures for RSG/SG Y-systems hold for any F.

- Key idea to overcome the difficulty: Using polygon realization.

(1). (by the referee of [N.& Tateo 10]): For $F = 2$, the cluster algebras realizing the RSG/SG Y-systems are of types A/D, respectively.

(2). It is known that the cluster algebras of types A/D are realized by polygons (without/with puncture). [Fock & Goncharov 07, Fomin-Shapiro-Thurston 08]

(3). So, we may try to realize the Y-systems for general F by polygons. This is indeed possible; and it turned out that the nature of RSG/SG Y-systems becomes most apparent in the polygon realization.

Outline

Y-seed of cluster algebras

[Fomin-Zelevinsky 02]

- Fix a positive integer n.

Start from initial (labeled) Y-seed (B, y):
a skew-symmetric matrix $B = (b_{ij})_{i,j=1}^{n}$ (initial exchange matrix),
algebraically independent variables $y = (y_1, \ldots, y_n)$ (initial coefficients/y-variables)

- The mutation $(B', y') = \mu_k(B, y)$ at $k = 1, \ldots, n$ is defined by the following rule.

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + b_{ik}[b_{kj}]_+ + [-b_{ik}]_+ b_{kj} & i, j \neq k, \end{cases}$$

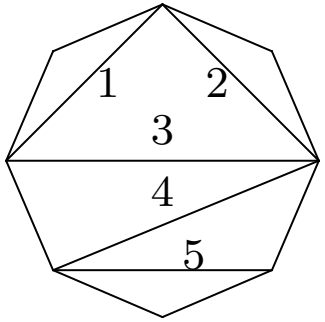
$$y'_i = \begin{cases} y_i^{-1} & i = k \\ y_i \dfrac{(1 + y_k)^{[-b_{ki}]_+}}{(1 + y_k^{-1})^{[b_{ki}]_+}} & i \neq k, \end{cases}$$

where $[a]_+ = a$ for $a > 0$ and 0 otherwise.

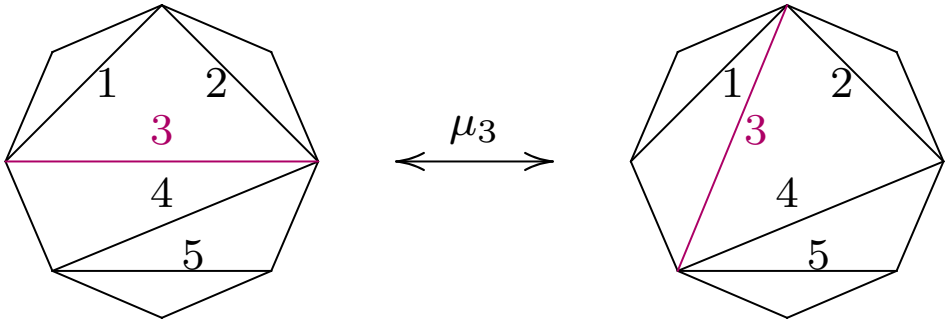
- Repeat mutations.

Mutation of a labeled triangulation of a polygon

- Fix a given (initial) labeled triangulation of $n+3$-gon.
There are n diagonals. The diagonals are labeled by $i = 1, \ldots, n$.
e.g., octagon, $8 - 3 = 5$ diagonals

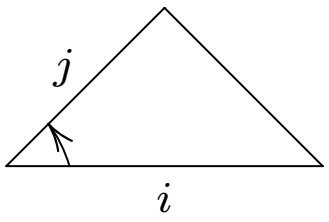


- mutation at k: flip the diagonal k



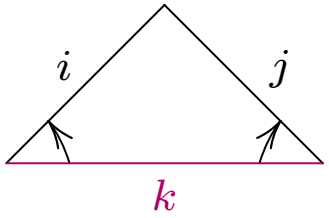
Polygon realization of Y-seeds of type A

- We associate a Y-seed (B, y) to each labeled triangulation Γ as follows.
- The matrix B: We set $b_{ij} = 1$, $b_{ji} = -1$ for the configuration



Otherwise $b_{ij} = 0$. This is compatible with the mutation of B-matrices.

- y-variables: For the initial triangulation Γ, attach the initial y-variable y_i to the diagonal i. For the labeled triangulation $\Gamma' = \mu_k(\Gamma)$, attach the y-variables y'_i to the diagonal i such that



$$y'_k = y_k^{-1}, \quad y'_i = y_i(1 + y_k^{-1})^{-1}, \quad y'_j = y_j(1 + y_k), \quad \text{for the above } k, i, j$$
$$y'_\ell = y_\ell, \quad \text{else.}$$

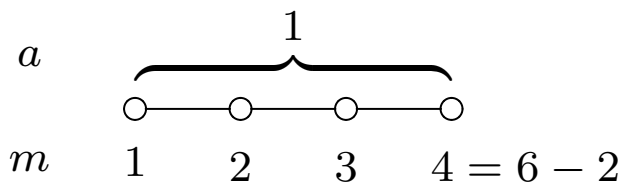
- Then we have a one-to-one correspondence (not obvious!):

$$\text{labeled triangulation} \quad \leftrightarrow \quad Y\text{-seed}$$

shown via cross-ratio [Fock-Goncharov], or via tropicalization [Fomin-Thurston] + [Plamondon] + [Inoue-Iyama-Keller-Kuniba-N.]

Example 1: $\mathbb{Y}_{\mathrm{RSG}}(6)$. Y-system.

The input data $F = 1$, $(n_1) = (6)$.
$\xi_1 = \frac{1}{6}$, $(p_1, q_1) = (1, 6)$, $r_1 = 7$, $2r_1 = 2 \times 7 = 14$ (period of Y-system).



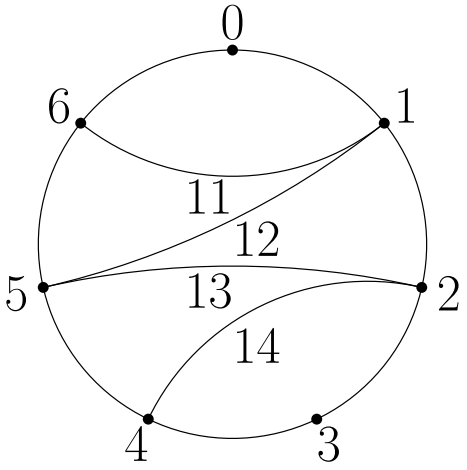
Y-variables: $Y_1^{(1)}(u), \ldots, Y_4^{(1)}(u)$
Y-system:

$$Y_m^{(1)}(u-1)Y_m^{(1)}(u+1) = \prod_{(1,k)\sim(1,m)} (1 + Y_k^{(1)}(u)).$$

This is the classic Y-system of type A_4, which is realized by
$7 = 4 + 3 = n_1 + 1 = r_1$-gon.

Example 1: $\mathbb{Y}_{\mathrm{RSG}}(6)$. Initial triangulation.

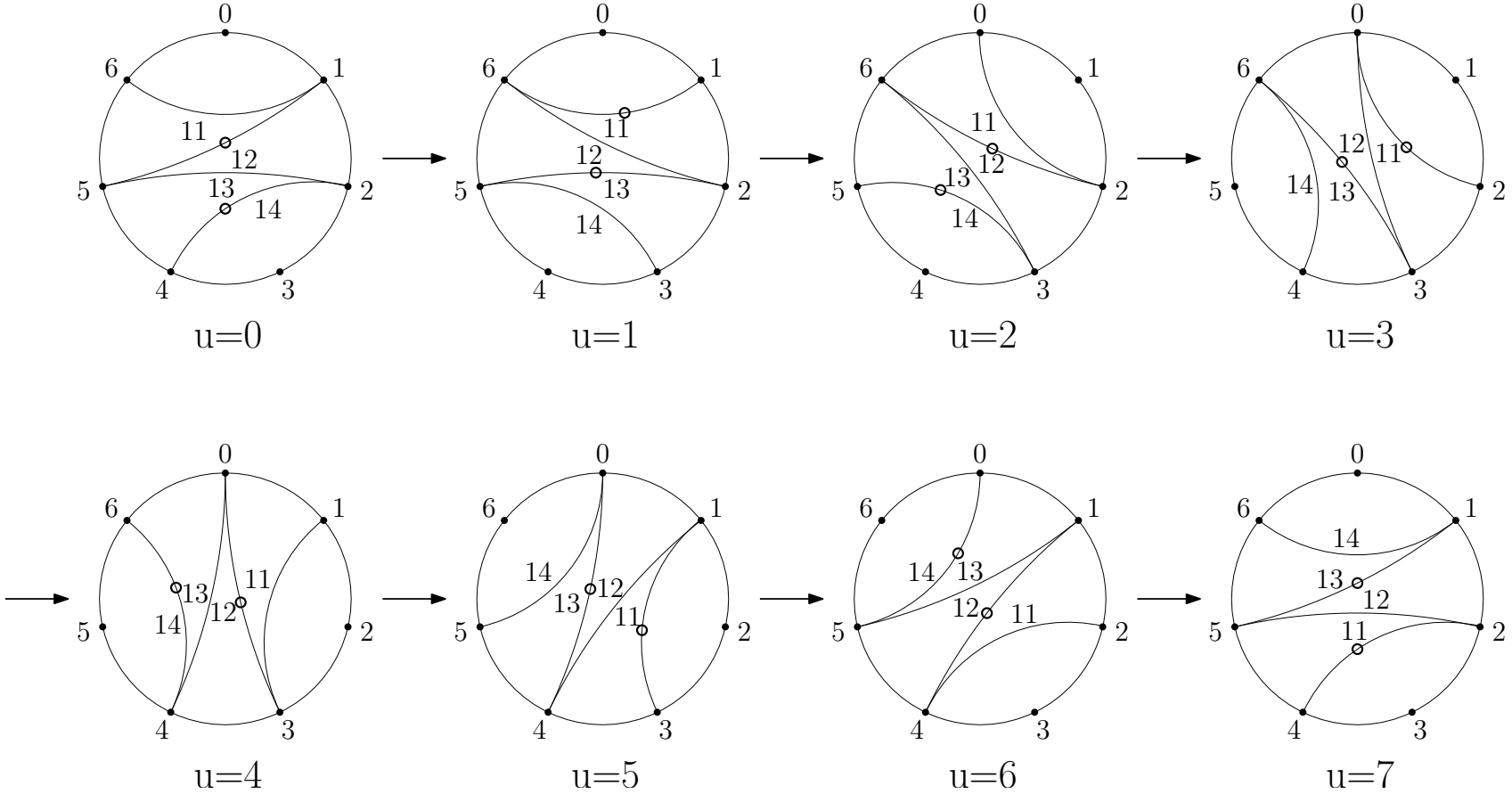
- Initial labeled triangulation of 7-gon. (zigzag/bipartite triangulation)



$11, \ldots 14$ are the abbreviation of the label $(a, m) = (1, 1), \ldots, (1, 4)$.
They correspond to the labels for Y-variables $Y_1^{(1)}(u), \ldots, Y_4^{(1)}(u)$.

Example 1: $\mathbb{Y}_{\mathrm{RSG}}(6)$. Sequence of mutations.

- Consider the following sequence of mutations:

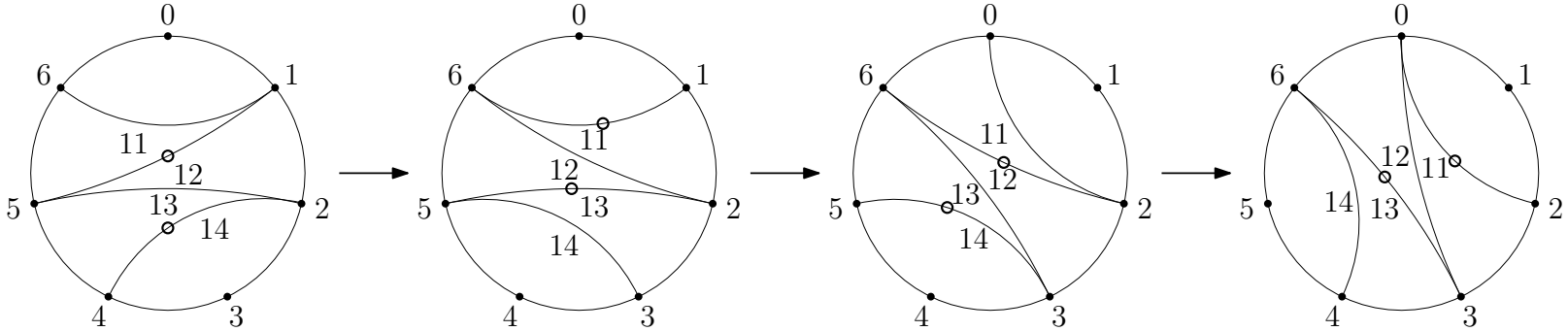


half period:7, full period: 14.
The labels marked by $\circ$ are called the forward mutation points at time u

Example 1: $\mathbb{Y}_{\mathrm{RSG}}(6)$. Realization of Y-system.

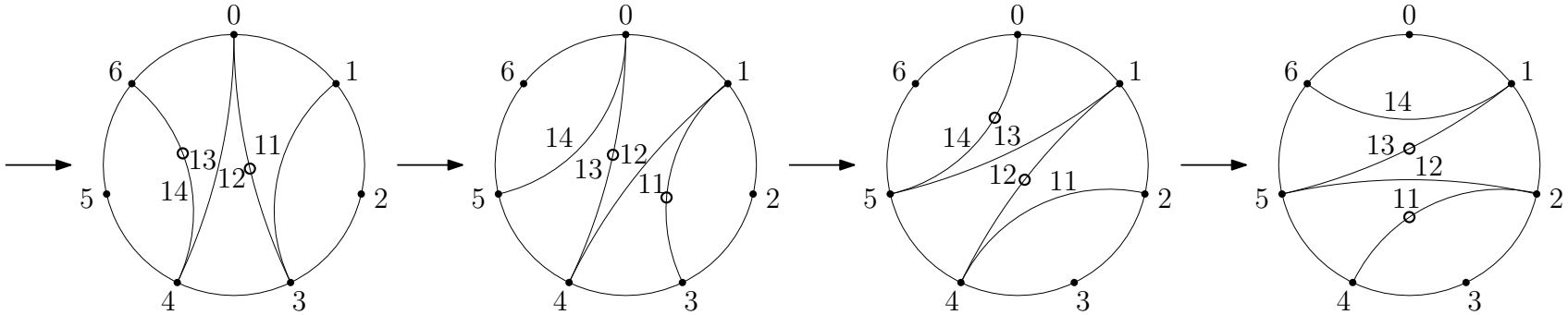
- y-variables at time u: $y_1^{(1)}(u)$, $y_2^{(1)}(u)$, $y_3^{(1)}(u)$, $y_4^{(1)}(u)$
- Identify $y_m^{(1)}(u)$ with $Y_m^{(1)}(u)$ only at the forward mutation points at u.

These Y-variables belongs to $\mathcal{Y}_+ := \{Y_m^{(1)}(u) \mid m+u \text{ is even}\}$.
They satisfies the Y-system for $\mathcal{Y}_+$.
e.g. $Y_2^{(1)}(0)Y_2^{(1)}(2) = (1+Y_1^{(1)}(1))(1+Y_3^{(1)}(1))$.



u=0 u=1 u=2 u=3

$y_2^{(1)}(0)$ $\quad y_2^{(1)}(1) = y_2^{(1)}(0)^{-1}$ $\quad y_2^{(1)}(2) = y_2^{(1)}(0)^{-1}(1+y_1^{(1)}(1))(1+y_3^{(1)}(1))$



u=4 u=5 u=6 u=7

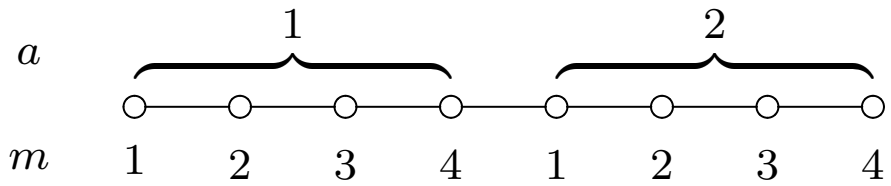
This proves the periodicity of the Y-system with period 14!

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Y-system.

- The input data $F=2$, $(n_1,n_2)=(6,4)$.

$$\xi_2=\frac{1}{4+\frac{1}{6}}=\frac{6}{25},\quad (p_2,q_2)=(6,25),\quad r_2=31.$$

$2r_2=2\times 31=62$ (period of Y-system).



Y-variables: $Y_1^{(1)}(u),\dots,Y_4^{(1)}(u)$; $Y_1^{(2)}(u),\dots,Y_4^{(2)}(u)$
Y-system: General relation for $(a,m)\neq(2,1)$,

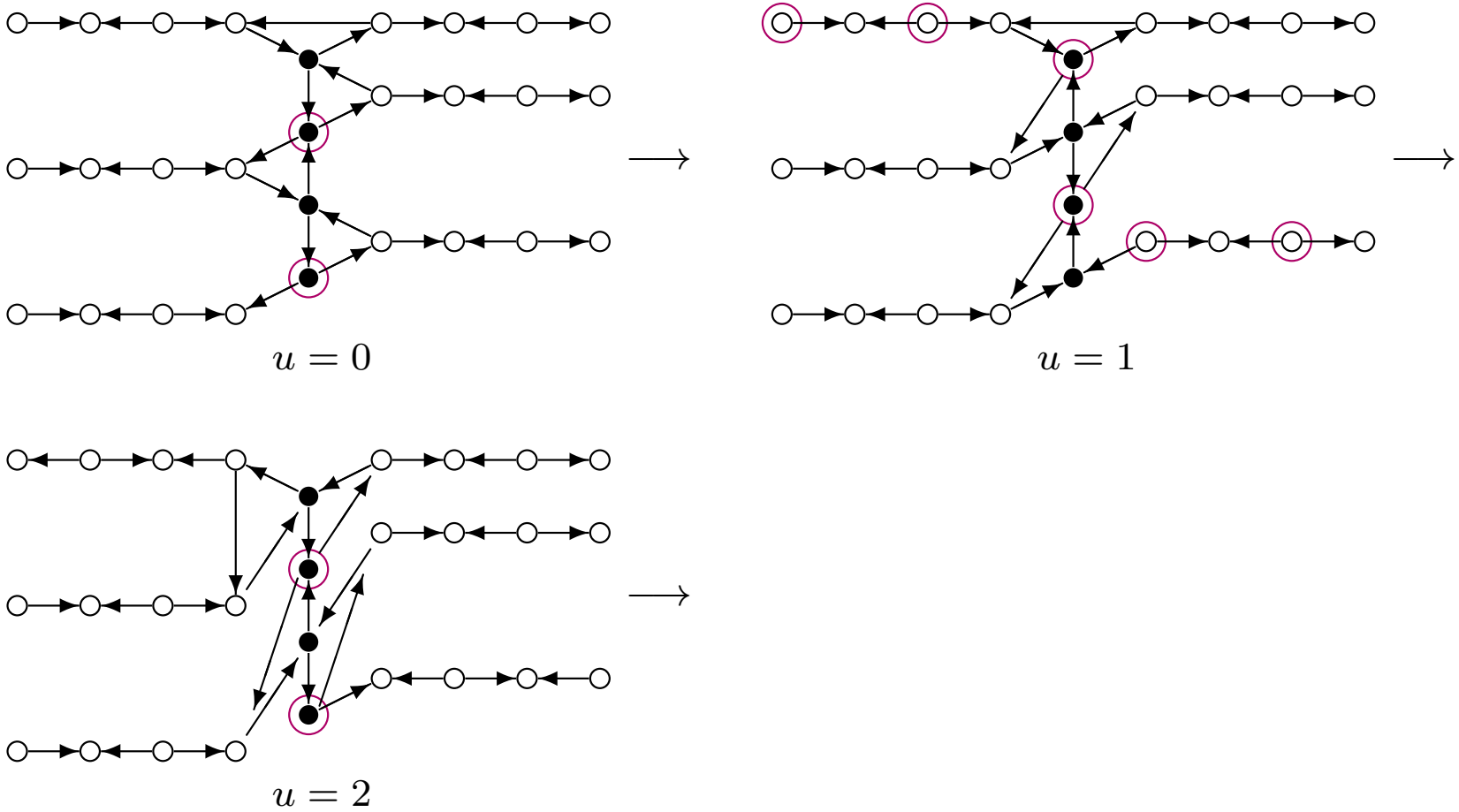
$$Y_m^{(a)}(u-p_a)Y_m^{(a)}(u+p_a)=\prod_{(b,k)\sim(a,m)}(1+Y_k^{(b)}(u)^{\varepsilon_b})^{\varepsilon_b}.$$

Exceptional relation for $(a,m)=(2,1)$,

$$\begin{aligned}Y_1^{(2)}(u-6)Y_1^{(2)}(u+6)&=(1+Y_2^{(2)}(u)^{-1})^{-1}(1+Y_1^{(1)}(u))\\&\times(1+Y_4^{(1)}(u-5)^{-1})^{-1}(1+Y_3^{(1)}(u-4)^{-1})^{-1}\\&\times(1+Y_2^{(1)}(u-3)^{-1})^{-1}(1+Y_1^{(1)}(u-2)^{-1})^{-1}\\&\times(1+Y_1^{(1)}(u+2)^{-1})^{-1}(1+Y_2^{(1)}(u+3)^{-1})^{-1}\\&\times(1+Y_3^{(1)}(u+4)^{-1})^{-1}(1+Y_4^{(1)}(u+5)^{-1})^{-1}.\end{aligned}$$

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Mutation sequence of quivers.

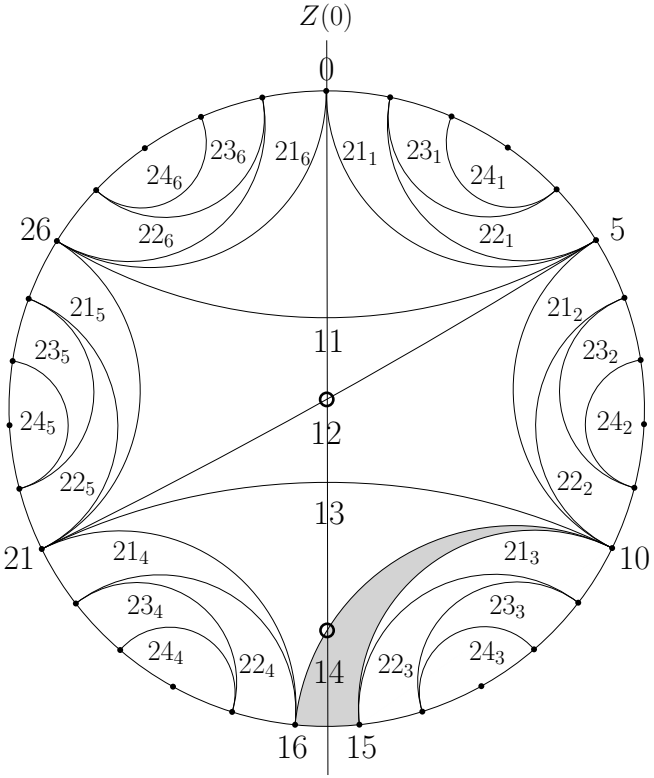
- Realization of $\mathbb{Y}_{\mathrm{RSG}}(6,4)$ by the the sequences of quivers with 28 vertices [N.-Tateo 10]. ($\bullet/\circ$: first/second generation)



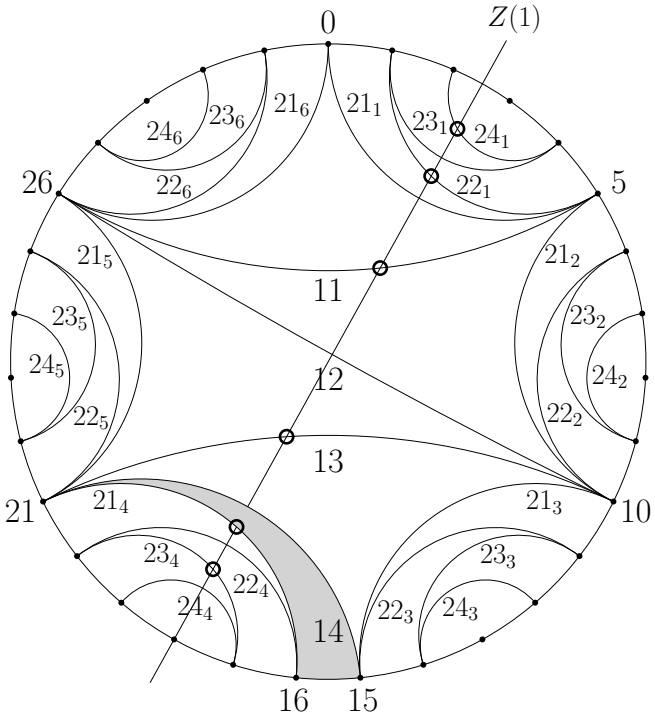
- The quivers at $u = 0$ and 2 are isomorphic by the relabeling for the second generation. (periodicity of quivers)
- They are mutation equivalent to the quiver of type A.
- The choice of quivers and forward mutation points are rather mysterious.

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Realization by triangulations.

- One can translate the quivers of [N.-Tateo 10] into the triangulations of the 31-gon.



$u = 0$



$u = 1$

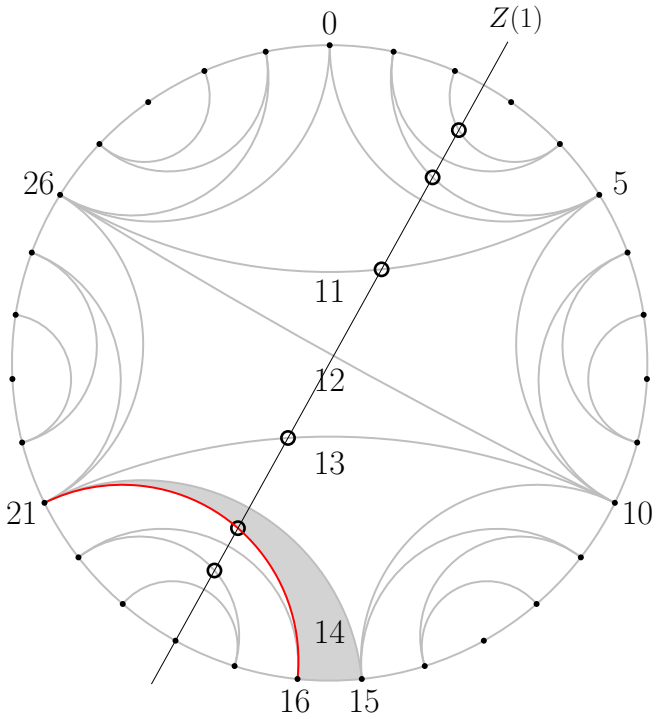
The $\circ$'s are the forward mutation points.

- Key observation: Each mutation is the reflection with respect to the axis $Z(u)$.
$\Longrightarrow$ Combining two reflections, we have the rotation by 5 units.

- Repeat the reflections with respect the images of these axes.
Since 31 and 5 are coprime, we have the (desired) period of 62 as an unlabeled triangulation. (As a labeled triangulation, the label $(2,m)_s$ is replaced by $(2,m)_{s+1}$.)

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(2,1)$:

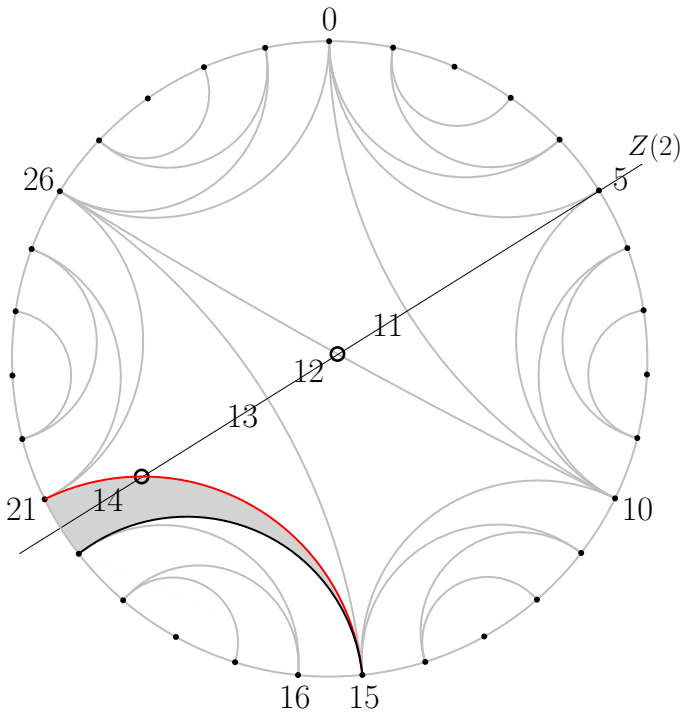


$$u=1$$

$$\begin{aligned}
Y_1^{(2)}(13) = {} & Y_1^{(2)}(1)^{-1}(1+Y_4^{(1)}(2)^{-1})^{-1}(1+Y_3^{(1)}(3)^{-1})^{-1} \\
& \times (1+Y_2^{(1)}(4)^{-1})^{-1}(1+Y_1^{(1)}(5)^{-1})^{-1} \\
& \times (1+Y_2^{(2)}(7)^{-1})^{-1}(1+Y_1^{(1)}(7)) \\
& \times (1+Y_1^{(1)}(9)^{-1})^{-1}(1+Y_2^{(1)}(10)^{-1})^{-1} \\
& \times (1+Y_3^{(1)}(11)^{-1})^{-1}(1+Y_4^{(1)}(12)^{-1})^{-1}.
\end{aligned}$$

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(2,1)$:

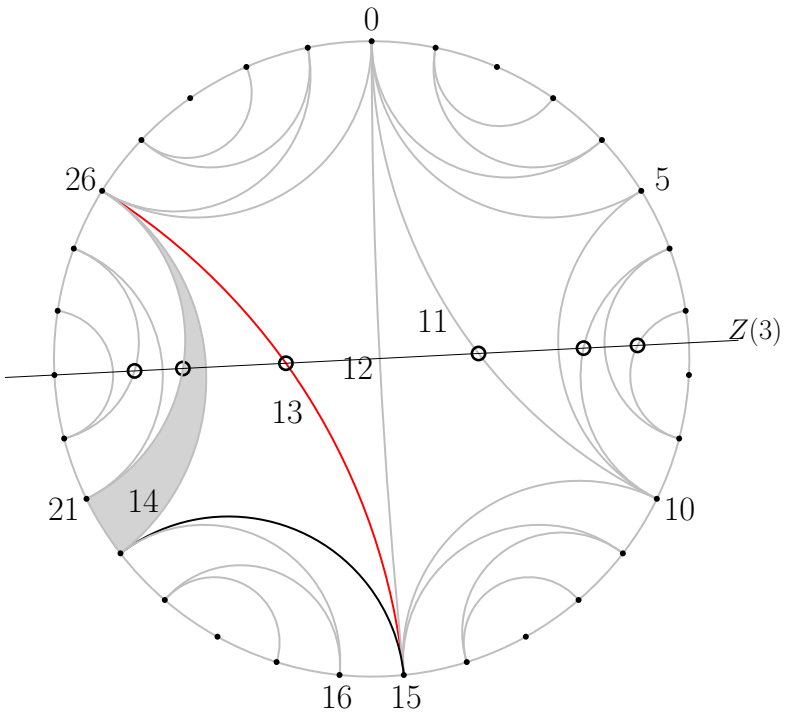


$$u=2$$

$$\begin{aligned}
Y_1^{(2)}(13) = {} & Y_1^{(2)}(1)^{-1}(1+Y_4^{(1)}(2)^{-1})^{-1}(1+Y_3^{(1)}(3)^{-1})^{-1} \\
& \times (1+Y_2^{(1)}(4)^{-1})^{-1}(1+Y_1^{(1)}(5)^{-1})^{-1} \\
& \times (1+Y_2^{(2)}(7)^{-1})^{-1}(1+Y_1^{(1)}(7)) \\
& \times (1+Y_1^{(1)}(9)^{-1})^{-1}(1+Y_2^{(1)}(10)^{-1})^{-1} \\
& \times (1+Y_3^{(1)}(11)^{-1})^{-1}(1+Y_4^{(1)}(12)^{-1})^{-1}.
\end{aligned}$$

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(2,1)$:

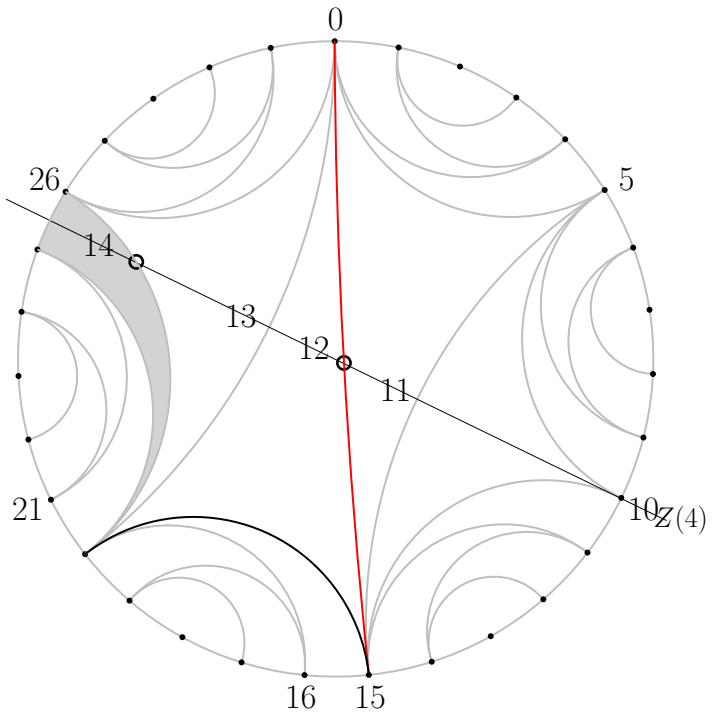


$$u=3$$

$$\begin{aligned}
Y_1^{(2)}(13) = {} & Y_1^{(2)}(1)^{-1}(1+Y_4^{(1)}(2)^{-1})^{-1}(1+Y_3^{(1)}(3)^{-1})^{-1} \\
& \times (1+Y_2^{(1)}(4)^{-1})^{-1}(1+Y_1^{(1)}(5)^{-1})^{-1} \\
& \times (1+Y_2^{(2)}(7)^{-1})^{-1}(1+Y_1^{(1)}(7)) \\
& \times (1+Y_1^{(1)}(9)^{-1})^{-1}(1+Y_2^{(1)}(10)^{-1})^{-1} \\
& \times (1+Y_3^{(1)}(11)^{-1})^{-1}(1+Y_4^{(1)}(12)^{-1})^{-1}.
\end{aligned}$$

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(2,1)$:

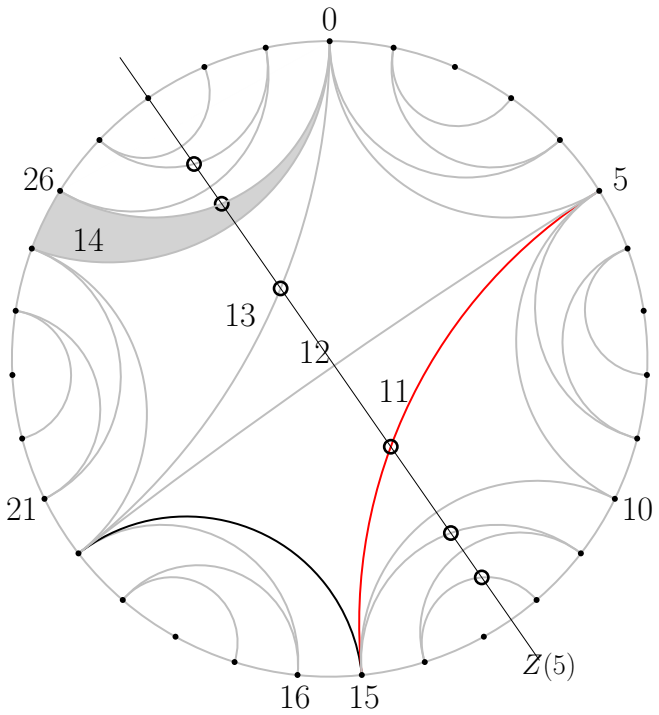


$$u=4$$

$$\begin{aligned}
Y_1^{(2)}(13) = {} & Y_1^{(2)}(1)^{-1}(1+Y_4^{(1)}(2)^{-1})^{-1}(1+Y_3^{(1)}(3)^{-1})^{-1} \\
& \times (1+Y_2^{(1)}(4)^{-1})^{-1}(1+Y_1^{(1)}(5)^{-1})^{-1} \\
& \times (1+Y_2^{(2)}(7)^{-1})^{-1}(1+Y_1^{(1)}(7)) \\
& \times (1+Y_1^{(1)}(9)^{-1})^{-1}(1+Y_2^{(1)}(10)^{-1})^{-1} \\
& \times (1+Y_3^{(1)}(11)^{-1})^{-1}(1+Y_4^{(1)}(12)^{-1})^{-1}.
\end{aligned}$$

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(2,1)$:

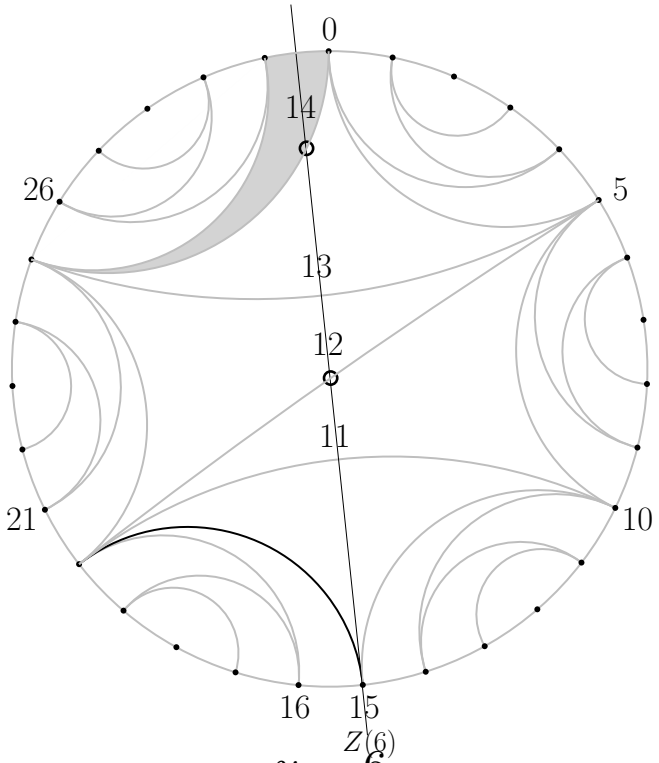


$$u=5$$

$$\begin{aligned}
Y_1^{(2)}(13) = {} & Y_1^{(2)}(1)^{-1}(1+Y_4^{(1)}(2)^{-1})^{-1}(1+Y_3^{(1)}(3)^{-1})^{-1} \\
& \times (1+Y_2^{(1)}(4)^{-1})^{-1}(1+Y_1^{(1)}(5)^{-1})^{-1} \\
& \times (1+Y_2^{(2)}(7)^{-1})^{-1}(1+Y_1^{(1)}(7)) \\
& \times (1+Y_1^{(1)}(9)^{-1})^{-1}(1+Y_2^{(1)}(10)^{-1})^{-1} \\
& \times (1+Y_3^{(1)}(11)^{-1})^{-1}(1+Y_4^{(1)}(12)^{-1})^{-1}.
\end{aligned}$$

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(2,1)$:

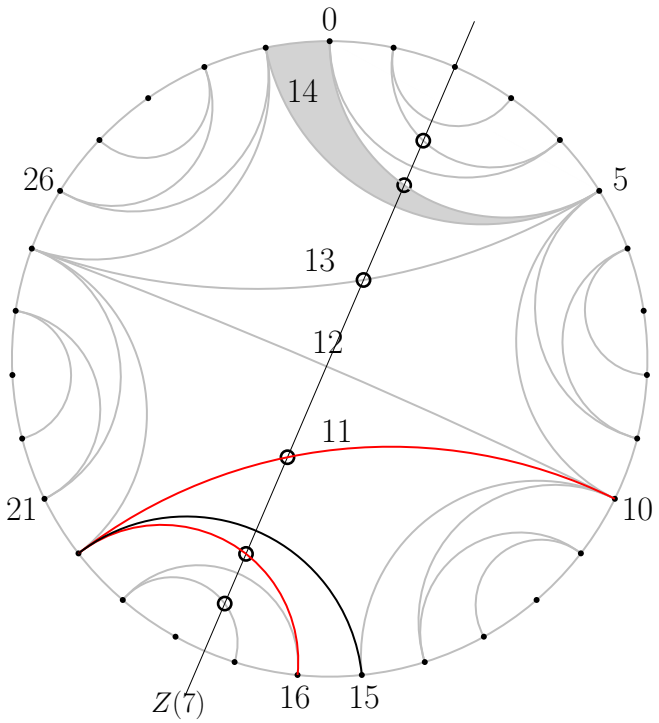


$u=6$

$$
\begin{aligned}
Y_1^{(2)}(13) = {} & Y_1^{(2)}(1)^{-1}(1+Y_4^{(1)}(2)^{-1})^{-1}(1+Y_3^{(1)}(3)^{-1})^{-1} \\
& \times (1+Y_2^{(1)}(4)^{-1})^{-1}(1+Y_1^{(1)}(5)^{-1})^{-1} \\
& \times (1+Y_2^{(2)}(7)^{-1})^{-1}(1+Y_1^{(1)}(7)) \\
& \times (1+Y_1^{(1)}(9)^{-1})^{-1}(1+Y_2^{(1)}(10)^{-1})^{-1} \\
& \times (1+Y_3^{(1)}(11)^{-1})^{-1}(1+Y_4^{(1)}(12)^{-1})^{-1}.
\end{aligned}
$$

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(2,1)$:

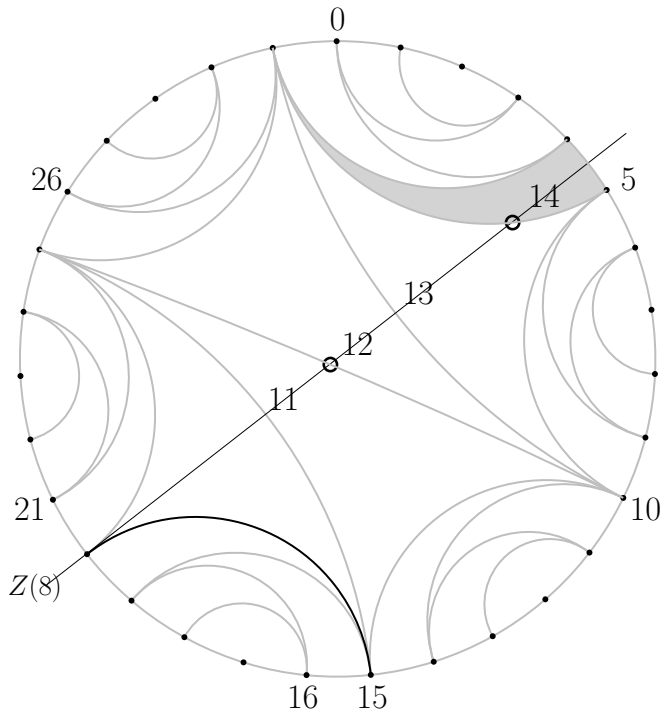


$$u=7$$

$$\begin{aligned}
Y_1^{(2)}(13) = {} & Y_1^{(2)}(1)^{-1}(1+Y_4^{(1)}(2)^{-1})^{-1}(1+Y_3^{(1)}(3)^{-1})^{-1} \\
& \times (1+Y_2^{(1)}(4)^{-1})^{-1}(1+Y_1^{(1)}(5)^{-1})^{-1} \\
& \times (1+Y_2^{(2)}(7)^{-1})^{-1}(1+Y_1^{(1)}(7)) \\
& \times (1+Y_1^{(1)}(9)^{-1})^{-1}(1+Y_2^{(1)}(10)^{-1})^{-1} \\
& \times (1+Y_3^{(1)}(11)^{-1})^{-1}(1+Y_4^{(1)}(12)^{-1})^{-1}.
\end{aligned}$$

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(2,1)$:

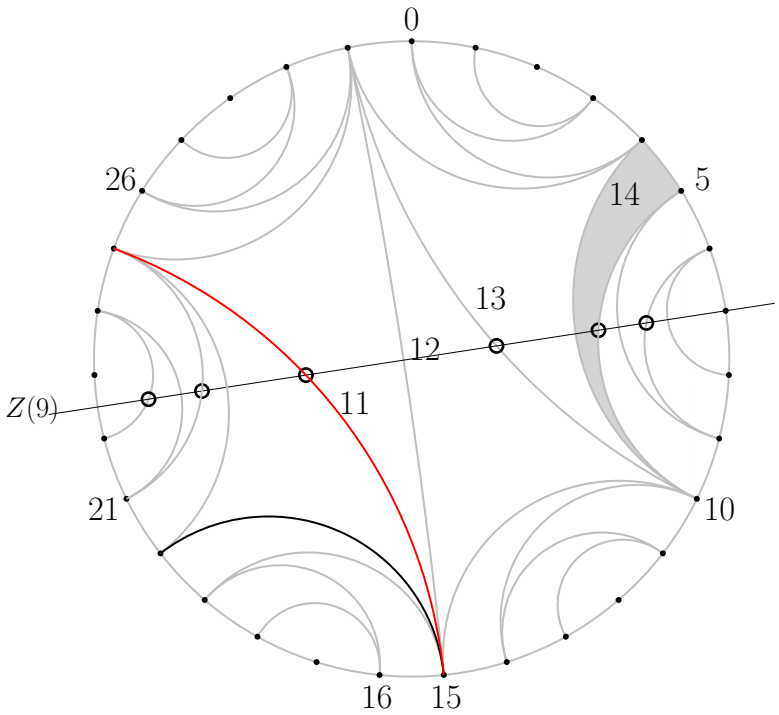


$$u=8$$

$$\begin{aligned}
Y_1^{(2)}(13) = {} & Y_1^{(2)}(1)^{-1}(1+Y_4^{(1)}(2)^{-1})^{-1}(1+Y_3^{(1)}(3)^{-1})^{-1} \\
& \times (1+Y_2^{(1)}(4)^{-1})^{-1}(1+Y_1^{(1)}(5)^{-1})^{-1} \\
& \times (1+Y_2^{(2)}(7)^{-1})^{-1}(1+Y_1^{(1)}(7)) \\
& \times (1+Y_1^{(1)}(9)^{-1})^{-1}(1+Y_2^{(1)}(10)^{-1})^{-1} \\
& \times (1+Y_3^{(1)}(11)^{-1})^{-1}(1+Y_4^{(1)}(12)^{-1})^{-1}.
\end{aligned}$$

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(2,1)$:

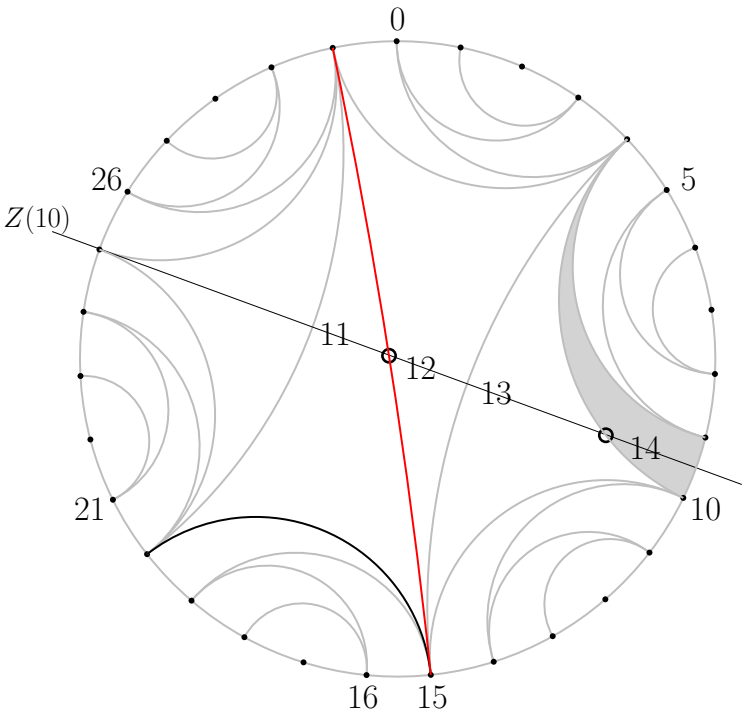


$$u=9$$

$$\begin{aligned}
Y_1^{(2)}(13) = {} & Y_1^{(2)}(1)^{-1}(1+Y_4^{(1)}(2)^{-1})^{-1}(1+Y_3^{(1)}(3)^{-1})^{-1} \\
& \times (1+Y_2^{(1)}(4)^{-1})^{-1}(1+Y_1^{(1)}(5)^{-1})^{-1} \\
& \times (1+Y_2^{(2)}(7)^{-1})^{-1}(1+Y_1^{(1)}(7)) \\
& \times (1+Y_1^{(1)}(9)^{-1})^{-1}(1+Y_2^{(1)}(10)^{-1})^{-1} \\
& \times (1+Y_3^{(1)}(11)^{-1})^{-1}(1+Y_4^{(1)}(12)^{-1})^{-1}.
\end{aligned}$$

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(2,1)$:

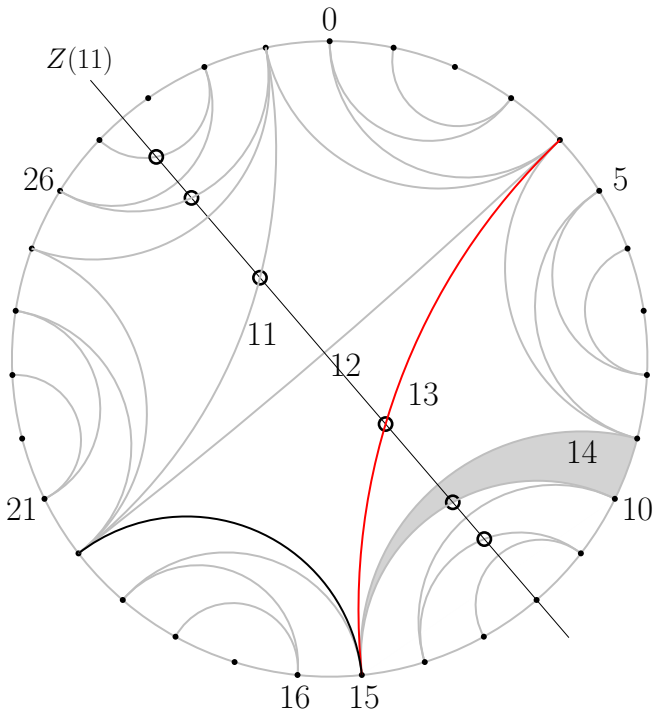


$$u=10$$

$$\begin{aligned}
Y_1^{(2)}(13) = {} & Y_1^{(2)}(1)^{-1}(1+Y_4^{(1)}(2)^{-1})^{-1}(1+Y_3^{(1)}(3)^{-1})^{-1} \\
& \times (1+Y_2^{(1)}(4)^{-1})^{-1}(1+Y_1^{(1)}(5)^{-1})^{-1} \\
& \times (1+Y_2^{(2)}(7)^{-1})^{-1}(1+Y_1^{(1)}(7)) \\
& \times (1+Y_1^{(1)}(9)^{-1})^{-1}(1+Y_2^{(1)}(10)^{-1})^{-1} \\
& \times (1+Y_3^{(1)}(11)^{-1})^{-1}(1+Y_4^{(1)}(12)^{-1})^{-1}.
\end{aligned}$$

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(2,1)$:

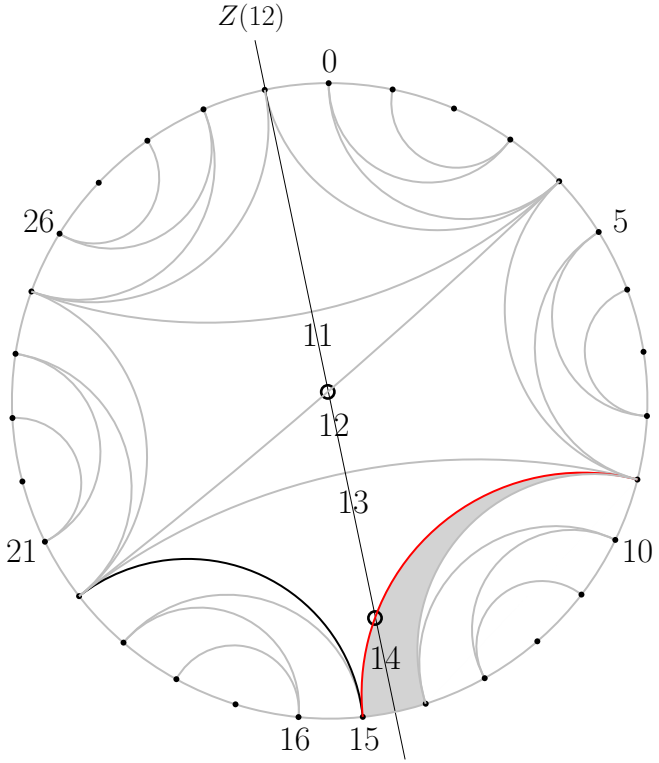


$$u=11$$

$$\begin{aligned}
Y_1^{(2)}(13) = {} & Y_1^{(2)}(1)^{-1}(1+Y_4^{(1)}(2)^{-1})^{-1}(1+Y_3^{(1)}(3)^{-1})^{-1} \\
& \times (1+Y_2^{(1)}(4)^{-1})^{-1}(1+Y_1^{(1)}(5)^{-1})^{-1} \\
& \times (1+Y_2^{(2)}(7)^{-1})^{-1}(1+Y_1^{(1)}(7)) \\
& \times (1+Y_1^{(1)}(9)^{-1})^{-1}(1+Y_2^{(1)}(10)^{-1})^{-1} \\
& \times (1+Y_3^{(1)}(11)^{-1})^{-1}(1+Y_4^{(1)}(12)^{-1})^{-1}.
\end{aligned}$$

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(2,1)$:

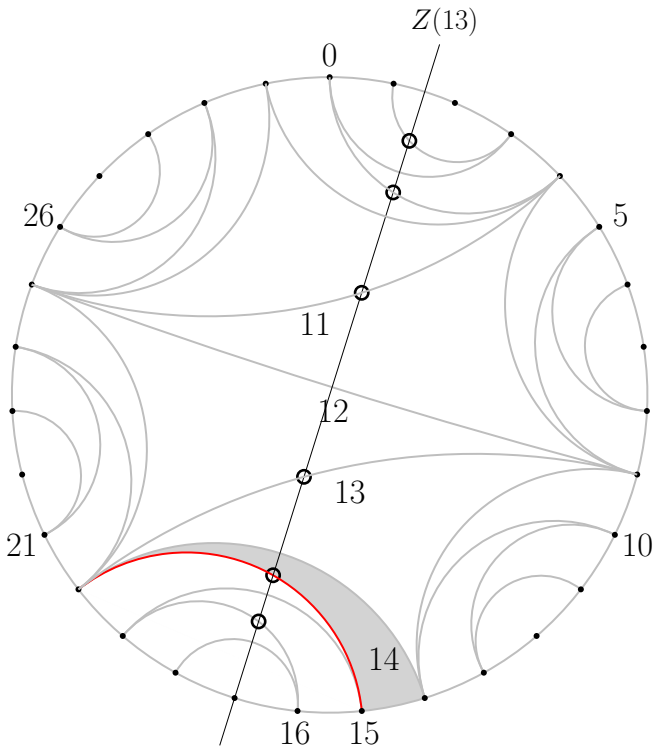


$u=12$

$$
\begin{aligned}
Y_1^{(2)}(13) = {} & Y_1^{(2)}(1)^{-1}(1+Y_4^{(1)}(2)^{-1})^{-1}(1+Y_3^{(1)}(3)^{-1})^{-1} \\
& \times (1+Y_2^{(1)}(4)^{-1})^{-1}(1+Y_1^{(1)}(5)^{-1})^{-1} \\
& \times (1+Y_2^{(2)}(7)^{-1})^{-1}(1+Y_1^{(1)}(7)) \\
& \times (1+Y_1^{(1)}(9)^{-1})^{-1}(1+Y_2^{(1)}(10)^{-1})^{-1} \\
& \times (1+Y_3^{(1)}(11)^{-1})^{-1}(1+Y_4^{(1)}(12)^{-1})^{-1}.
\end{aligned}
$$

Example 2: $\mathbb{Y}_{\mathrm{RSG}}(6,4)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(2,1)$:



$$u=13$$

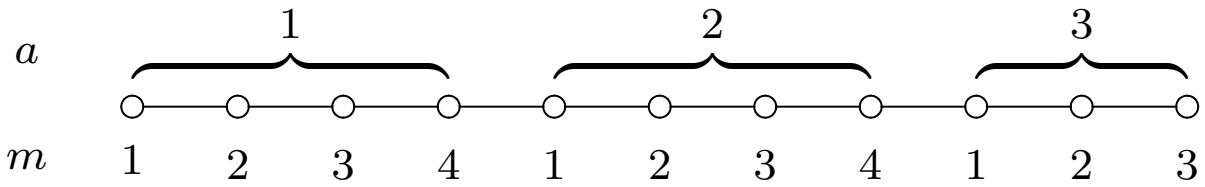
$$\begin{aligned}
Y_1^{(2)}(13) = {} & Y_1^{(2)}(1)^{-1}(1+Y_4^{(1)}(2)^{-1})^{-1}(1+Y_3^{(1)}(3)^{-1})^{-1} \\
& \times (1+Y_2^{(1)}(4)^{-1})^{-1}(1+Y_1^{(1)}(5)^{-1})^{-1} \\
& \times (1+Y_2^{(2)}(7)^{-1})^{-1}(1+Y_1^{(1)}(7)) \\
& \times (1+Y_1^{(1)}(9)^{-1})^{-1}(1+Y_2^{(1)}(10)^{-1})^{-1} \\
& \times (1+Y_3^{(1)}(11)^{-1})^{-1}(1+Y_4^{(1)}(12)^{-1})^{-1}.
\end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Y-system.

- The input data $F=3$, $(n_1,n_2,n_3)=(6,4,3)$.

$$\xi_3=\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{6}}}=\frac{25}{81},\quad (p_3,q_3)=(25,81),\quad r_3=106.$$

$2r_3=2\times 106=212$ (period of Y-system).

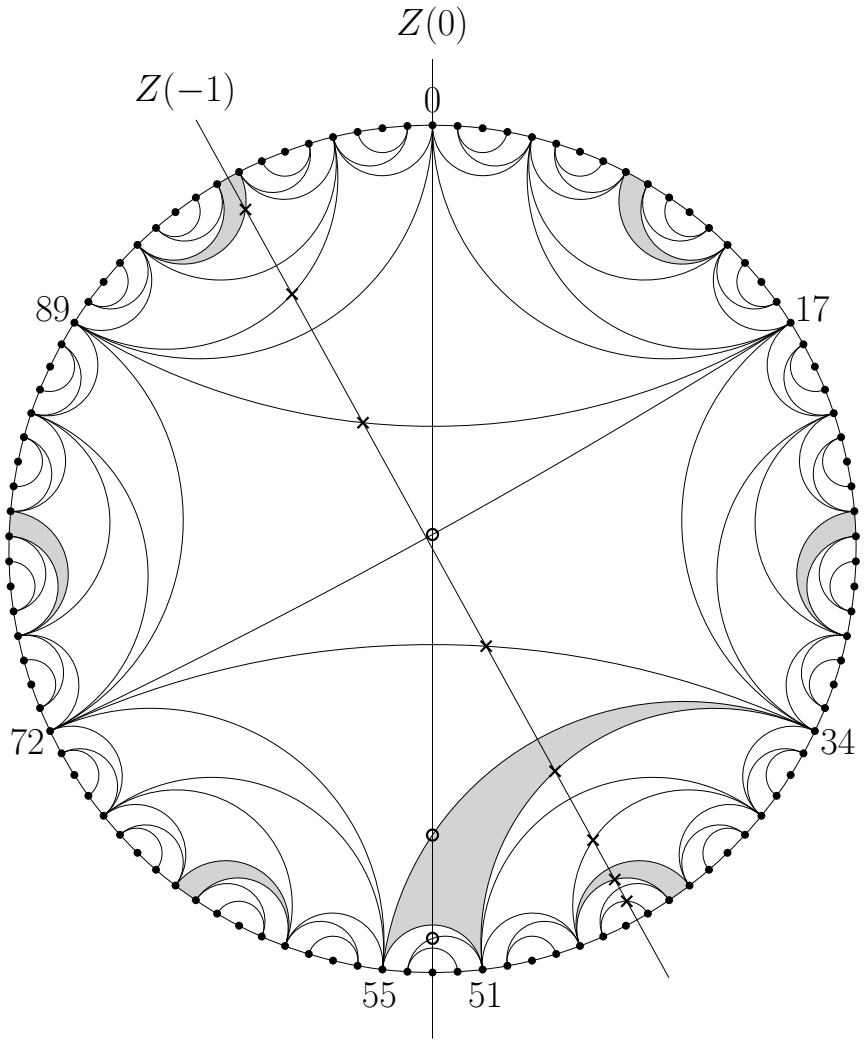


Y-variables: $Y_1^{(1)}(u),\dots,Y_4^{(1)}(u)$; $Y_1^{(2)}(u),\dots,Y_4^{(2)}(u)$; $Y_1^{(3)}(u),\dots,Y_3^{(3)}(u)$
Y-system: General relation for $(a,m)\neq(2,1),(3,1)$ and the exceptional relation for $(a,m)=(2,1)$ are the ones as before.
The exceptional relation for $(a,m)=(3,1)$:

$$\begin{aligned}Y_1^{(3)}(u-25)Y_1^{(3)}(u+25)=&(1+Y_2^{(3)}(u))(1+Y_4^{(1)}(u))\\&\times(1+Y_4^{(2)}(u-19))(1+Y_3^{(2)}(u-13))\\&\times(1+Y_2^{(2)}(u-7))(1+Y_1^{(2)}(u-1))\\&\times(1+Y_1^{(2)}(u+1))(1+Y_2^{(2)}(u+7))\\&\times(1+Y_3^{(2)}(u+13))(1+Y_4^{(2)}(u+19)),\end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Initial triangulation.

- Initial triangulation of an 106-gon.

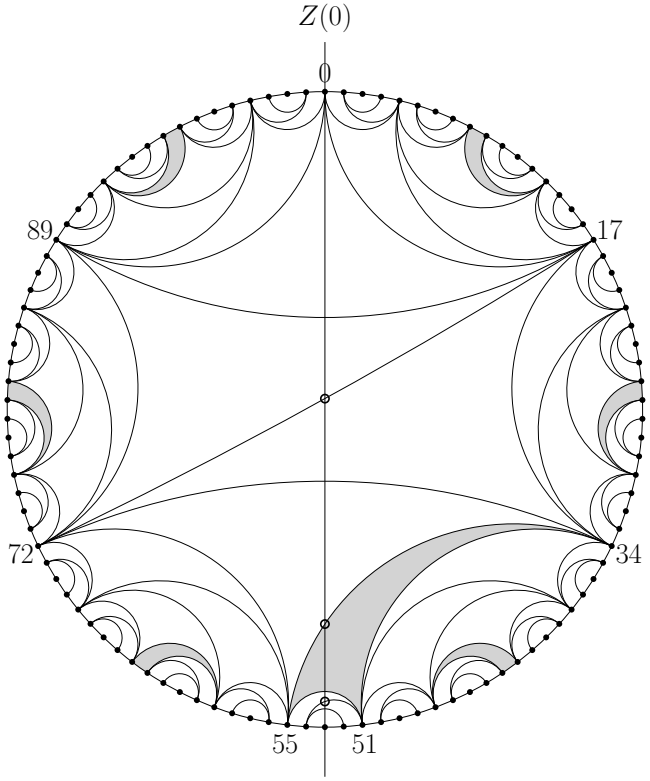


- Quasi-symmetry with respect to $Z(u)$ $(u = 0, -1)$.

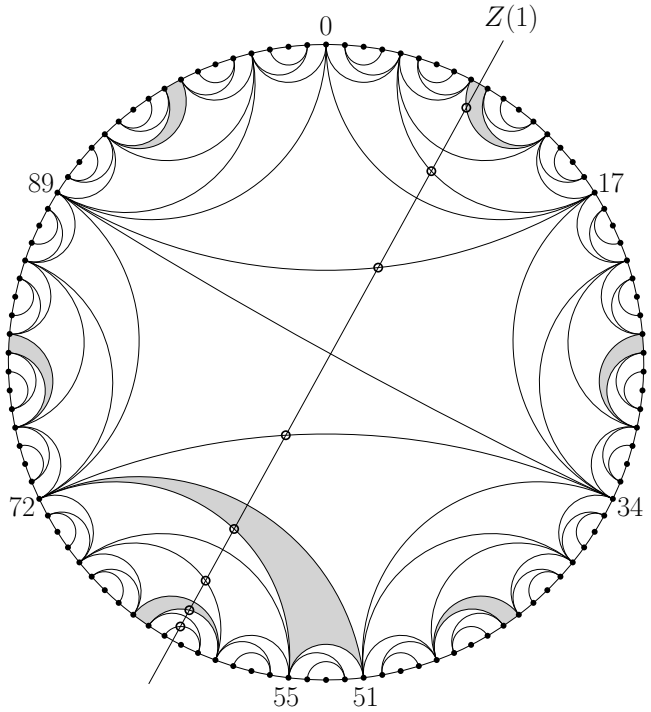
= It is symmetric w.r.t. $Z(u)$ except for the diagonals which intersects with $Z(u)$.

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Sequence of mutations.

- Consider a sequence of mutations:



$u = 0$



$u = 1$

The $\circ$'s are the forward mutation points.

- Combining two reflections, we have the rotation by 17 units.

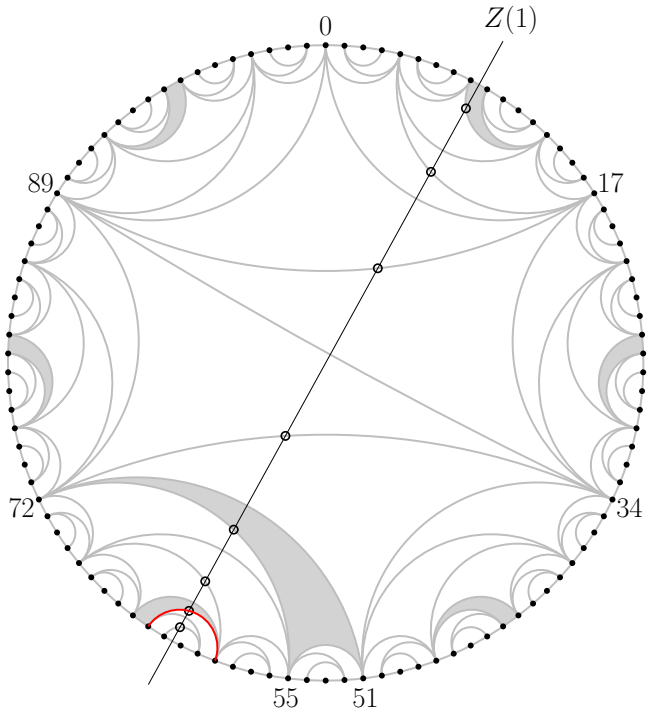
- Repeat the reflections with respect the images of these axes.

Since 106 and 17 are coprime, we have the (desired) period of 212 as an unlabeled triangulation.

- Furthermore, they realize the Y-system $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$.

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

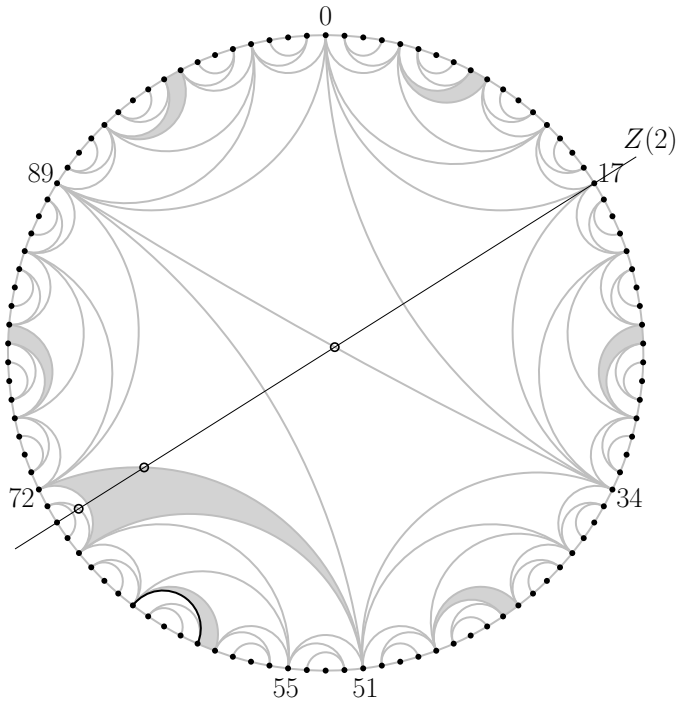


$$u=1$$

$$\begin{aligned}
Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\
& \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\
& \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\
& \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\
& \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)).
\end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

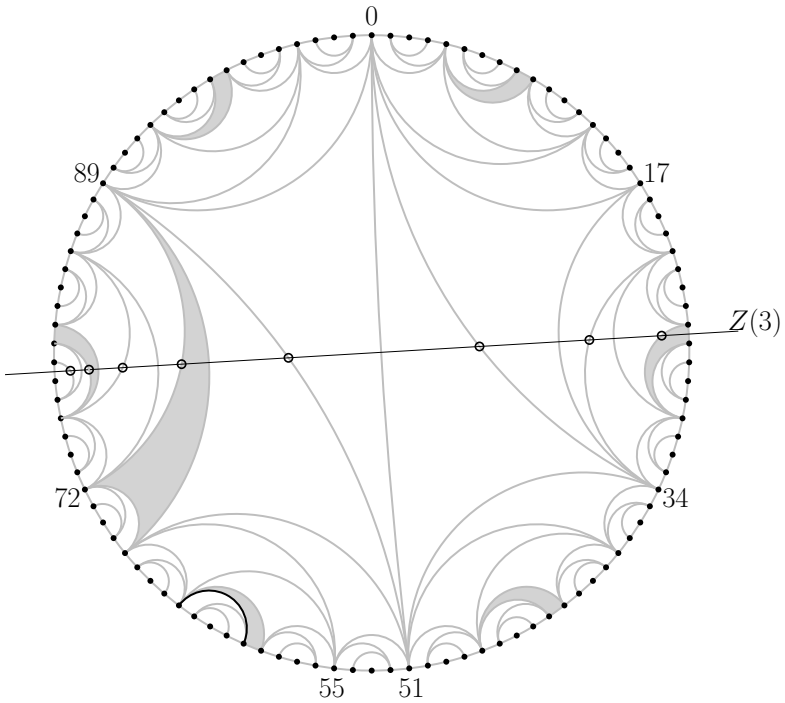


$$u=2$$

$$\begin{aligned}
Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13))\\
&\times(1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25))\\
&\times(1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26))\\
&\times(1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33))\\
&\times(1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)).
\end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

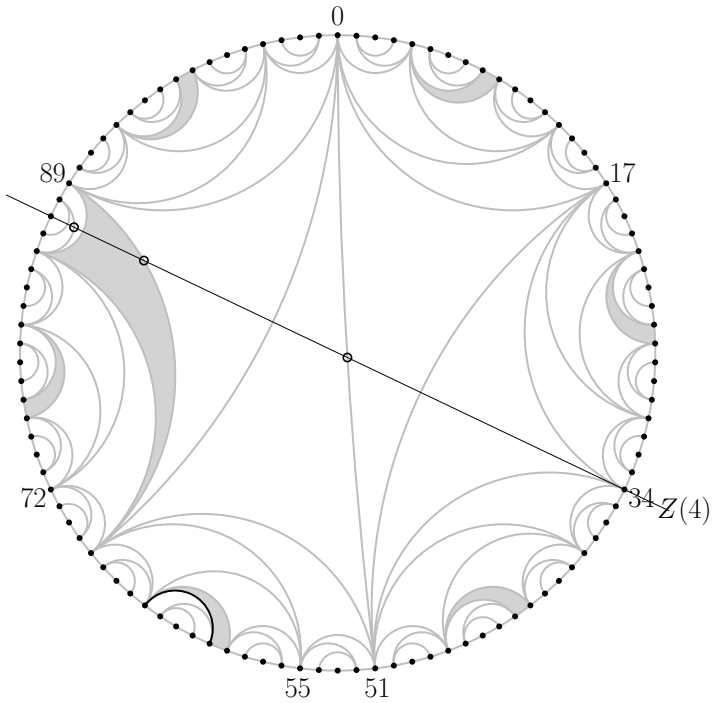


$$u=3$$

$$\begin{aligned}
Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\
& \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\
& \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\
& \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\
& \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)).
\end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

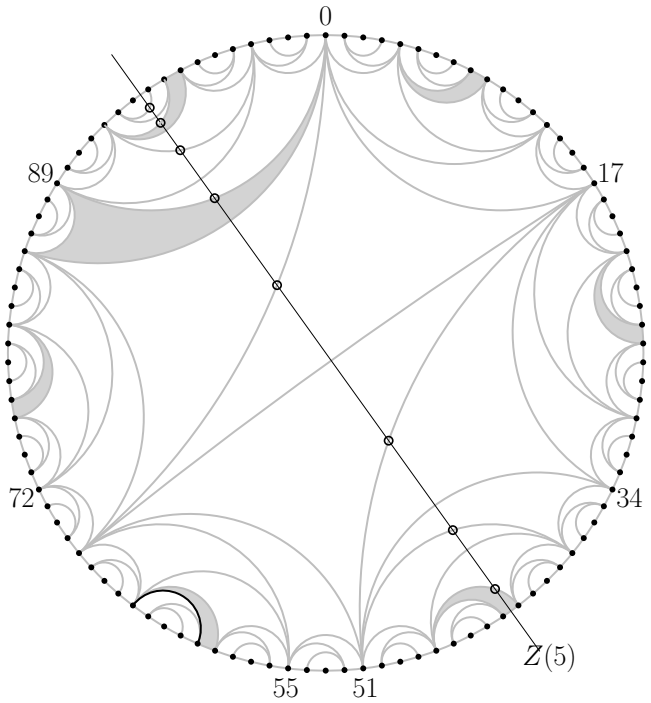


$$u=4$$

$$\begin{aligned}
Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13))\\
&\times(1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25))\\
&\times(1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26))\\
&\times(1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33))\\
&\times(1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)).
\end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

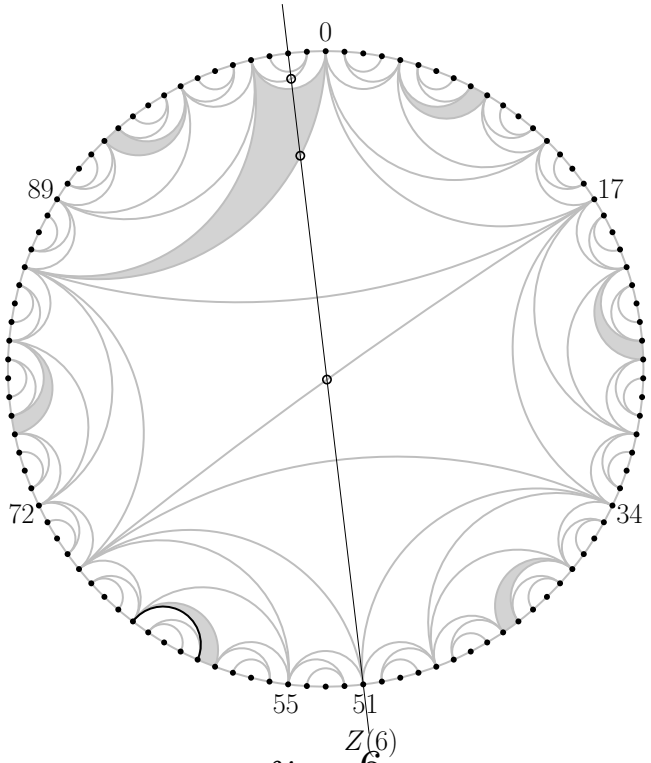


$$u=5$$

$$\begin{aligned}
Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13))\\
&\times(1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25))\\
&\times(1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26))\\
&\times(1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33))\\
&\times(1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)).
\end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

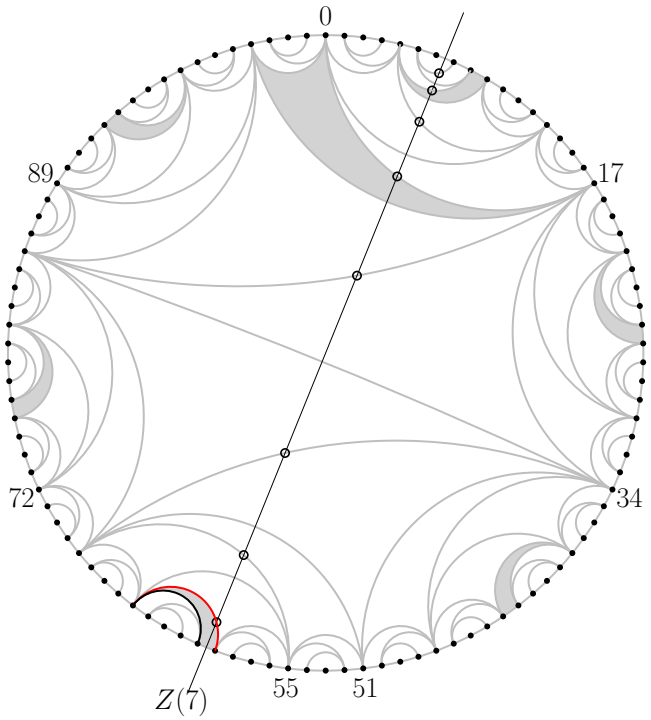


$u=6$

$$
\begin{aligned}
Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13))\\
&\times(1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25))\\
&\times(1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26))\\
&\times(1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33))\\
&\times(1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)).
\end{aligned}
$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

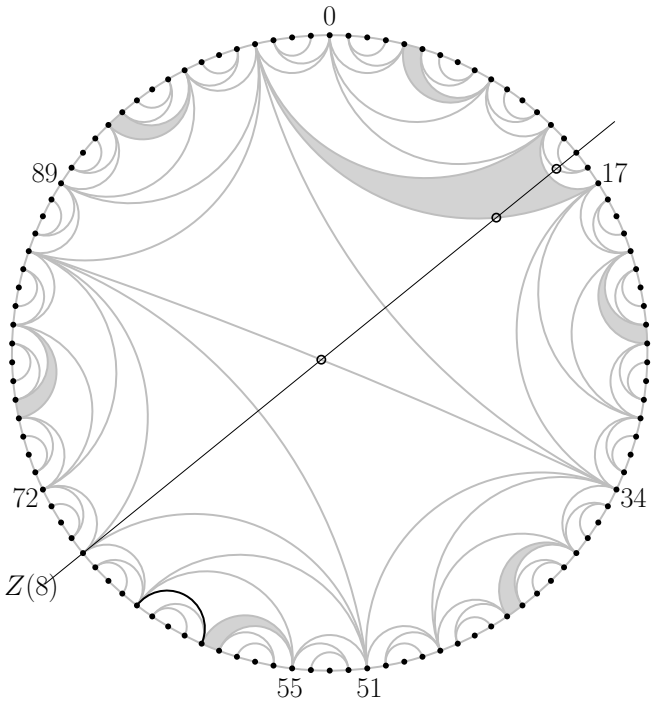


$$u=7$$

$$\begin{aligned}
Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13))\\
&\times(1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25))\\
&\times(1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26))\\
&\times(1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33))\\
&\times(1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)).
\end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

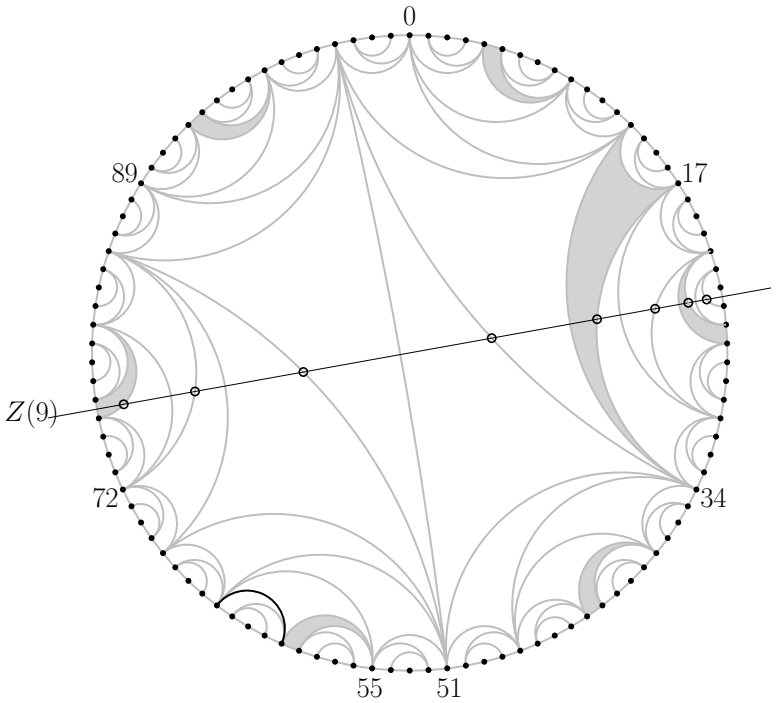


$$u=8$$

$$\begin{aligned}
Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13))\\
&\times(1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25))\\
&\times(1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26))\\
&\times(1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33))\\
&\times(1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)).
\end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

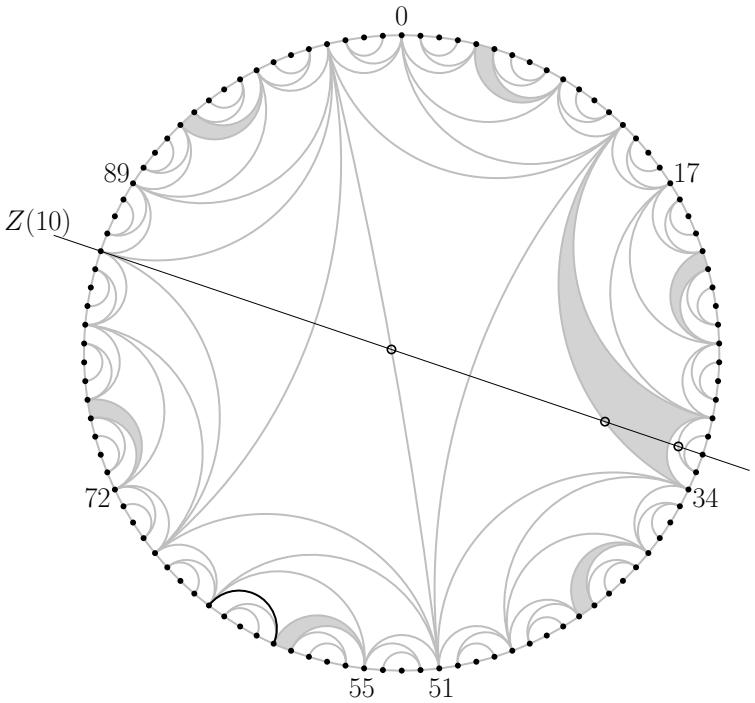


$$u=9$$

$$\begin{aligned}
Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13))\\
&\times(1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25))\\
&\times(1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26))\\
&\times(1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33))\\
&\times(1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)).
\end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

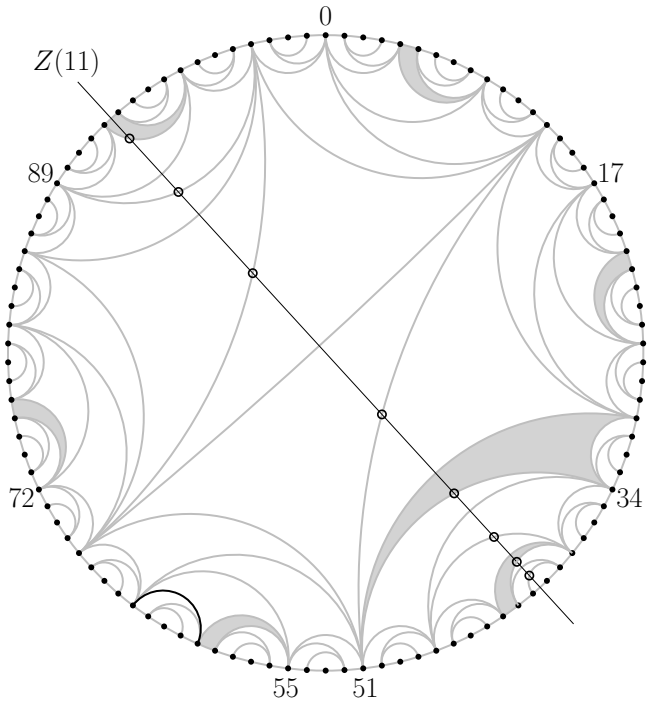


$$u = 10$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

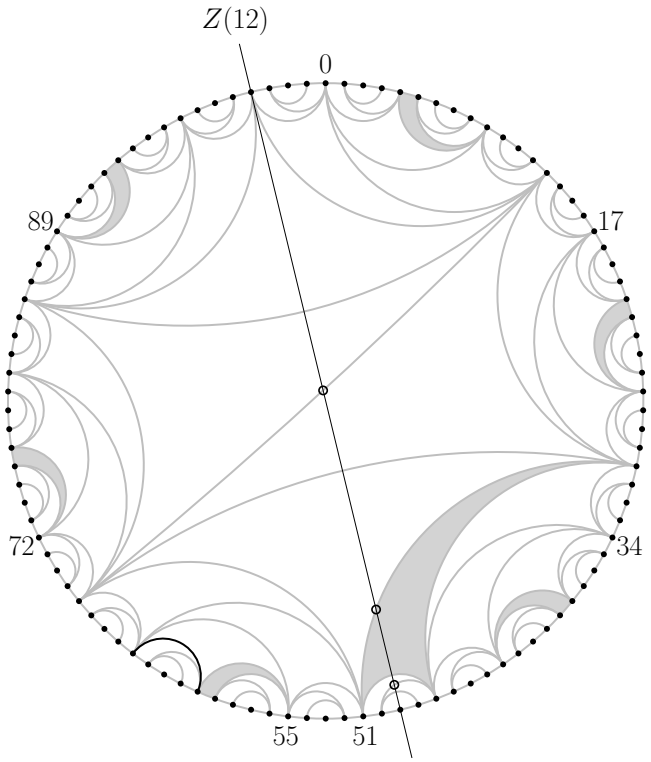


$$u = 11$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

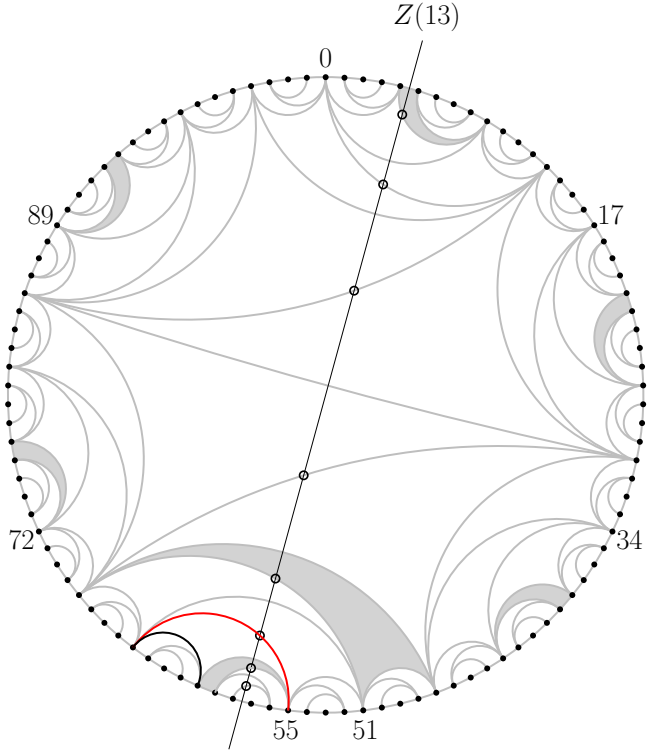


$$u=12$$

$$\begin{aligned}
Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\
& \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\
& \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\
& \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\
& \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)).
\end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

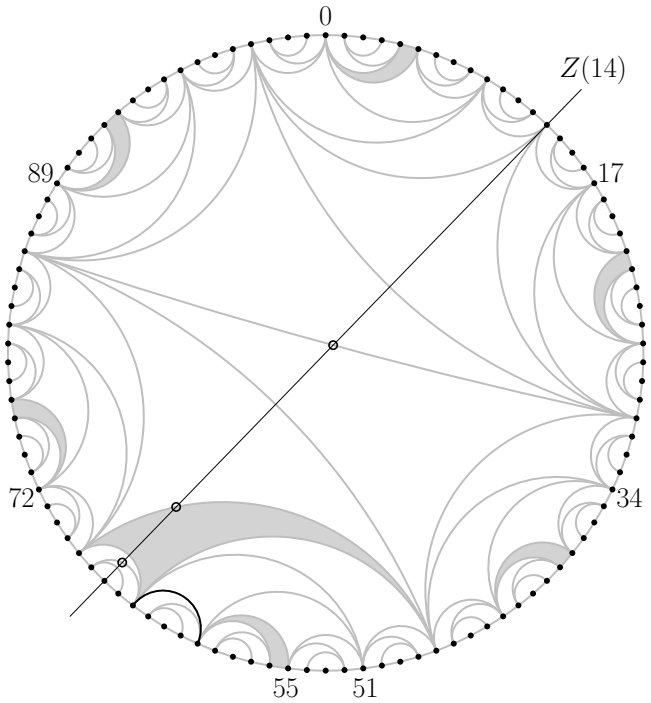


$$u = 13$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

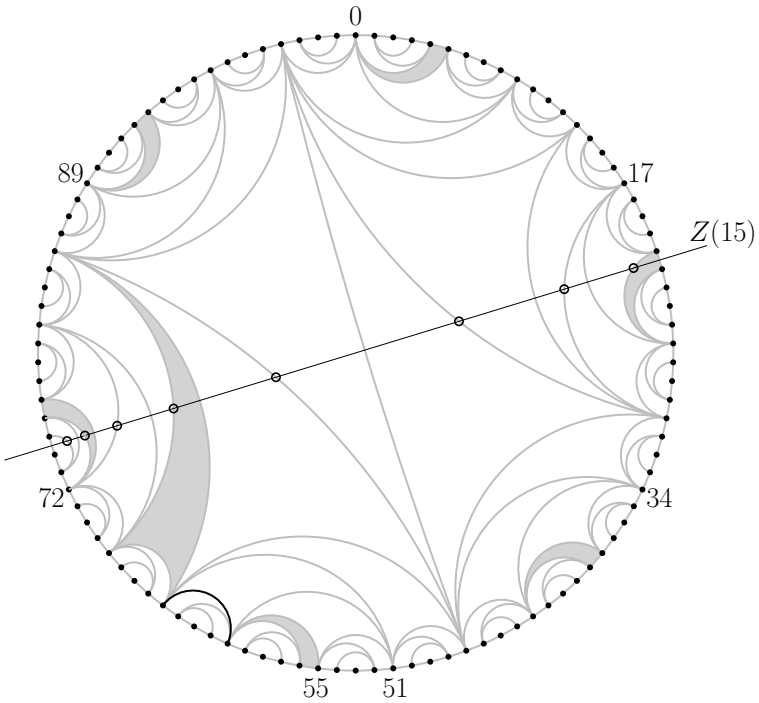


$$u = 14$$

$$\begin{aligned} Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ & \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ & \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ & \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ & \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

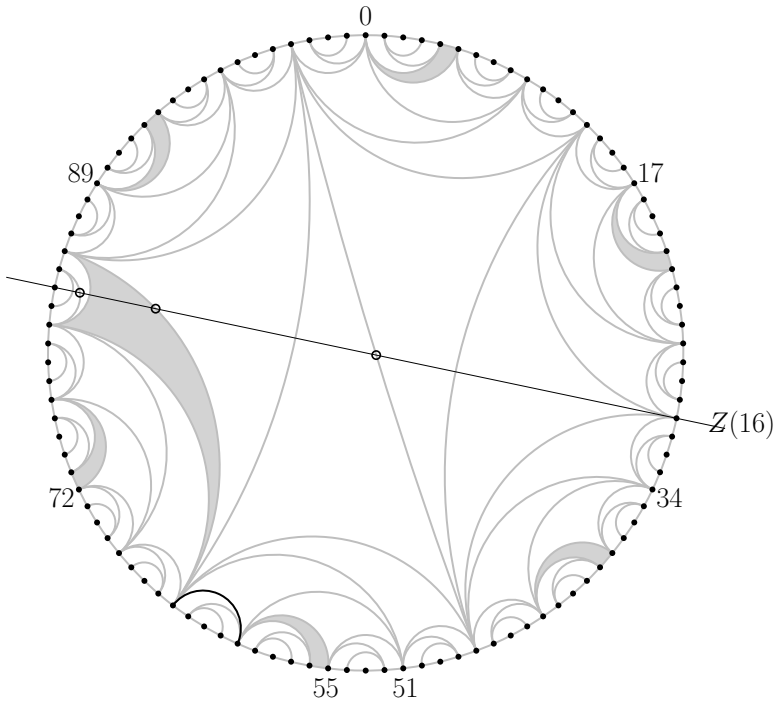


$$u = 15$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

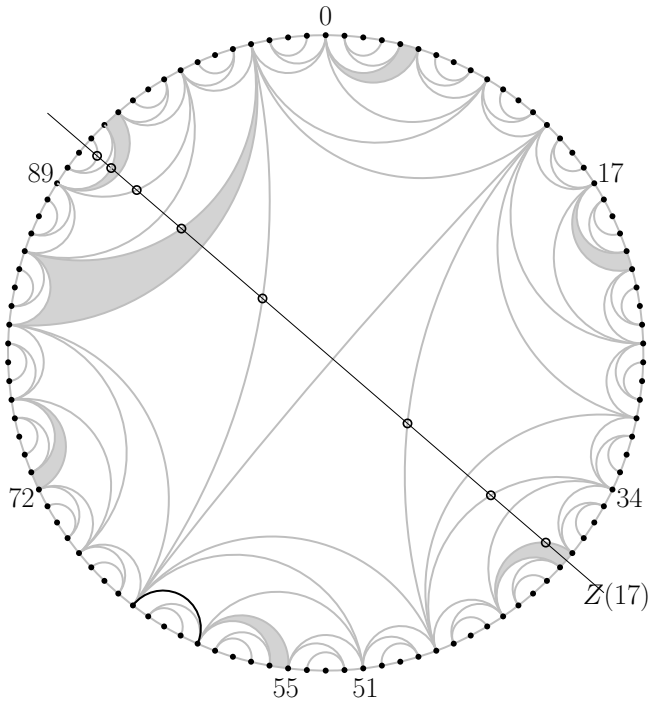


$$u=16$$

$$\begin{aligned}
Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\
& \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\
& \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\
& \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\
& \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)).
\end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

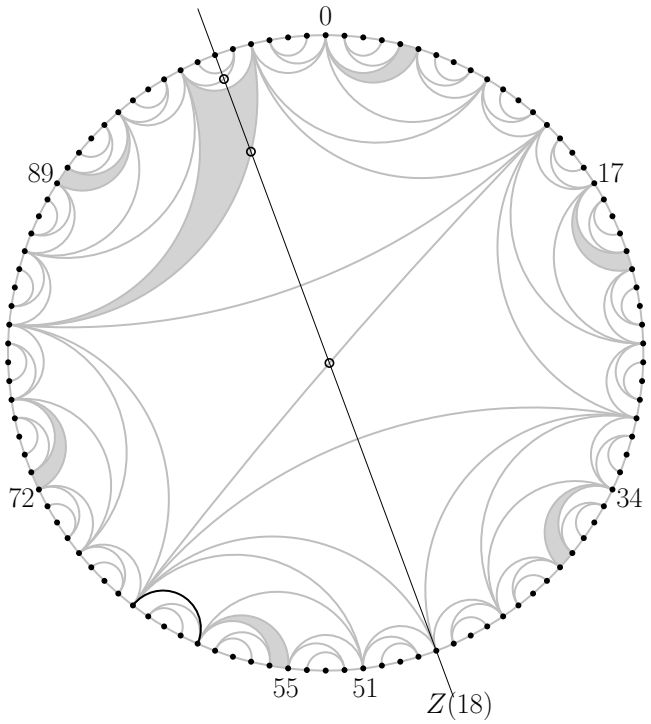


$$u = 17$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

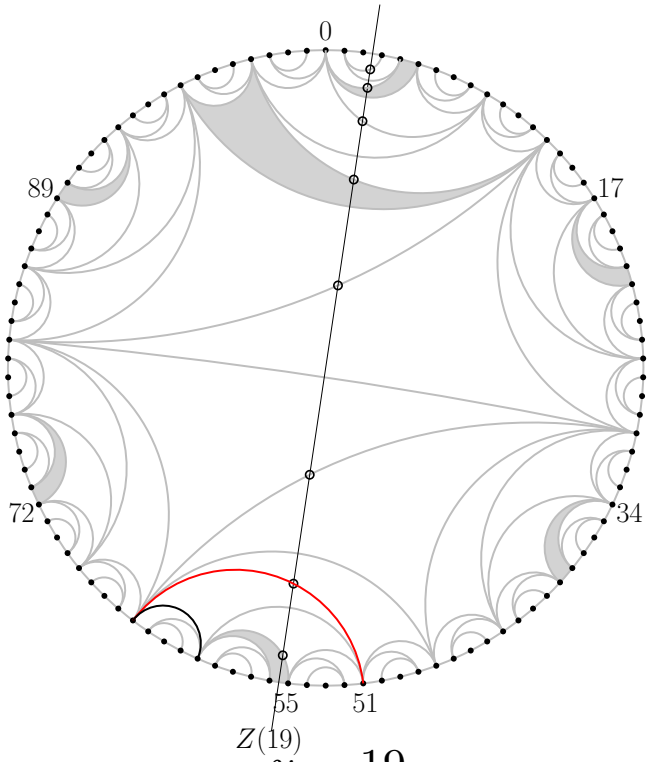


$$u = 18$$

$$\begin{aligned} Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ & \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ & \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ & \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ & \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

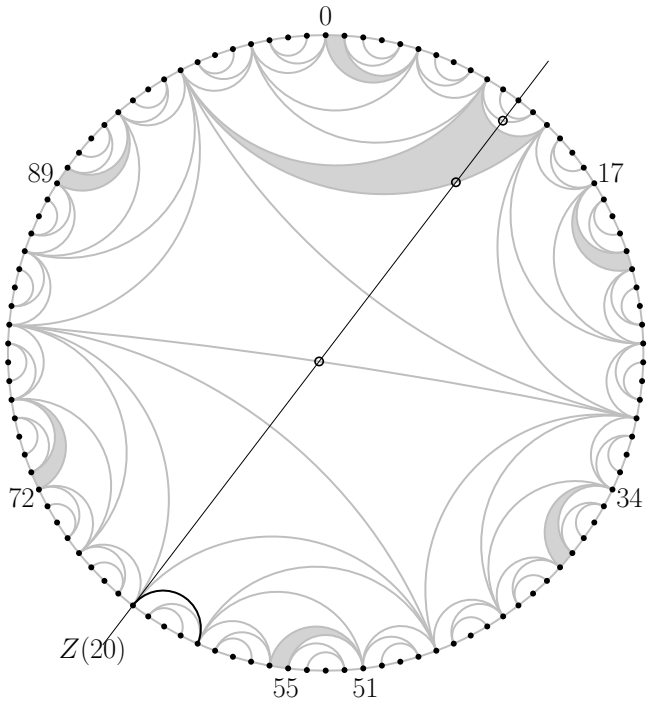


$u=19$

$$
\begin{aligned}
Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\
& \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\
& \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\
& \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\
& \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)).
\end{aligned}
$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

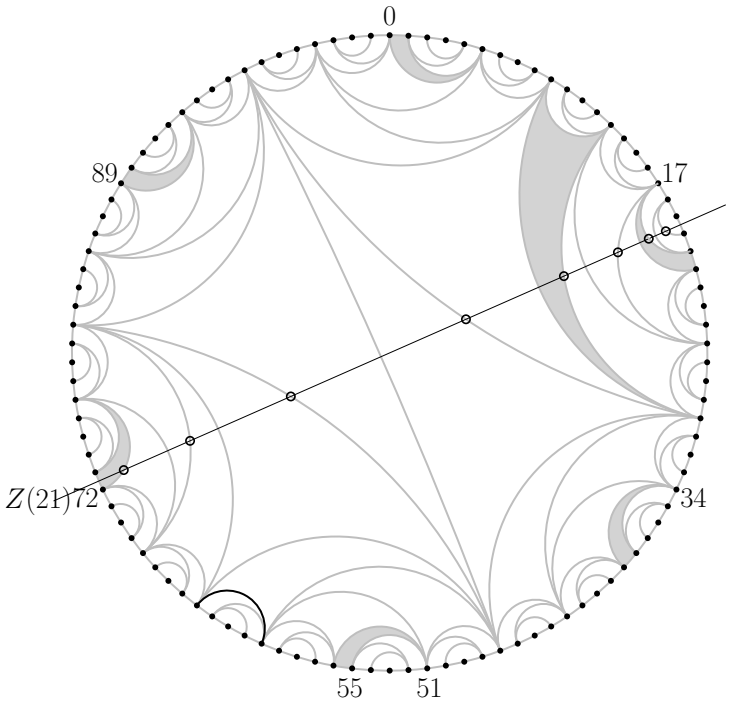


$$u = 20$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

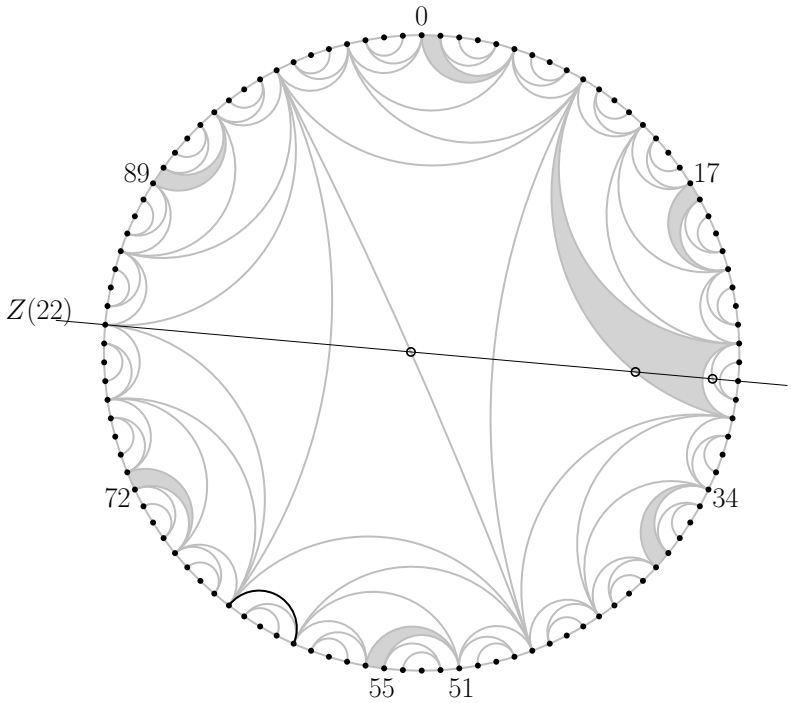


$$u = 21$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

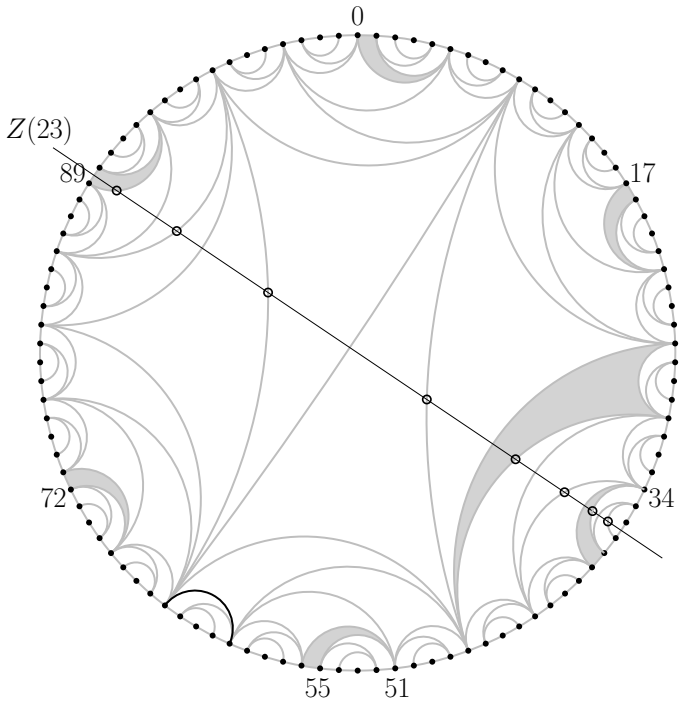


$$u = 22$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

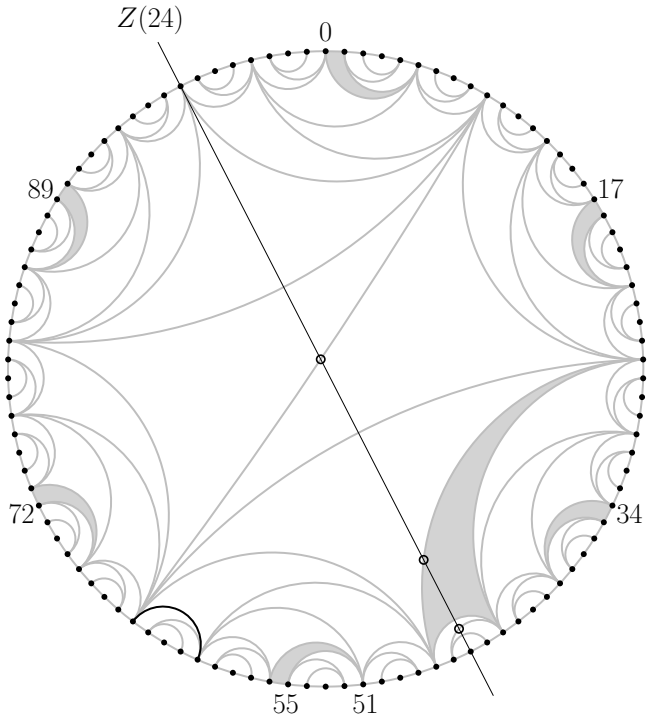


$$u = 23$$

$$\begin{aligned} Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ & \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ & \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ & \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ & \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

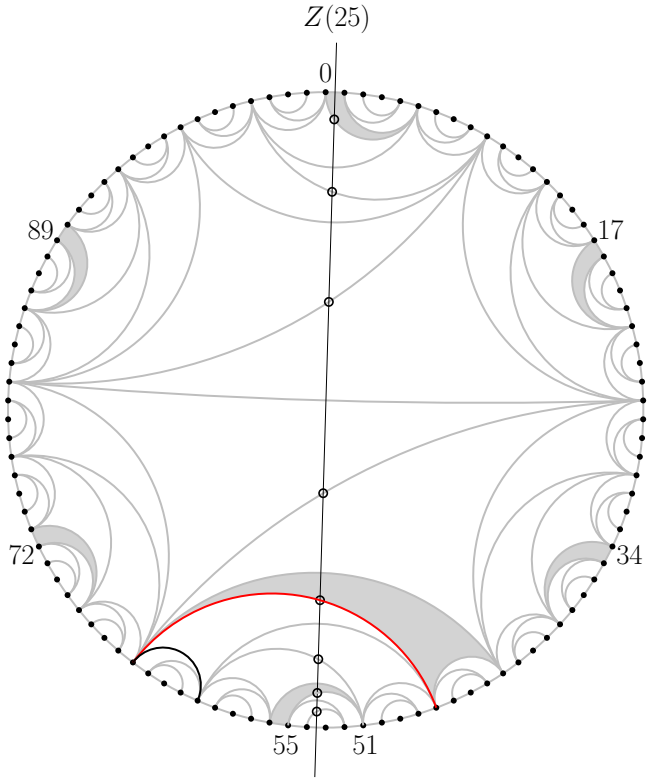


$$u = 24$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

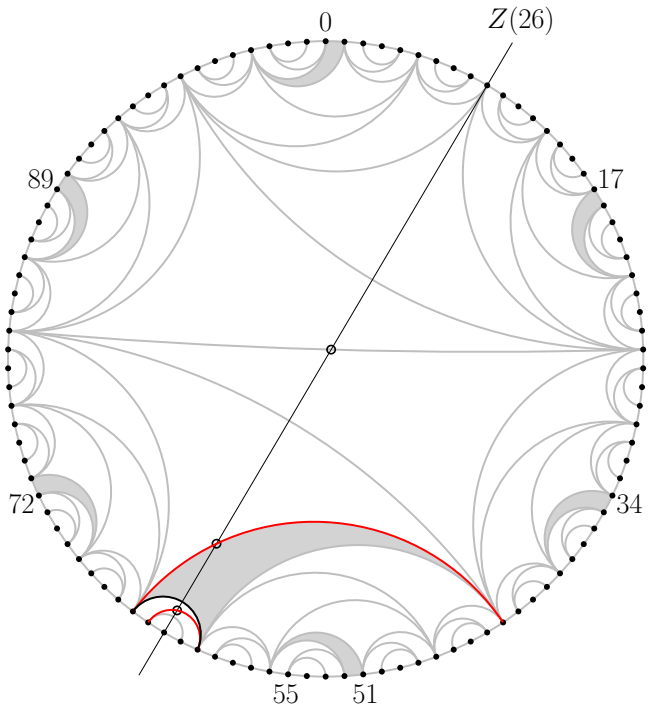


$$u = 25$$

$$\begin{aligned} Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ & \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ & \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ & \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ & \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

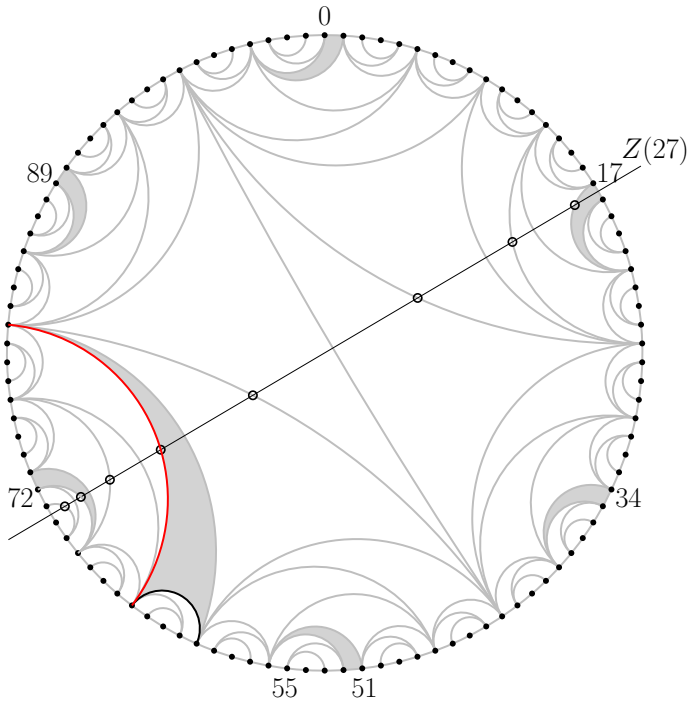


$$u = 26$$

$$\begin{aligned} Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ & \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ & \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ & \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ & \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

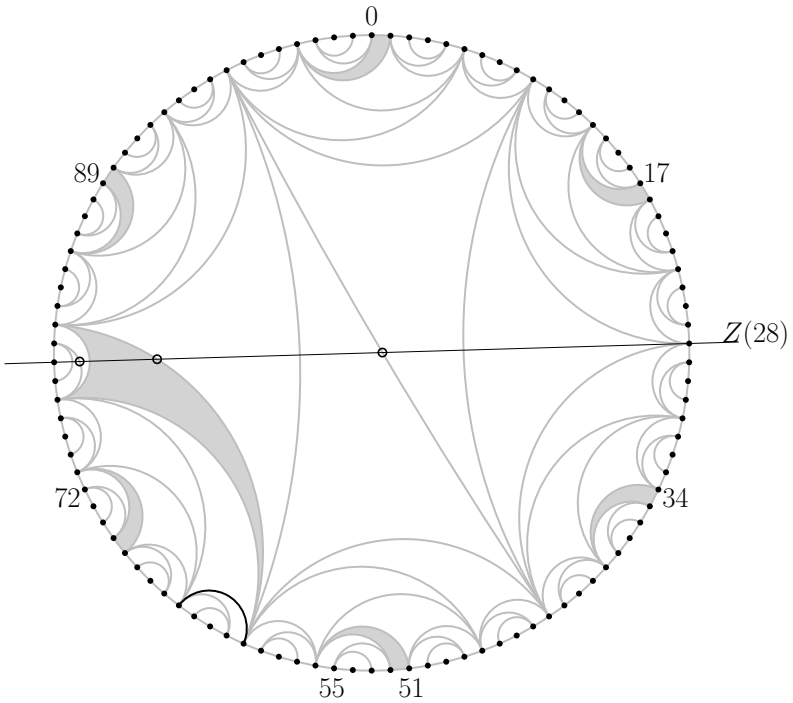


$$u = 27$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

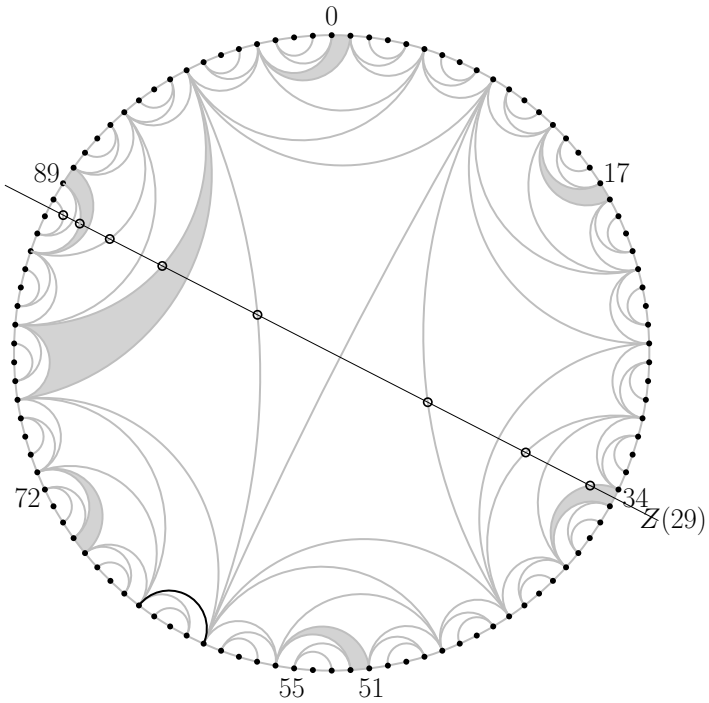


$$u = 28$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

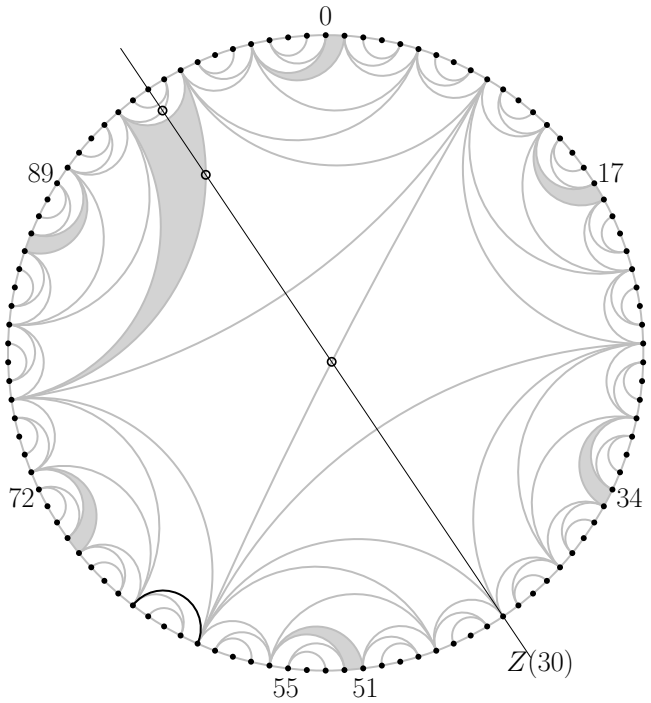


$$u = 29$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

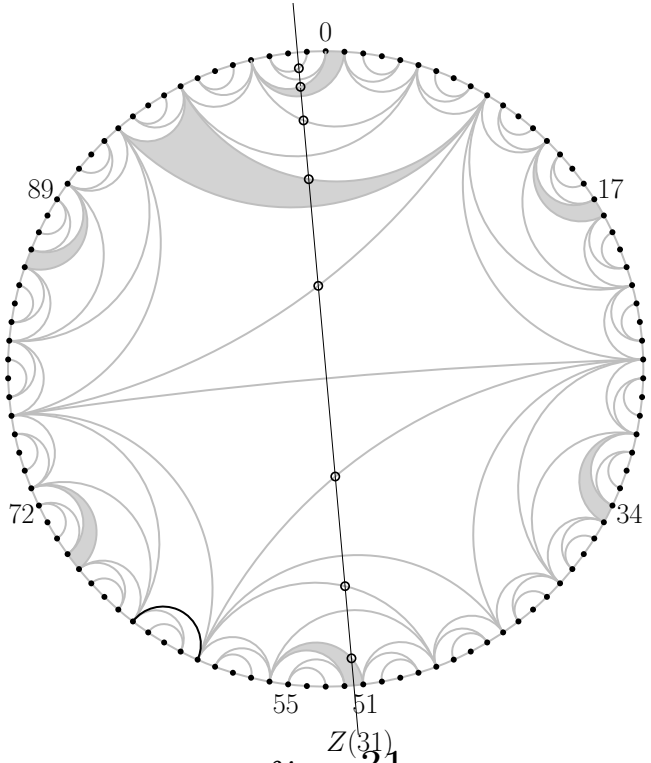


$$u = 30$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

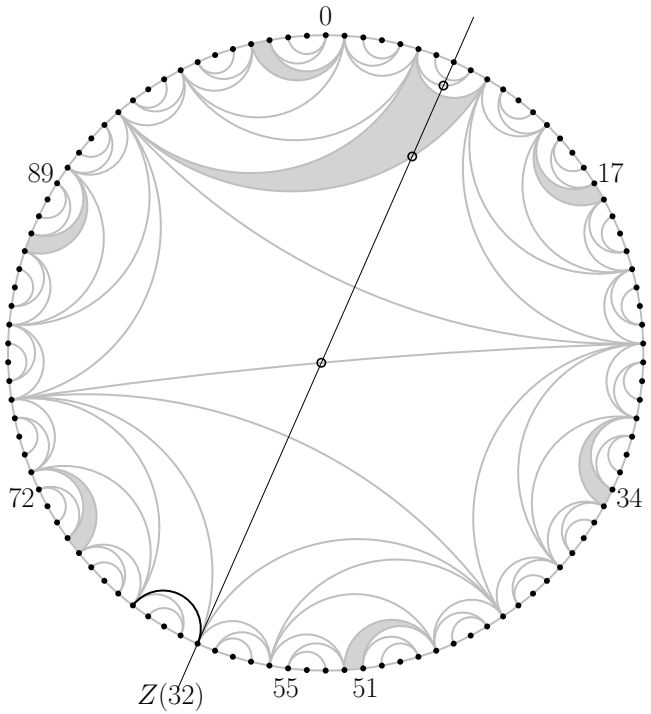


$$u = 31$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

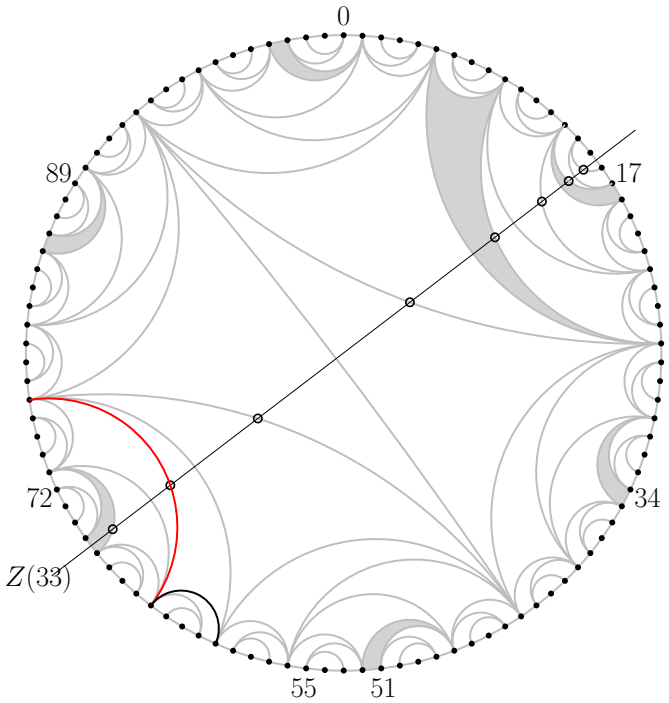


$$u = 32$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

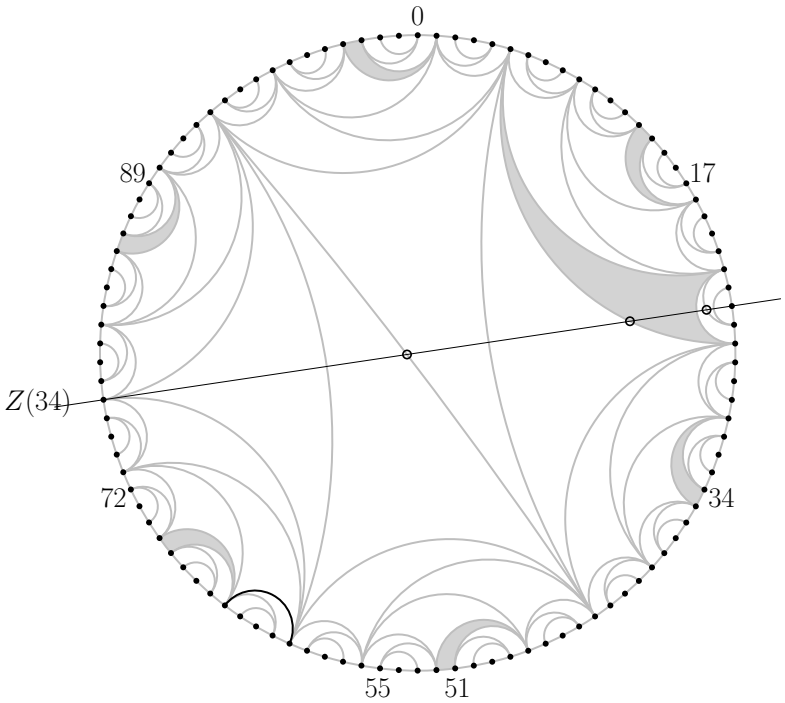


$$u = 33$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

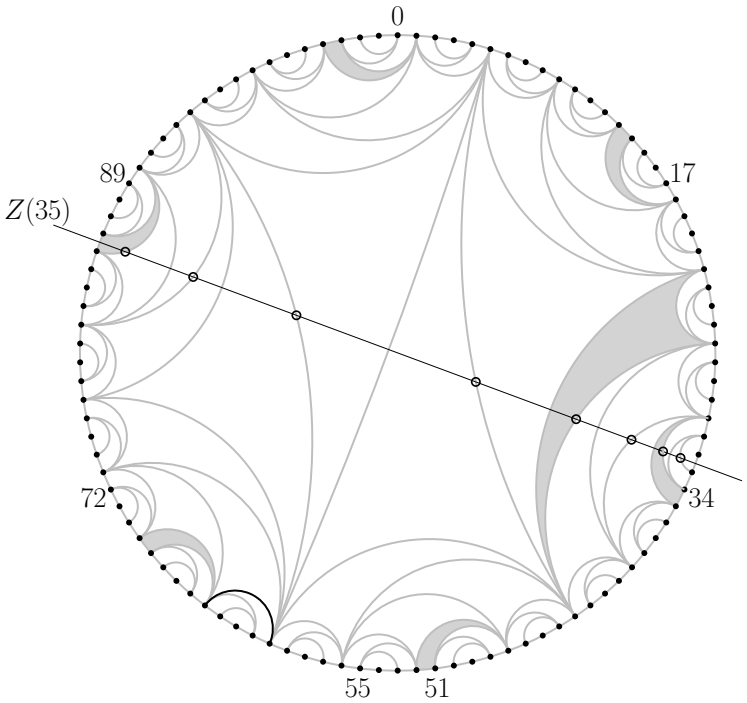


$$u = 34$$

$$\begin{aligned} Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ & \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ & \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ & \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ & \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

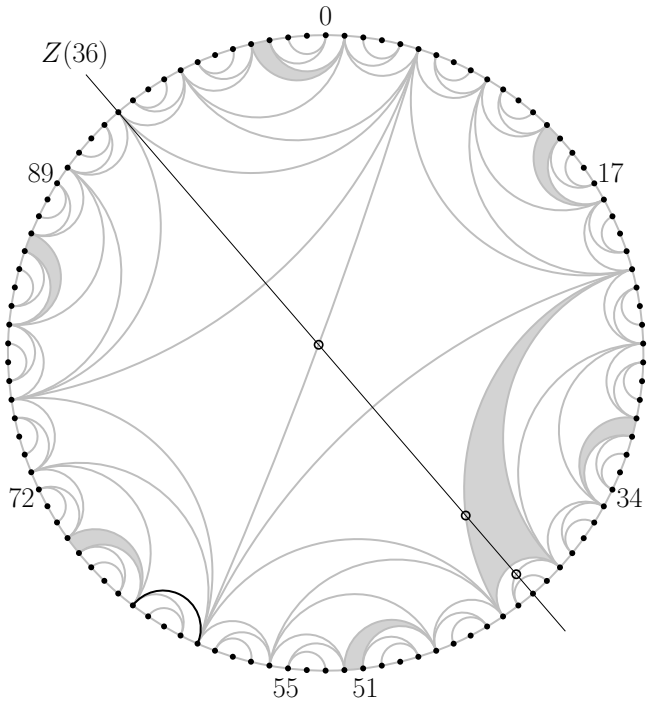


$$u = 35$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

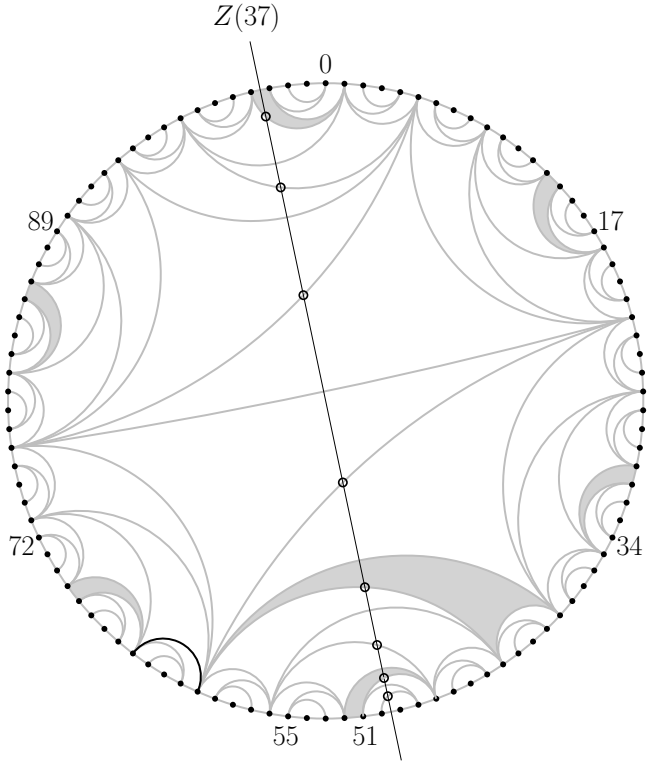


$$u = 36$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

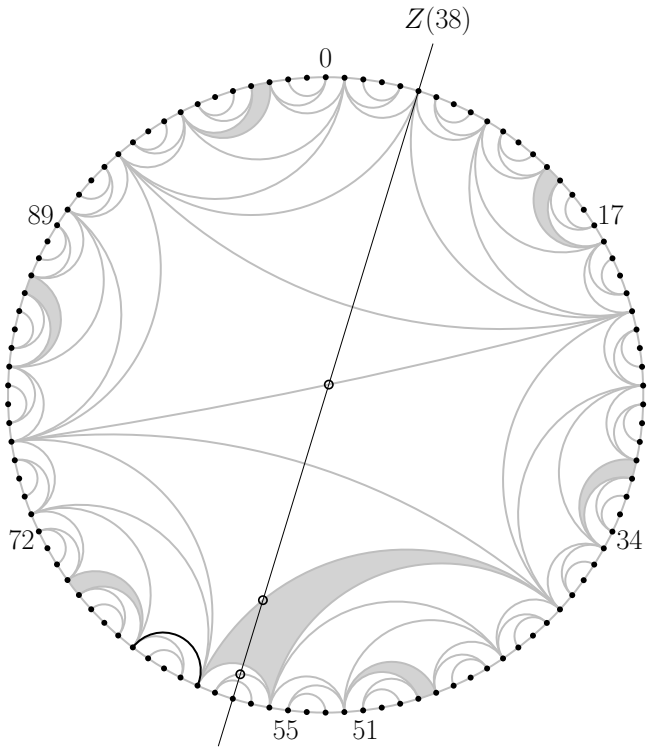


$$u=37$$

$$\begin{aligned}
Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\
& \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\
& \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\
& \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\
& \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)).
\end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

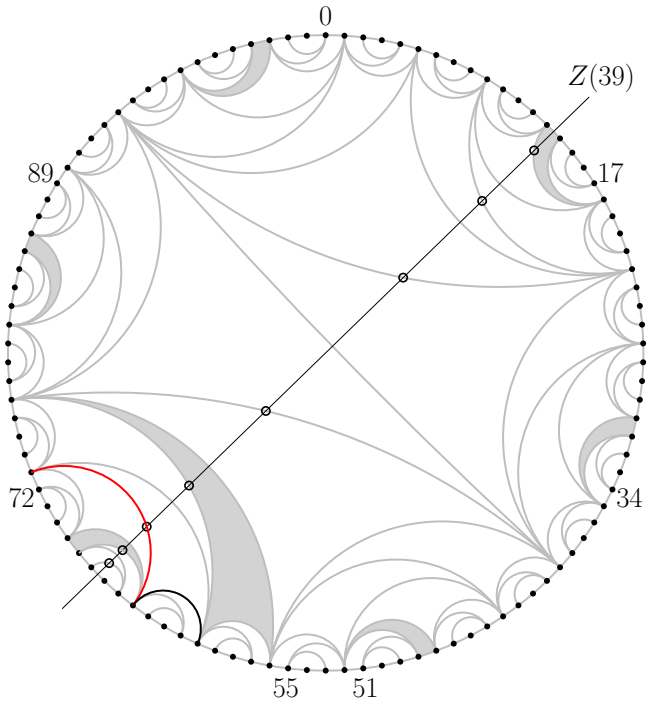


$$u = 38$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

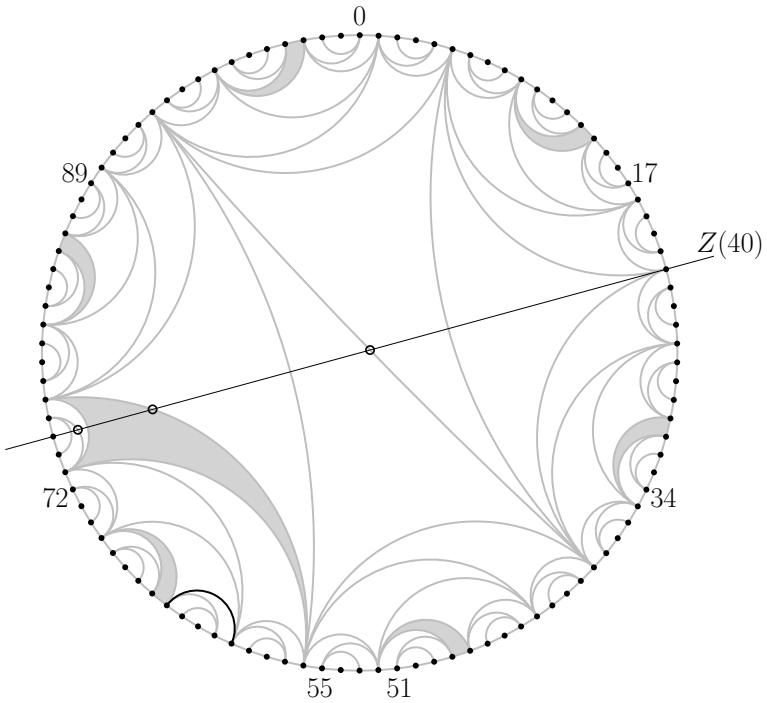


$$u = 39$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

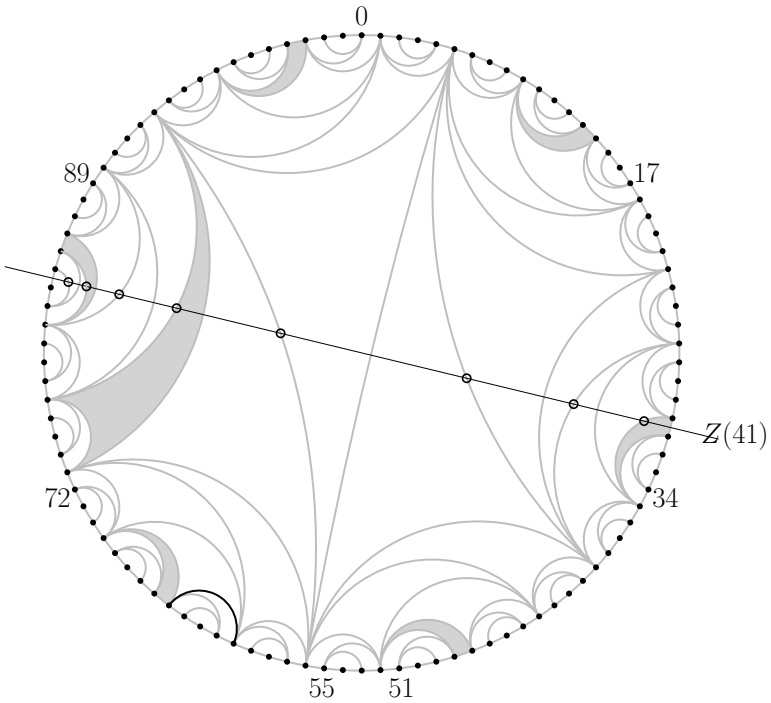


$$u = 40$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

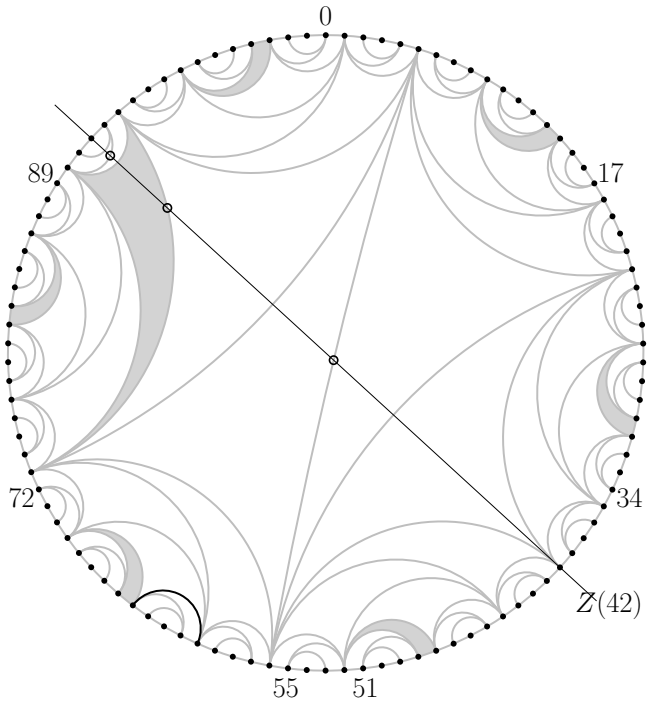


$$u = 41$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

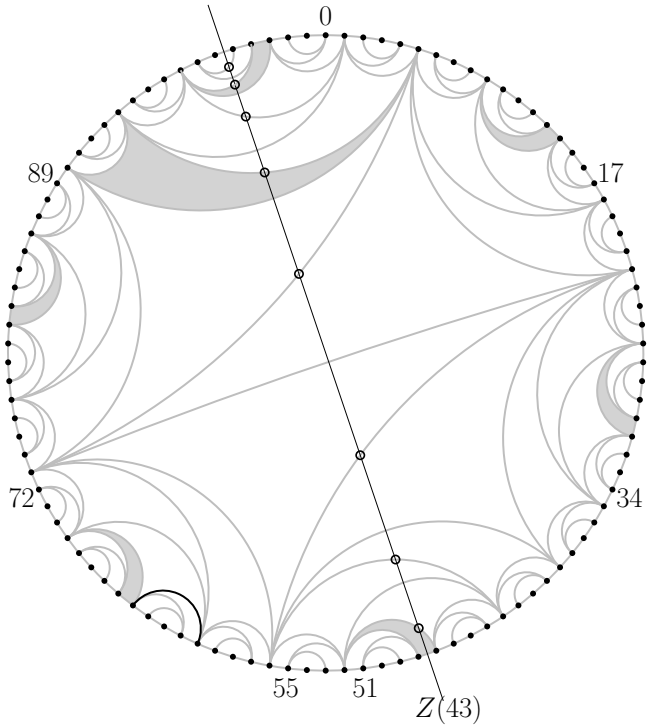


$$u = 42$$

$$\begin{aligned} Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ & \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ & \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ & \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ & \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

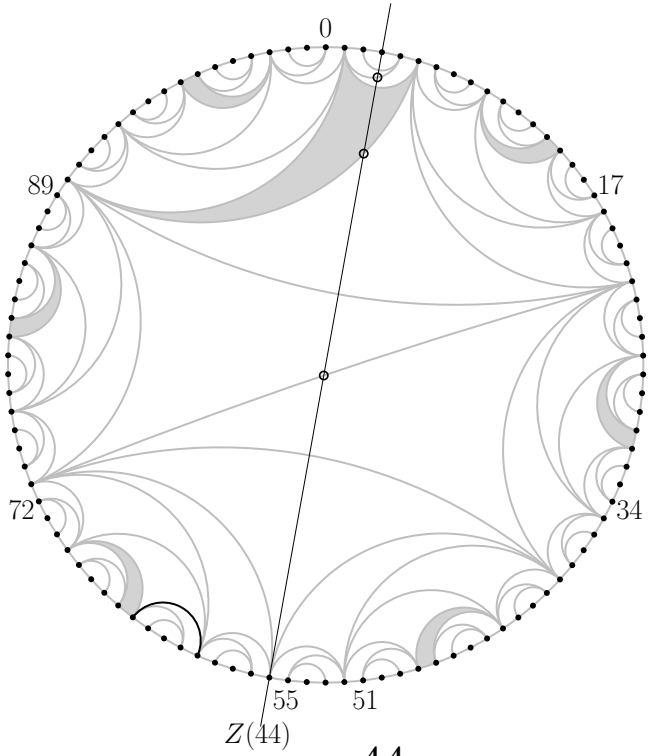


$$u = 43$$

$$\begin{aligned} Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ & \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ & \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ & \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ & \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

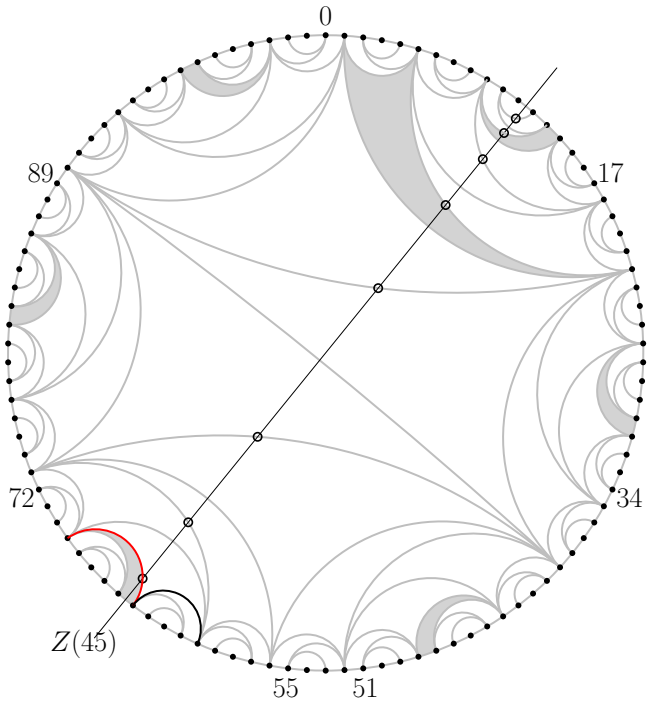


$u = 44$

$$
\begin{aligned}
Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13))\\
&\times(1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25))\\
&\times(1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26))\\
&\times(1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33))\\
&\times(1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)).
\end{aligned}
$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

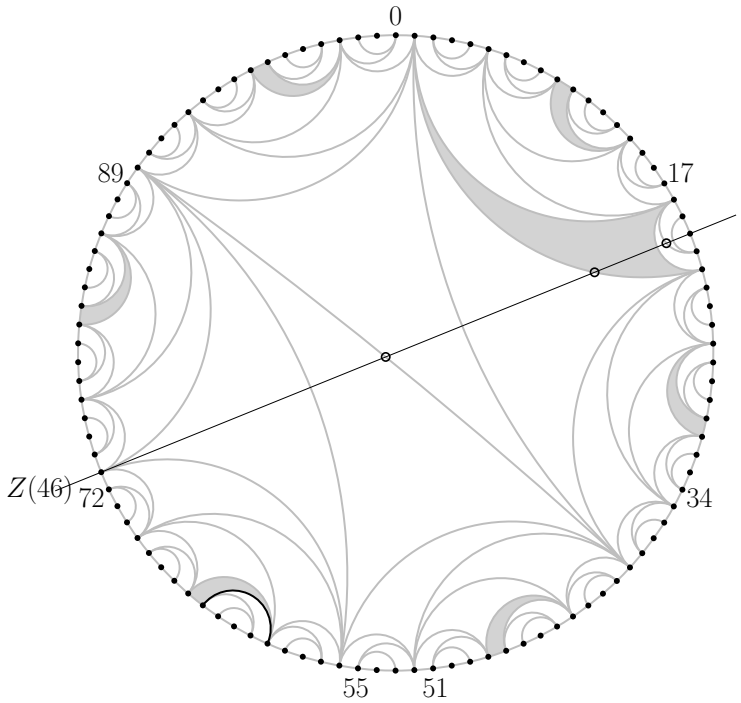


$$u = 45$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

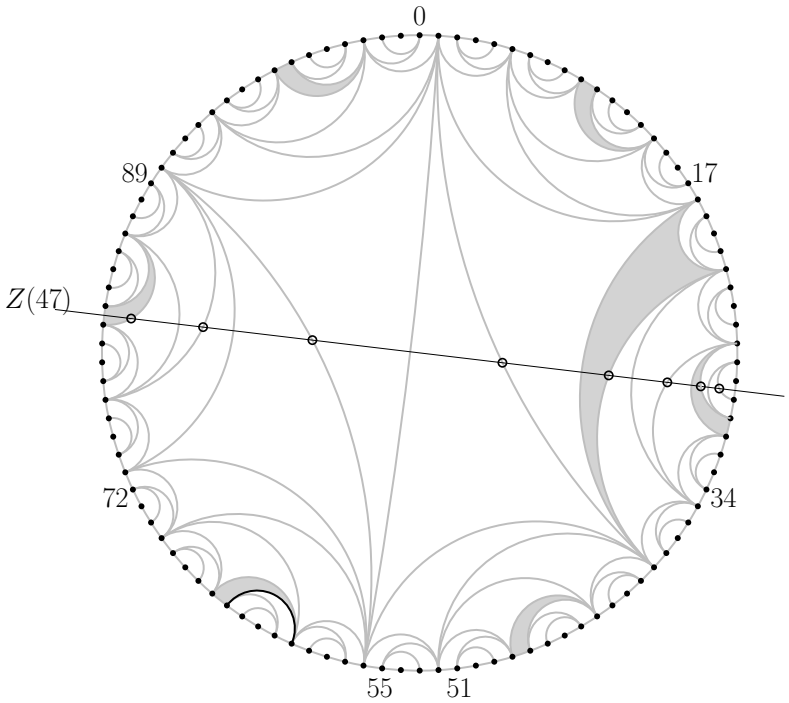


$$u = 46$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

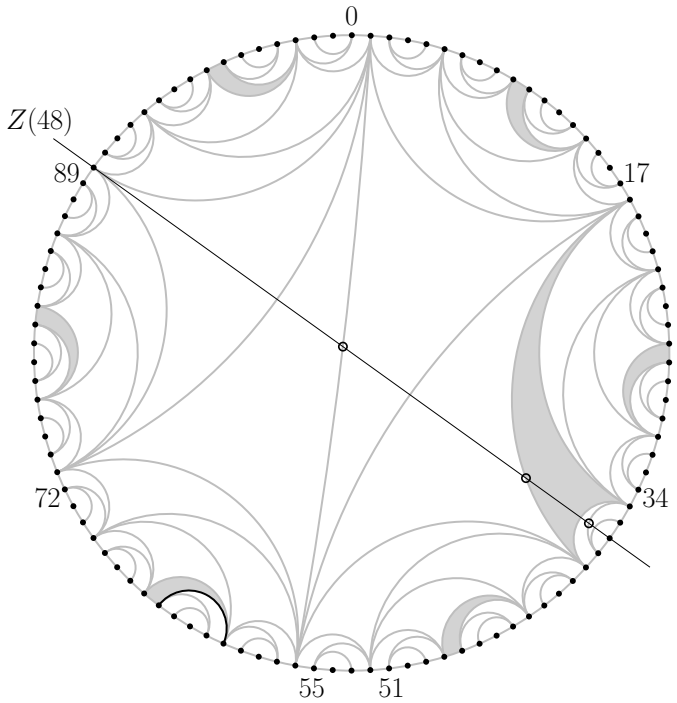


$$u = 47$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:

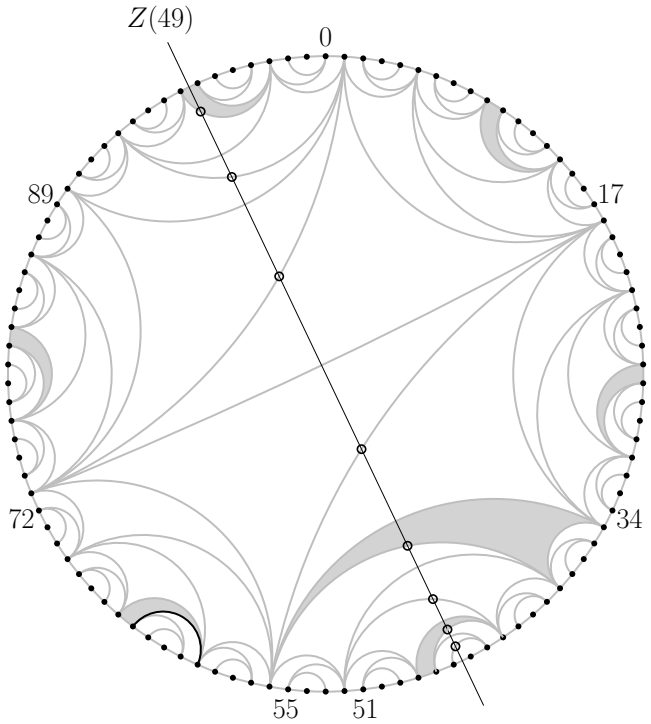


$$u = 48$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Example 3: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Realization of Y-system.

- Let us focus on the exceptional relation for $(a,m)=(3,1)$:



$$u = 49$$

$$\begin{aligned} Y_1^{(3)}(51) = Y_1^{(3)}(1)^{-1}&(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ &\times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ &\times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ &\times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ &\times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

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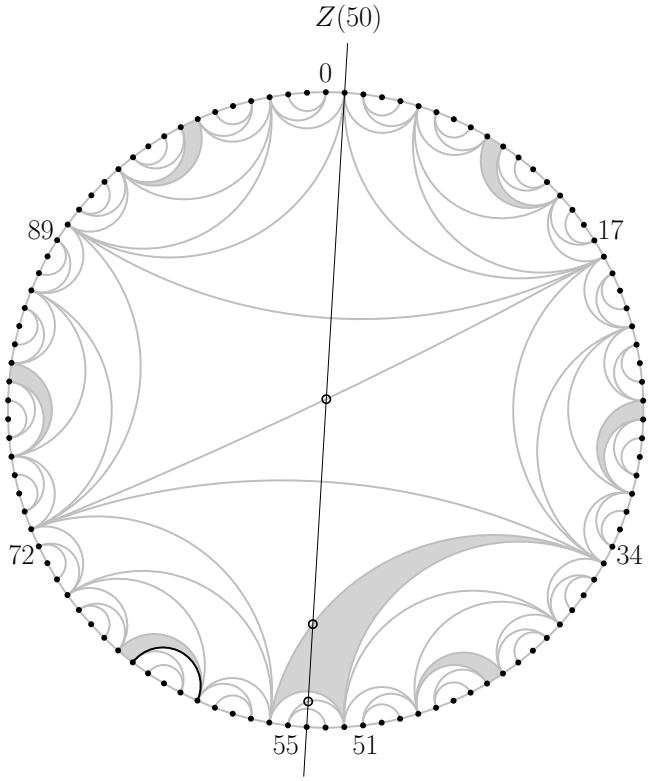

$$u=50$$

$$\begin{aligned}
Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\
& \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\
& \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\
& \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\
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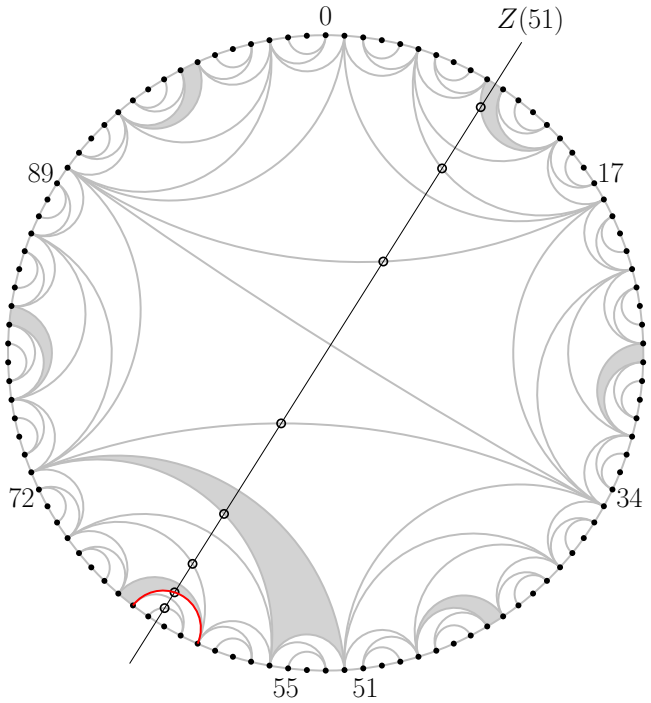
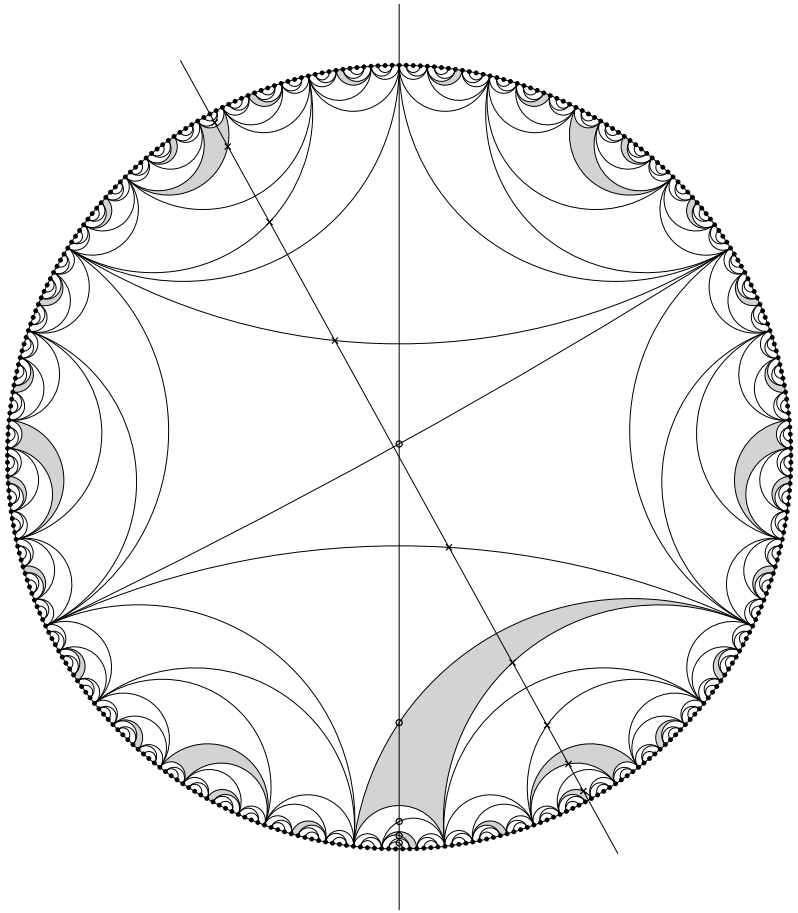


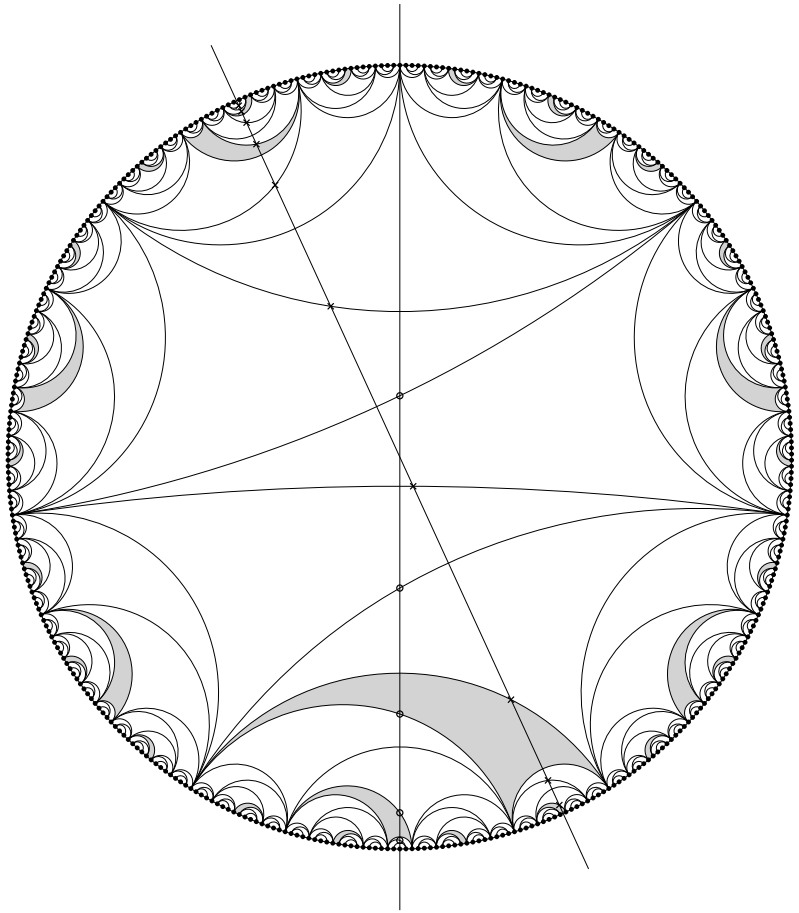
$$u = 51$$

$$\begin{aligned} Y_1^{(3)}(51) = {} & Y_1^{(3)}(1)^{-1}(1+Y_4^{(2)}(7))(1+Y_3^{(2)}(13)) \\ & \times (1+Y_2^{(2)}(19))(1+Y_1^{(2)}(25)) \\ & \times (1+Y_2^{(3)}(26))(1+Y_4^{(1)}(26)) \\ & \times (1+Y_1^{(2)}(27))(1+Y_2^{(2)}(33)) \\ & \times (1+Y_3^{(2)}(39))(1+Y_4^{(2)}(45)). \end{aligned}$$

Further examples



$(6,4,3,3)$, 349-gon



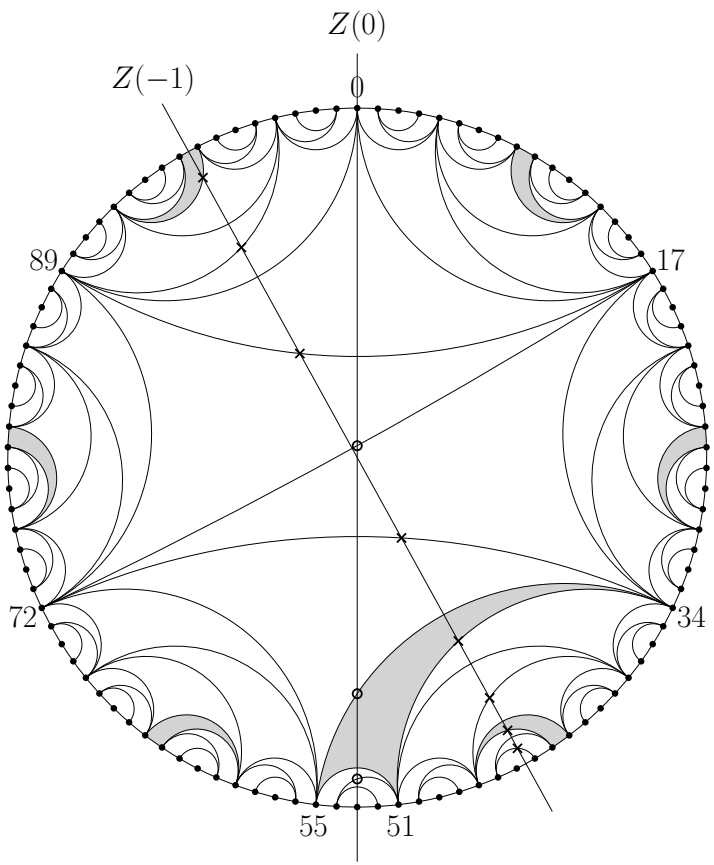
$(7,3,4,3)$, 402-gon

They are directly associated with continued fractions.

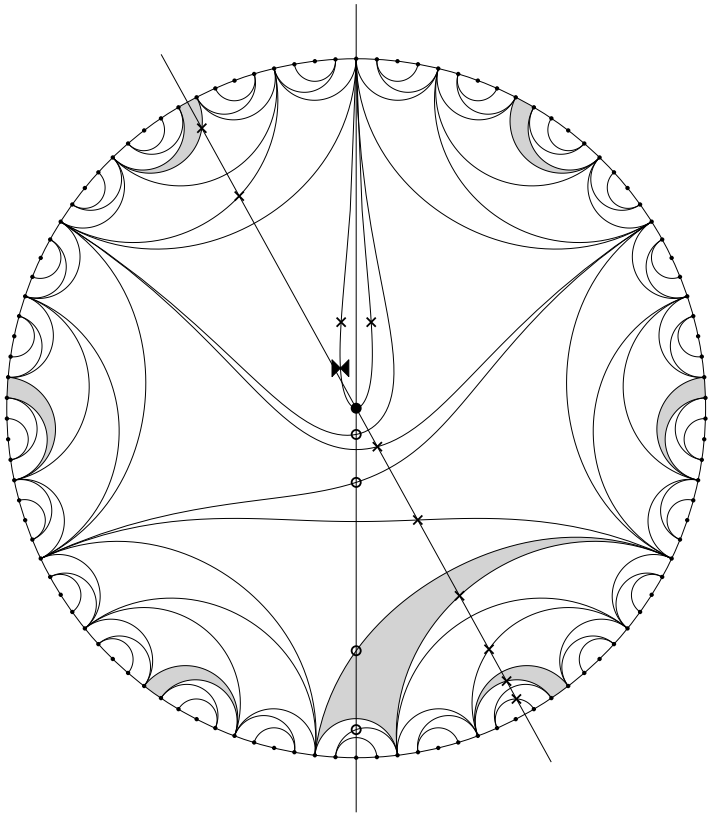
RSG vs. SG Y-systems

- RSG Y-system: type A, triangulation of r_F-gon (without puncture)
  SG Y-system: type D, (tagged) triangulation of r_F-gon (with a puncture)

The difference is only the diagonals of the first generation.



RSG



SG

Dilogarithm identities in general form

Suppose we have a (partial) period of Y-seed:

$$(B(0), y(0)) \overset{k_0}{\longleftrightarrow} (B(1), y(1)) \overset{k_1}{\longleftrightarrow} \cdots \overset{k_N}{\longleftrightarrow} (B(N), y(N)) \equiv (B(0), y(0)),$$

where $\equiv$ is modulo permutation of labels.

Dilogarithm identity in general form [N11]

For any real positive evaluation of $y(0)$, we have

$$\frac{6}{\pi^2} \sum_{t=0}^{N-1} L\left(\frac{y_{k_t}(t)}{1 + y_{k_t}(t)}\right) = N_-, \tag{1}$$

where N_- is the total number of $y_{k_t}(t)$ $(t = 0, \ldots, N = 1)$ whose tropical sign (= the sign of the corresponding c-vector) is negative.

Once the periodicity is established, Tateo's conjecture reduces to the formula for N_-:

$$N_- = \begin{cases} r_F \left(\sum_{a:\mathrm{odd}} n_a - 4 + \sum_{a=1}^{F-1} (-1)^{a+1} \frac{6}{p_a q_a} + (-1)^{F+1} \frac{6}{p_F r_F} \right) \\ = r_F \left(\sum_{a:\mathrm{odd}} n_a - 4 \right) + 6 r_F^{(2)}, r_F^{(2)} := p_F^{(2)} + q_F^{(2)}, p_F^{(2)}/q_F^{(2)} = [n_F, \ldots, n_2] & \text{for RSG} \\ r_F \left(\sum_{a:\mathrm{odd}} n_a \right), & \text{for SG.} \end{cases}$$

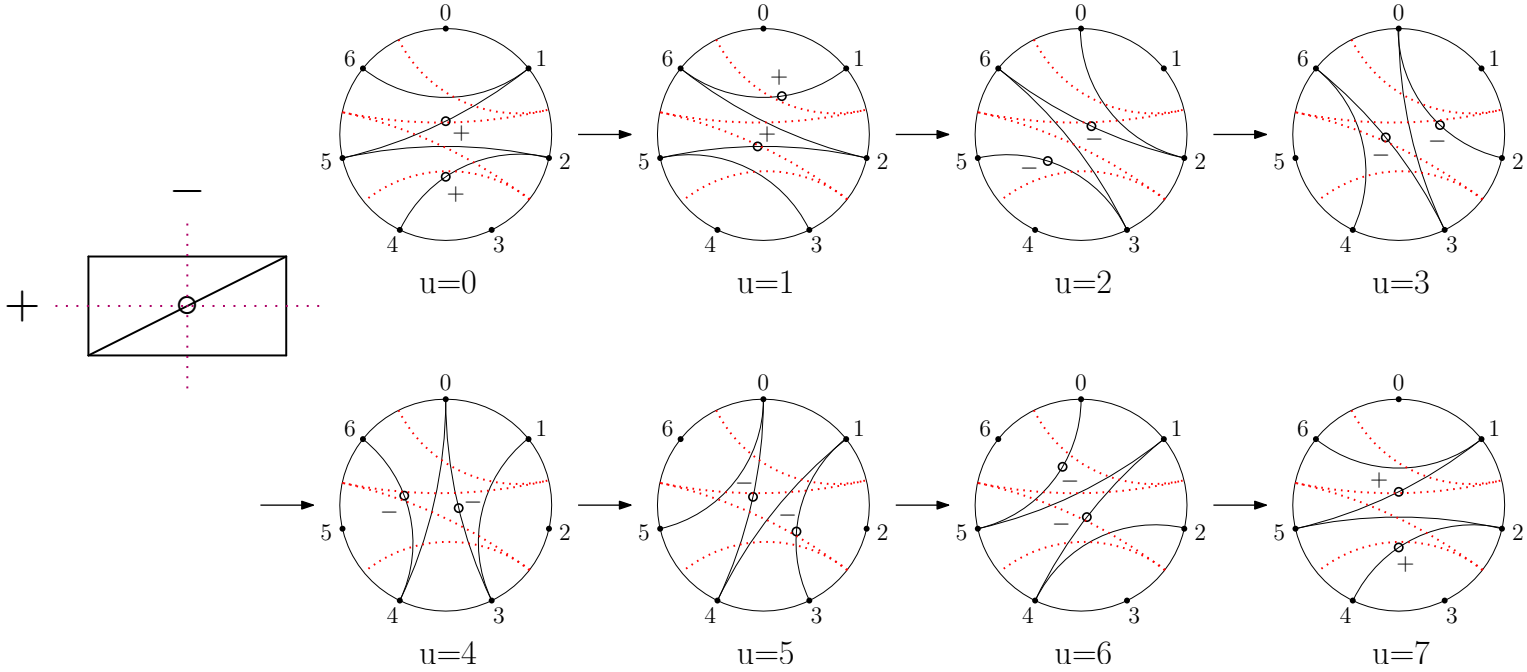
Counting of N_- by lamination technique

- We can count N_- by analyzing the triangulations, using the lamination technique due to [Fock-Goncharov 07], [Fomin-Thurston 08].
- Example. RSG, $F = 1$, $(n_1) = (6)$. Data: $p_1 = 1$, $r_1 = 7$.

Tateo's conjecture:

$$N_- = r_F \left(\sum_{a:\mathrm{odd}} n_a - 4 + \sum_{a=1}^{F-1} (-1)^{a+1} \frac{6}{p_a q_a} + (-1)^{F+1} \frac{6}{p_F r_F} \right)$$
$$= 7 \left(6 - 4 + \frac{6}{1 \times 7} \right) = 20.$$

Counting N_- using the (initial) lamination:



Summary

- The sine-Gordon (SG) Y-systems are some (very complicated) systems of algebraic functional equations introduced by Roberto Tateo in 90's.
- They originate from the SG equation, which is a famous soliton equation.
- To be more precise, there are two subfamilies:
  SG Y-systems and reduced SG (RSG) Y-systems.

Both are associated with continued fractions.

Main Message

There is a wonderful interplay among

- continued fractions
- SG/RSG Y-systems
- triangulations of polygons
- cluster algebras of types A and D

- Since the SG equation is associated with affine sl_2, this should be a tip of iceberg of some gigantic thing!

Important remark

Don't be discouraged by the horrible appearance of the SG/RSG Y-systems, since they are beautiful in nature.

Outline

More on continued fractions

- Input $(n_1, \ldots, n_F)$. We introduced continued fractions $(1 \le a \le F)$

$$\xi_a = \frac{p_a}{q_a} = [n_a, \ldots, n_1] := \cfrac{1}{n_a + \cfrac{1}{n_{a-1} + \cfrac{1}{\ddots + \cfrac{1}{n_1}}}},$$

and $r_a := p_a + q_a$.
We further introduce continued fractions $2 \le k \le a \le F$

$$\xi_a^{(k)} = \frac{p_a^{(k)}}{q_a^{(k)}} = [n_a, \ldots, n_k] = \cfrac{1}{n_a + \cfrac{1}{n_{a-1} + \cfrac{1}{\ddots + \cfrac{1}{n_k}}}},$$

and $r_a^{(k)} := p_a^{(k)} + q_a^{(k)}$.

- We will construct a triangulation of an r_F-gon associated with $(n_1, \ldots, n_F)$.

The numbers $r := r_F$, $r^{(k)} := r_F^{(k)}$ are especially important for the construction.
Recursion relation: $r^{(k)} = n_k r^{(k+1)} + r^{(k+2)}$.

Example

- Running example: $F = 3$, $(n_1, n_2, n_3) = (6, 4, 3)$. We have

$$\xi_1 = \frac{1}{6}, \qquad (p_1, q_1) = (1, 6), \qquad r_1 = 7,$$

$$\xi_2 = \frac{1}{4 + \frac{1}{6}} = \frac{6}{25}, \qquad (p_2, q_2) = (6, 25), \qquad r_2 = 31,$$

$$\xi_3 = \frac{1}{3 + \frac{1}{4 + \frac{1}{6}}} = \frac{25}{81}, \quad (p_3, q_3) = (25, 81), \quad r = r_3 = 106.$$

We also have

$$\xi_2^{(2)} = \frac{1}{4}, \qquad (p_2^{(2)}, q_2^{(2)}) = (1, 4), \qquad r_2^{(2)} = 5,$$

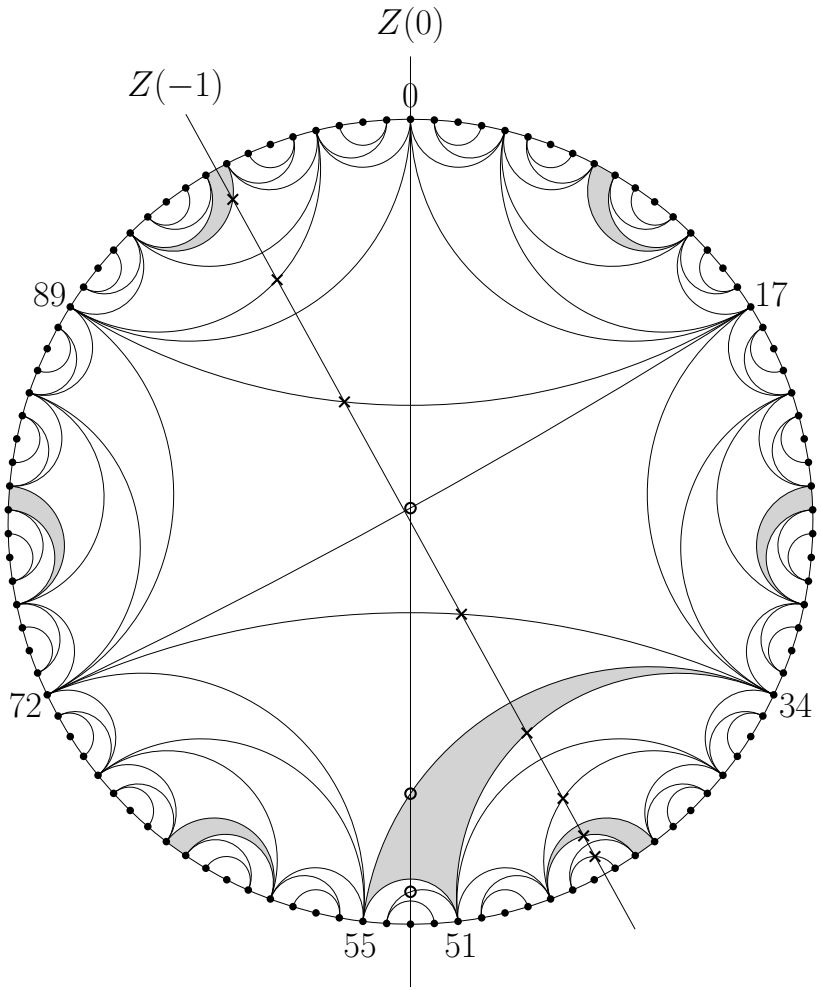
$$\xi_3^{(2)} = \frac{1}{3 + \frac{1}{4}} = \frac{4}{13}, \quad (p_3^{(2)}, q_3^{(2)}) = (4, 13), \quad r^{(2)} = r_3^{(2)} = 17,$$

$$\xi_3^{(3)} = \frac{1}{3}, \qquad (p_3^{(3)}, q_3^{(3)}) = (1, 3), \quad r^{(3)} = r_3^{(3)} = 4.$$

Recursion relation: $106 = 6 \times 17 + 4$.

Example: $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$. Initial triangulation.

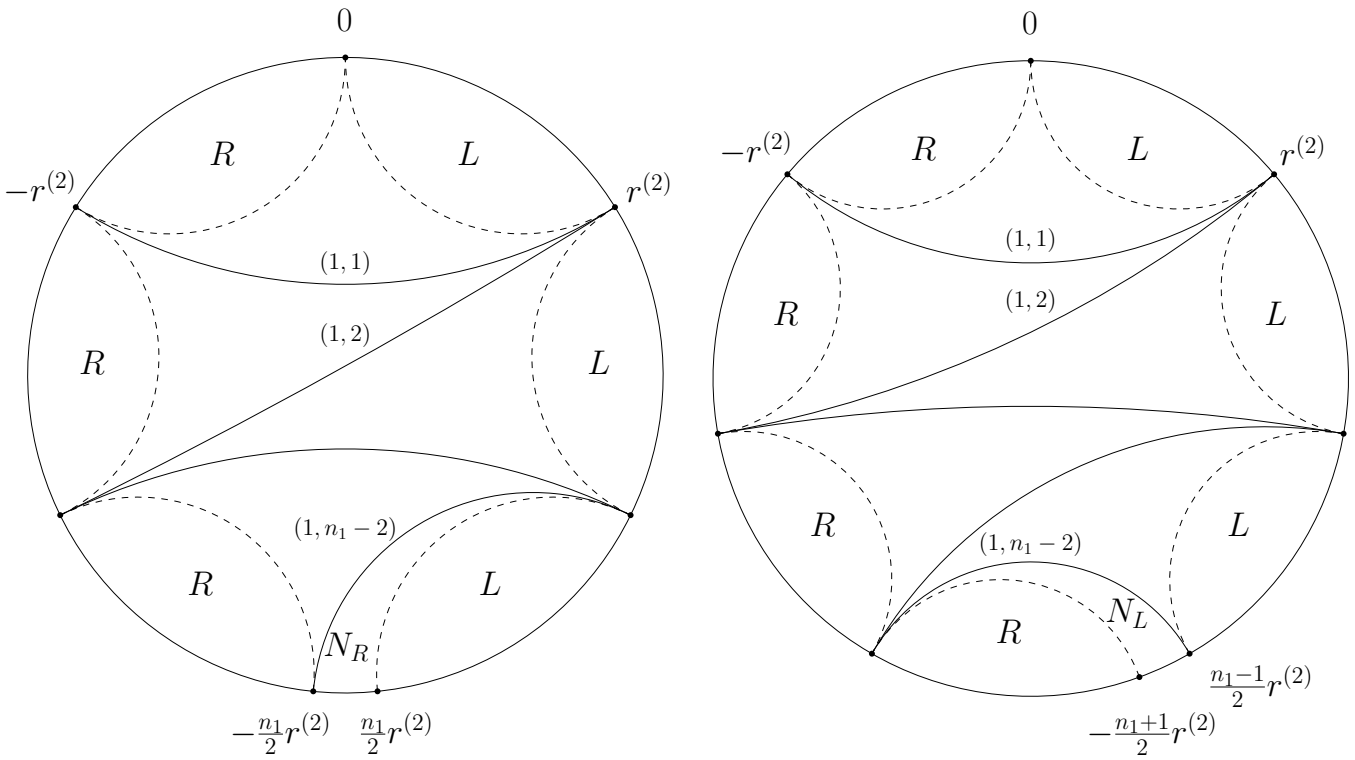
- Recall: Initial triangulation of an 106-gon for $\mathbb{Y}_{\mathrm{RSG}}(6,4,3)$.



$r = 106$: The number of vertices.
$r^{(2)} = 17$: The width of the zigzag of the second generation
$r^{(3)} = 4$: The width of the zigzag of the third generation

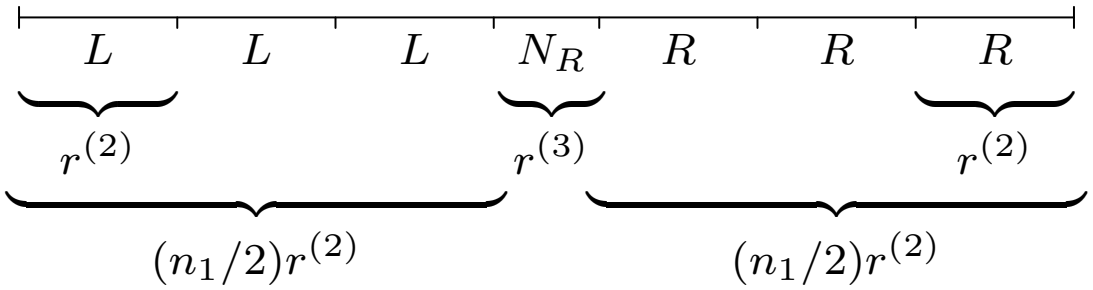
General construction: Step 1. The intervals of the second generation

- The diagonals of the first generation and the intervals of the second generation



(a) even n_1 (b) odd n_1

The intervals of the second generation for even n_1, schematically:



General construction: Step 2. Subdivision of the intervals

The intervals of the a-th generation is subdivided into the intervals of the $a+1$-th generation:
For even n_a,

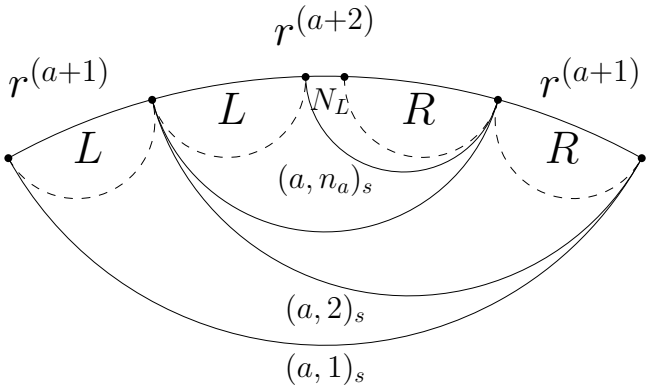
$$\overbrace{\underbrace{\overbrace{\underbrace{\overbrace{L}^{r^{(a+1)}}\ \overbrace{L}}_{(n_a/2)r^{(a+1)}}\ \underbrace{\overbrace{N_L}^{r^{(a+2)}}\ \overbrace{R}}_{(n_a/2)r^{(a+1)}}}^{L}\ \overbrace{R}^{r^{(a+1)}}}_{L}\ \overbrace{N_T}^{r^{(a+1)}}\ \overbrace{\underbrace{\overbrace{\underbrace{\overbrace{L}^{r^{(a+1)}}\ \overbrace{L}}_{(n_a/2)r^{(a+1)}}\ \underbrace{\overbrace{N_R}^{r^{(a+2)}}\ \overbrace{R}}_{(n_a/2)r^{(a+1)}}}^{L}\ \overbrace{R}^{r^{(a+1)}}}_{R}}^{R}$$

For odd n_a,

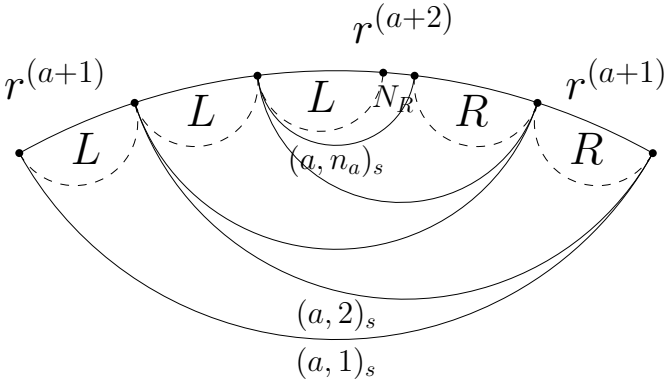
$$\overbrace{\underbrace{\overbrace{\underbrace{\overbrace{L}^{r^{(a+1)}}\ \overbrace{L}\ \overbrace{L}}_{((n_a+1)/2)r^{(a+1)}}\ \underbrace{\overbrace{N_R}^{r^{(a+2)}}\ \overbrace{R}}_{((n_a-1)/2)r^{(a+1)}}}^{L}\ \overbrace{R}^{r^{(a+1)}}}_{L}\ \overbrace{N_T}^{r^{(a+1)}}\ \overbrace{\underbrace{\overbrace{\underbrace{\overbrace{L}^{r^{(a+1)}}\ \overbrace{L}\ \overbrace{N_L}}_{((n_a-1)/2)r^{(a+1)}}\ \underbrace{\overbrace{R}\ \overbrace{R}\ \overbrace{R}}_{((n_a+1)/2)r^{(a+1)}}}^{L}}_{R}}^{R}$$

General construction: Step 3. Draw the diagonals

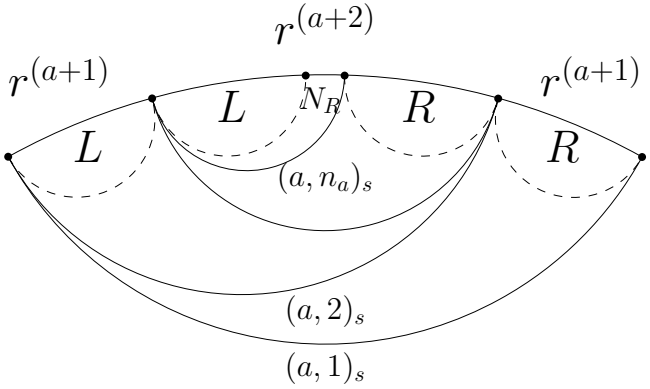
- For each interval of type R or L of the a-th generation, draw the diagonals of the a-th generation:



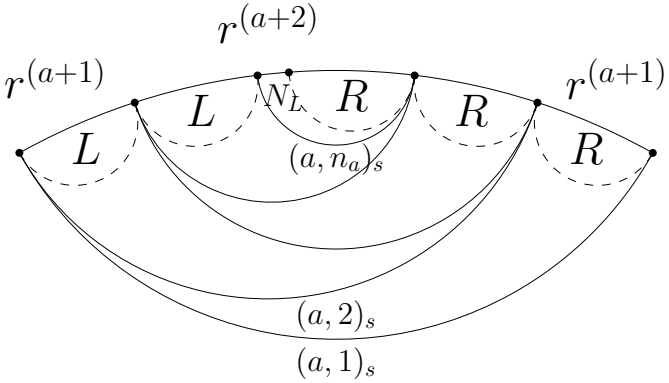
type L, even n_a



type L, odd n_a



type R, even n_a



type R, odd n_a