Cluster algebras and applications

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Talk presented at JMS meeting, Shinshu Univ., September 2011

Based on joint works with
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The pdf file of this talk will be available on my web site.

Summary

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- In 90's the systems of discrete functional equations called T-systems and Y-systems were introduced and studied in the Bethe ansatz method for integrable models.
- After the introduction of cluster algebras by Fomin and Zelevinsky around 2000 it has been gradually recognized that T-systems and Y-systems are a part of cluster algebra structure.
- In particular, the long standing conjecture of periodicities of Y-systems by Zamolodchikov et al. is proved by the tropicalization method in cluster algebras.
- One can associate classical and quantum dilogarithm identities with any period of a cluster algebra.
- As a further consequence, the long standing conjecture of the central charge identities in conformal field theory by Kirillov et al. is proved.

Plan of Talk

Outline

Mutation of matrix/quiver

★ mutation of matrix

I: finite index set

$B = (b_{ij})_{i,j \in I}$: a skew symmetric (integer) matrix

mutation of B at $k \in I$, $B' = \mu_k(B)$:

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + [-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+ & i, j \neq k \end{cases}$$

where $[a]_+ := \max(a, 0)$. B' is again skew symmetric, and $\mu_k^2 = \mathrm{id}$.

★ correspondence to quiver

skew symmetric matrix B $\leftrightarrow$ quiver Q (with no loop and 2-cycle)

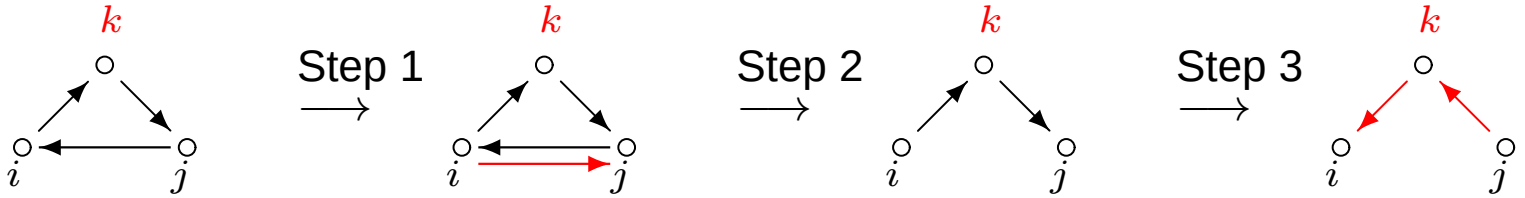
$$b_{ij} = t > 0 \quad \leftrightarrow \quad \overset{i}{\circ} \xrightarrow{t \text{ arrows}} \overset{j}{\circ}$$

★ mutation of quiver at k

Step 1. For each pair of an incoming arrow $i \to k$ and an outgoing arrow $k \to j$, add a new arrow $i \to j$.

Step 2. Remove a maximal set of pairwise disjoint 2-cycles.

Step 3. Reverse all arrows incident with k.



Semifield

★ Semifield

Definition: A *semifield* $(\mathbb{P}, \oplus, \cdot)$ is an abelian multiplicative group endowed with a binary operation of addition $\oplus$ which is commutative, associative, and distributive with respect to the multiplication in $\mathbb{P}$.

★ Three important examples:

(a) Universal semifield $\mathbb{P}_{\mathrm{univ}}(y)$. For an I-tuple of variables $y = (y_i)_{i \in I}$, it consists of all the rational functions with subtraction-free rational expressions (i.e., $P(y)/Q(y)$ with $P(y)$ and $Q(y)$ being polynomials in y_i's with positive rational coefficients).

(b) Tropical semifield $\mathbb{P}_{\mathrm{trop}}(y)$. For an I-tuple of variables $y = (y_i)_{i \in I}$, it is the abelian multiplicative group freely generated by the variables y_i's endowed with the addition $\oplus$

$$\prod_i y_i^{a_i} \oplus \prod_i y_i^{b_i} = \prod_i y_i^{\min(a_i, b_i)}. \tag{1}$$

(c) Trivial semifield $\mathbf{1}$. It consists of only one element 1 with $1 \cdot 1 = 1 \oplus 1 = 1$.

★ Sequence of surjections:

$$\begin{array}{ccccc} \mathbb{P}_{\mathrm{univ}}(y) & \to & \mathbb{P}_{\mathrm{trop}}(y) & \to & \mathbf{1} \\ y_i & \mapsto & y_i & \mapsto & 1 \\ c\ (>0) & \mapsto & 1 & \mapsto & 1 \end{array}$$

Cluster algebra with coefficients [Fomin-Zelevinsky 02]

★ initial seed (B, x, y):
initial exchange matrix $B = (b_{ij})_{i,j \in I}$: a skew symmetric (integer) matrix
initial cluster $x = (x_i)_{i \in I}$: an I-tuple of formal variables
initial coefficient tuple $y = (y_i)_{i \in I}$: an I-tuple of formal variables
$\mathbb{P}_{\mathrm{univ}}(y)$: universal semifield of y

★ mutation of (B, x, y) at $k \in I$ $(B', x', y') = \mu_k(B, x, y)$:

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + [-b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+ & i, j \neq k \end{cases}$$

$$y'_i = \begin{cases} y_i^{-1} & i = k \\ y_i y_k^{[b_{ki}]_+} (1 \oplus y_k)^{-b_{ki}} & i \neq k, \end{cases}$$

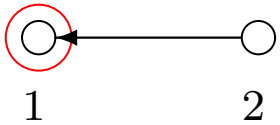
$$x'_i = \begin{cases} x_i^{-1} \left(\dfrac{y_k}{1 \oplus y_k} \prod_{j \in I} x_j^{[b_{jk}]_+} + \dfrac{1}{1 \oplus y_k} \prod_{j \in I} x_j^{[-b_{jk}]_+} \right) & i = k. \\ x_i & i \neq k, \end{cases}$$

Again, $\mu_k^2 = \mathrm{id}$.

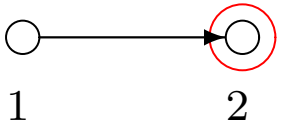
★ Iterate mutations and collect all the resulted triplets (B'', x'', y'').
The cluster algebra (with coefficients) $\mathcal{A}(B, x, y)$ is the $\mathbb{Z}(\mathbb{P}_{\mathrm{univ}}(y))$-subalgebra of the rational function field $\mathbb{Q}(\mathbb{P}_{\mathrm{univ}}(y))(x)$ generated by all the cluster variables x''_i.

Example: Cluster algebra of type A_2

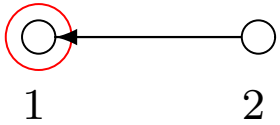
$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Q = \underset{1}{\circ} \longleftarrow \underset{2}{\circ}$$

$Q(0)$ 

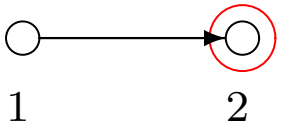
$$\begin{cases} x_1(0) = x_1 \\ x_2(0) = x_2, \end{cases} \qquad \begin{cases} y_1(0) = y_1 \\ y_2(0) = y_2, \end{cases}$$

$Q(1)$ 

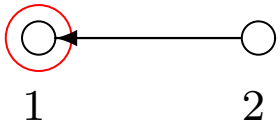
$$\begin{cases} x_1(1) = \dfrac{1 + y_1 x_2}{(1 \oplus y_1) x_1} \\ x_2(1) = x_2, \end{cases} \qquad \begin{cases} y_1(1) = y_1^{-1} \\ y_2(1) = y_2(1 \oplus y_1), \end{cases}$$

$Q(2)$ 

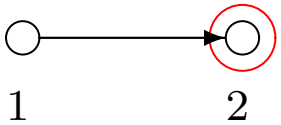
$$\begin{cases} x_1(2) = \dfrac{1 + y_1 x_2}{(1 \oplus y_1) x_1} \\ x_2(2) = \dfrac{x_1 + y_2 + y_1 y_2 x_2}{(1 \oplus y_2 \oplus y_1 y_2) x_1 x_2}, \end{cases} \qquad \begin{cases} y_1(2) = y_1^{-1}(1 \oplus y_2 \oplus y_1 y_2) \\ y_2(2) = y_2^{-1}(1 \oplus y_1)^{-1}, \end{cases}$$

$Q(3)$ 

$$\begin{cases} x_1(3) = \dfrac{y_2 + x_1}{(1 \oplus y_2) x_2} \\ x_2(3) = \dfrac{x_1 + y_2 + y_1 y_2 x_2}{(1 \oplus y_2 \oplus y_1 y_2) x_1 x_2}, \end{cases} \qquad \begin{cases} y_1(3) = y_1(1 \oplus y_2 \oplus y_1 y_2)^{-1} \\ y_2(3) = y_1^{-1} y_2^{-1}(1 \oplus y_2), \end{cases}$$

$Q(4)$ 

$$\begin{cases} x_1(4) = \dfrac{x_1 + y_2}{(1 \oplus y_2) x_2} \\ x_2(4) = x_1, \end{cases} \qquad \begin{cases} y_1(4) = y_2^{-1} \\ y_2(4) = y_1 y_2 (1 \oplus y_2)^{-1}, \end{cases}$$

$Q(5)$ 

$$\begin{cases} x_1(5) = x_2 \\ x_2(5) = x_1. \end{cases} \qquad \begin{cases} y_1(5) = y_2 \\ y_2(5) = y_1. \end{cases}$$

$\hat{y}$-variables

Proposition [Fomin-Zelevinsky 07]

For each seed (B', x', y'), set

$$\hat{y}'_i := y'_i \prod_{j \in I} {x'_j}^{b'_{ji}}.$$

Then, $\hat{y}$-variables satisfy the same exchange relation as y-variables. Namely,

$$\hat{y}''_i = \begin{cases} \hat{y}'^{-1}_i & i = k \\ \hat{y}'_i \hat{y}'^{[b'_{ki}]_+}_k (1 + \hat{y}'_k)^{-b'_{ki}} & i \neq k, \end{cases}$$

Remark. Recall the exchange relation of x':

$$x''_i = \begin{cases} x'^{-1}_i \left(\dfrac{y'_k}{1 \oplus y'_k} \prod_{j \in I} {x'_j}^{[b'_{jk}]_+} + \dfrac{1}{1 \oplus y'_k} \prod_{j \in I} {x'_j}^{[-b_{jk}]_+} \right) & i = k \\ x'_i & i \neq k. \end{cases}$$

$$\hat{y}'_k = \frac{\text{1st term for } i = k \text{ in the above}}{\text{2nd term for } i = k \text{ in the above}}$$

Alternatvie expression of exchange relations

The exchage relations are also written as

$$y_i'' = \begin{cases} {y_i'}^{-1} & i = k \\ y_i' {y_k'}^{[b_{ki}']_+} (1 \oplus y_k')^{-b_{ki}'} & i \neq k, \end{cases}$$

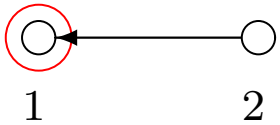
$$x_i'' = \begin{cases} {x_i'}^{-1} \left(\prod_{j \in I} {x_j'}^{[-b_{jk}']_+} \right) \dfrac{1 + \hat{y}_k'}{1 \oplus y_k'} & i = k \\ x_i' & i \neq k, \end{cases}$$

where

$$\hat{y}_i' := y_i' \prod_{j \in I} {x_j'}^{b_{ji}'}.$$

Example: Cluster algebra of type A_2 (revisited)

$$\hat{y}_1 = y_1 x_2, \quad \hat{y}_2 = y_2 x_1^{-1}.$$

$Q(0)$ 

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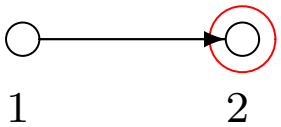
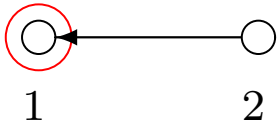
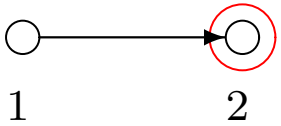
$Q(1)$ 

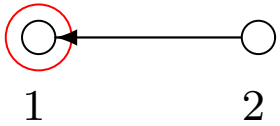
$$\begin{cases} x_1(1) = x_1^{-1} \dfrac{1+\hat{y}_1}{1 \oplus y_1} \\ x_2(1) = x_2, \end{cases} \qquad \begin{cases} y_1(1) = y_1^{-1} \\ y_2(1) = y_2(1 \oplus y_1), \end{cases}$$

$Q(2)$ 

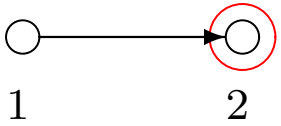
$$\begin{cases} x_1(2) = x_1^{-1} \dfrac{1+\hat{y}_1}{1 \oplus y_1} \\ x_2(2) = x_2^{-1} \dfrac{1+\hat{y}_2+\hat{y}_1\hat{y}_2}{1 \oplus y_2 \oplus y_1 y_2}, \end{cases} \qquad \begin{cases} y_1(2) = y_1^{-1}(1 \oplus y_2 \oplus y_1 y_2) \\ y_2(2) = y_2^{-1}(1 \oplus y_1)^{-1}, \end{cases}$$

$Q(3)$ 

$$\begin{cases} x_1(3) = x_1 x_2^{-1} \dfrac{1+\hat{y}_2}{1 \oplus y_2} \\ x_2(3) = x_2^{-1} \dfrac{1+\hat{y}_2+\hat{y}_1\hat{y}_2}{1 \oplus y_2 \oplus y_1 y_2}, \end{cases} \qquad \begin{cases} y_1(3) = y_1(1 \oplus y_2 \oplus y_1 y_2)^{-1} \\ y_2(3) = y_1^{-1} y_2^{-1}(1 \oplus y_2), \end{cases}$$

$Q(4)$ 

$$\begin{cases} x_1(4) = x_1 x_2^{-1} \dfrac{1+\hat{y}_2}{1 \oplus y_2} \\ x_2(4) = x_1, \end{cases} \qquad \begin{cases} y_1(4) = y_2^{-1} \\ y_2(4) = y_1 y_2 (1 \oplus y_2)^{-1}, \end{cases}$$

$Q(5)$ 

$$\begin{cases} x_1(5) = x_2 \\ x_2(5) = x_1. \end{cases} \qquad \begin{cases} y_1(5) = y_2 \\ y_2(5) = y_1. \end{cases}$$

Separation formulas

Theorem (Separation formulas [Fomin-Zelevinsky 07])

For each seed (B', x', y'), there exist some

$$F'_i(y)\ (i \in I) \quad \text{polynomial of } y,$$
$$C' = (c'_{ij})_{i,j \in I} \quad \text{integer matrix},$$
$$G' = (g'_{ij})_{i,j \in I} \quad \text{integer matrix}$$

such that

$$y'_i = \left(\prod_{j \in I} y_j^{c'_{ji}} \right) \prod_{j \in I} F'_j(y)_{\oplus}^{b'_{ji}},$$
$$x'_i = \left(\prod_{j \in I} x_j^{g'_{ji}} \right) \frac{F'_i(\hat{y})}{F'_i(y)_{\oplus}}, \quad \hat{y}_i = y_i \prod_{j \in I} x_j^{b_{ji}}.$$

Basic data of seed: B-matrix, C-matrix, G-matrix, F-polynomials

Properties of C-matrix, G-matrix, F-polynomials

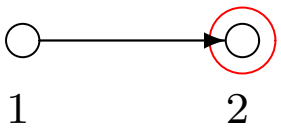
Theorem [Derksen-Weyman-Zelevinsky10, Plamondon10, Nagao10]

(a) $F_i'(y)$ has the constant term 1.
(b) *Sign-coherence:* Each column of C' is a nonzero vector and nonzero components are either *all positive* or *all negative*.

Theorem (Duality [N 10])

The transposition of G' is inverse to C'.

Example: type A_2

$Q(3)$ 

$$\begin{cases} x_1(3) = x_1 x_2^{-1} \dfrac{1+\hat{y}_2}{1 \oplus y_2} \\ x_2(3) = x_2^{-1} \dfrac{1+\hat{y}_2+\hat{y}_1\hat{y}_2}{1 \oplus y_2 \oplus y_1 y_2}, \end{cases} \qquad \begin{cases} y_1(3) = y_1(1 \oplus y_2 \oplus y_1 y_2)^{-1} \\ y_2(3) = y_1^{-1} y_2^{-1}(1 \oplus y_2), \end{cases}$$

$$C' = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad G' = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad (G')^T C' = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} = I.$$

Outline

T- and Y-systems of simply laced type

Fix (X, ℓ), where X: Dynkin diagram of type A, D, E with index set $\mathcal{I}$. $\ell \geq 2$ integer.

★ Y-system: For formal variables $\{Y_m^{(a)}(u) \mid a \in \mathcal{I}; m = 1, \ldots, \ell - 1; u \in \mathbb{Z}\}$

$$Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{\prod_{b \in \mathcal{I}, b \sim a} (1 + Y_m^{(b)}(u))}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})}$$

with $Y_0^{(a)}(u)^{-1} = Y_\ell^{(a)}(u)^{-1} = 0$
Origin: thermodynamic Bethe ansatz (TBA) equation for integrable models

★ T-system: For formal variables $\{T_m^{(a)}(u) \mid a \in \mathcal{I}; m = 1, \ldots, \ell - 1; u \in \mathbb{Z}\}$

$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = \prod_{b \in \mathcal{I}, b \sim a} T_m^{(b)}(u) + T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)$$

with $T_0^{(a)}(u) = T_\ell^{(a)}(u) = 1$
Origin: relation among transfer matrices of integrable models
(= relation among q-characters of KR modules of quantum groups)

★ Relation between Y- and T-systems: Set

$$\hat{Y}_m^{(a)}(u) := \frac{\prod_{b \in I, b \sim a} T_m^{(b)}(u)}{T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)}.$$

Then, $\{\hat{Y}_m^{(a)}(u)\}$ satisfies the Y-system.

Periodicity conjecture of Y-systems

$h^\vee$: dual Coxeter number of X, $h^\vee = h$ (Coxeter number) for $X = ADE$.
t: tier number of X, $t = 1$ for ADE, $t = 2$ for BCF, $t = 3$ for G.

Periodicity conjecture of Y-systems

$$Y_m^{(a)}(u + 2t(h^\vee + \ell)) = Y_m^{(a)}(u),$$

Conjectured by:
[Zamolodchikov 91] X: simply laced, $\ell = 2$.
[Ravanini-Tateo-Valleriani 93] X: simply laced, ℓ general.
[Kuniba-Nakanishi-Suzuki 94] X: nonsimply laced, ℓ general

Proved by:
[Frenkel-Szenes 95], [Gliozzi-Tateo 96] $X = A_1$, by explicit solution
[Fomin-Zelevinsky 03] X: simply laced, $\ell = 2$, by cluster algebra
[Volkov 07] $X = A_n$, by explicit solution
[Keller 10] X: simply laced, by cluster algebra + Auslander-Reiten theory
[Inoue-Iyama-Keller-Kuniba-N 10] X: general, by cluster algebra + tropicalization

Corollary

$$T_m^{(a)}(u + 2t(h^\vee + \ell)) = T_m^{(a)}(u),$$

Example 1. $(X, \ell) = (A_2, 2)$

Initial quiver

$$Q = \quad \underset{1}{\circ} \longleftarrow \underset{2}{\circ}$$

Periodicity of quivers under the sequence of mutations μ_1, μ_2:

$$\underset{Q}{\circ \longleftarrow \circ} \quad \overset{\mu_1}{\leftrightarrow} \quad \underset{Q^{\mathrm{op}}}{\circ \longrightarrow \circ} \quad \overset{\mu_2}{\leftrightarrow} \quad \underset{Q}{\circ \longleftarrow \circ}$$

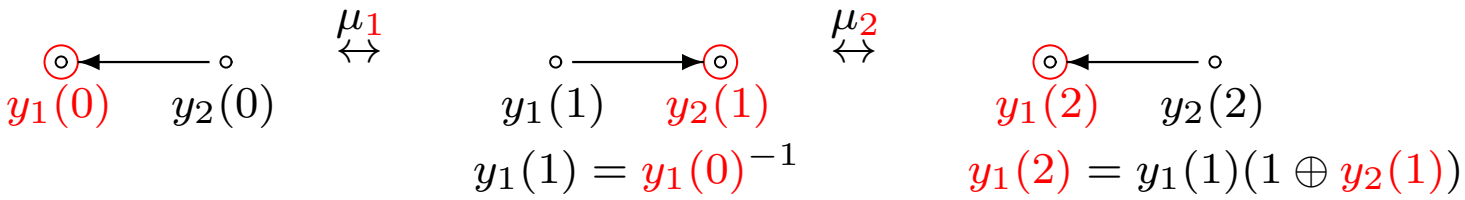
$\bigcirc$: forward mutation points

Set $(x(0), y(0)) = (x, y)$, and define $(x(u), y(u))$ ($u \in \mathbb{Z}$) by

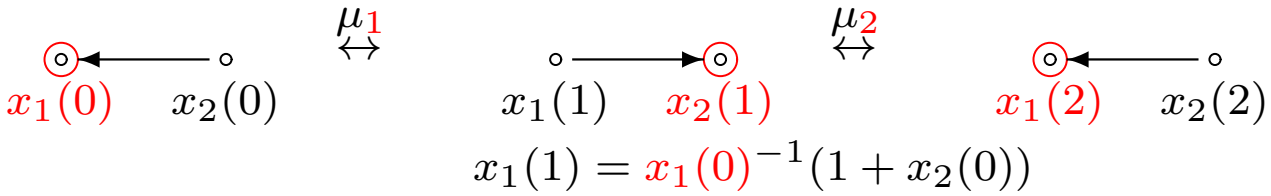
$$\cdots \overset{\mu_1}{\leftrightarrow} (Q^{\mathrm{op}}, x(-1), y(-1)) \overset{\mu_2}{\leftrightarrow} (Q, x(0), y(0)) \overset{\mu_1}{\leftrightarrow} (Q^{\mathrm{op}}, x(1), y(1)) \overset{\mu_2}{\leftrightarrow} (Q, x(2), y(2)) \overset{\mu_1}{\leftrightarrow} \cdots$$

Example 1. $(X, \ell) = (A_2, 2)$

Example of Y-system: $y_1(0)y_1(2) = 1 \oplus y_2(1)$.



Example of T-system: $x_1(0)x_1(2) = 1 + x_2(1)$.



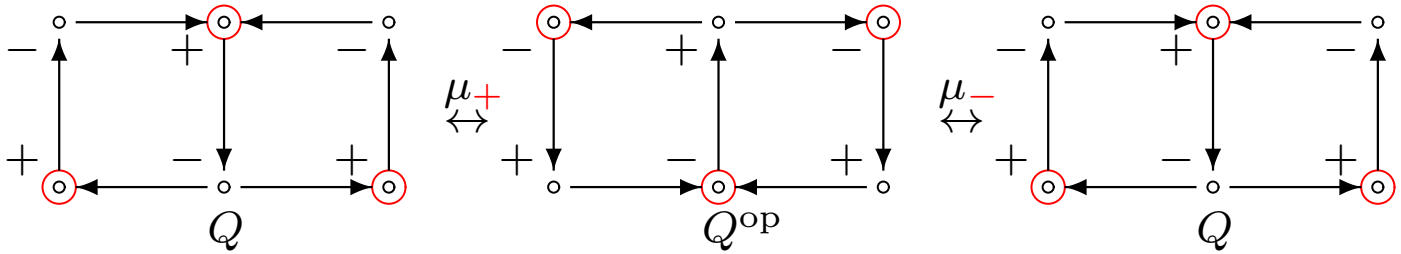
Summary:
(1) $\{y_a(u) \mid (u, a)$: forward mutation point$\}$ satisfies the Y-system.
(2) $\{x_a(u) \mid (u, a)$: forward mutation point$\}$ satisfies the T-system (by trivializing coefficients).
(3) The relation between x and $\hat{y}$ gives the relation between T- and Y-systems.
(5) The half period 5 of y-variables gives the half period of Y-systems $h + \ell = 3 + 2 = 5$.

Example 2. $(X, \ell) = (A_3, 3)$

Consider the initial quiver

$$Q = \begin{tikzcd}
(1,2) \arrow[r] & (2,2) \arrow[d, "+"] & (3,2) \arrow[l] \\
(a,m) = (1,1) \arrow[u, "-"] & (2,1) \arrow[l] \arrow[r] & (3,1) \arrow[u, "-"] \arrow[u, "+"']
\end{tikzcd}$$

Periodicity of quivers under the sequence of mutations $\mu_- \mu_+$:



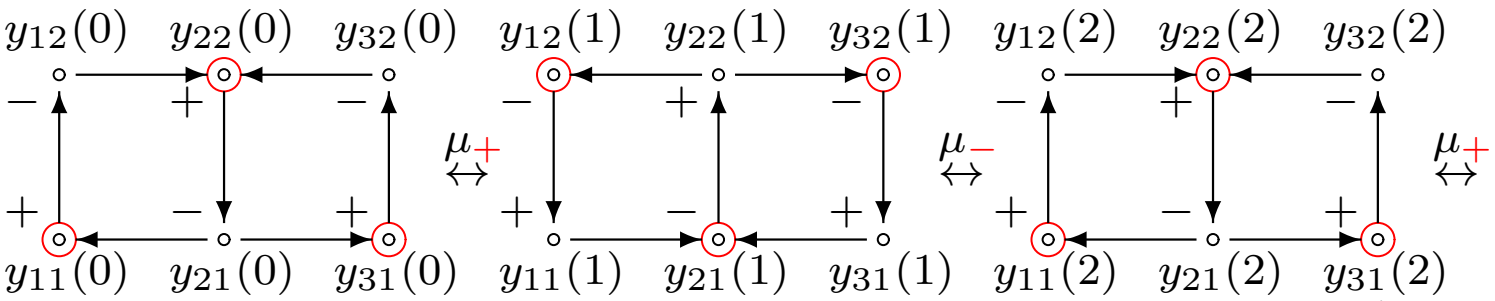
○ : forward mutation points

Set $(x(0), y(0)) = (x, y)$, and one can define $(x(u), y(u))$ ($u \in \mathbb{Z}$) by

$$\cdots \overset{\mu_+}{\leftrightarrow} (Q^{\mathrm{op}}, x(-1), y(-1)) \overset{\mu_-}{\leftrightarrow} (Q, x(0), y(0)) \overset{\mu_+}{\leftrightarrow} (Q^{\mathrm{op}}, x(1), y(1)) \overset{\mu_-}{\leftrightarrow} (Q, x(2), y(2)) \overset{\mu_+}{\leftrightarrow} \cdots$$

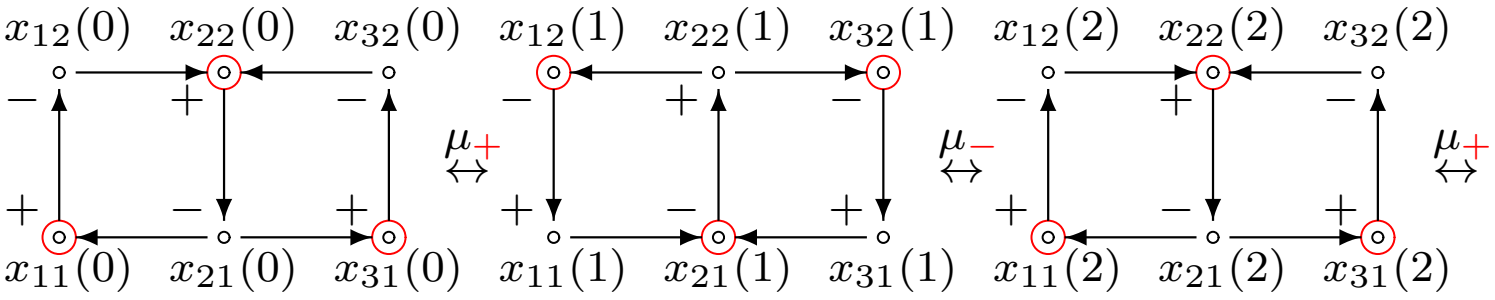
Example 2. $(X, \ell) = (A_3, 3)$

Example of Y-system: $y_{11}(0)y_{11}(2) = \dfrac{1 \oplus y_{21}(1)}{1 \oplus y_{12}(1)^{-1}}$.



$$y_{11}(1) = y_{11}(0)^{-1} \qquad y_{11}(2) = y_{11}(1)\frac{1 \oplus y_{21}(1)}{1 \oplus y_{12}(1)^{-1}}$$

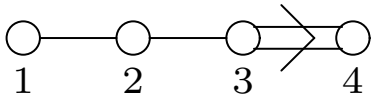
Example of T-systems: $x_{11}(0)x_{11}(2) = x_{21}(1) + x_{12}(1)$.



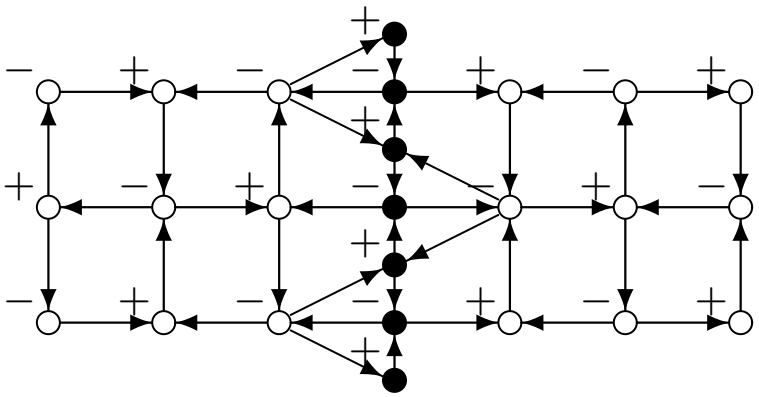
$$x_{11}(0)x_{11}(1) = x_{21}(0) + x_{12}(0)$$

Example 3. $(X, \ell) = (B_4, 4)$ [IIKKN 10]

Dynkin diagram of type B_4



Initial quiver Q:



Periodicity of quivers Q:

$$Q \overset{\mu^{\bullet}_{+}\mu^{\circ}_{+}}{\leftrightarrow} Q_1 \overset{\mu^{\bullet}_{-}}{\leftrightarrow} Q_2 \overset{\mu^{\bullet}_{+}\mu^{\circ}_{-}}{\leftrightarrow} Q_3 \overset{\mu^{\bullet}_{-}}{\leftrightarrow} Q$$

T- and Y-systems of type B_n

★ Y-system:

$$Y_m^{(a)}(u-2)Y_m^{(a)}(u+2) = \frac{(1+Y_m^{(a-1)}(u))(1+Y_m^{(a+1)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})} \quad (1 \le a \le n-2),$$

$$Y_m^{(n-1)}(u-2)Y_m^{(n-1)}(u+2) = \frac{\begin{array}{c}(1+Y_m^{(n-2)}(u))(1+Y_{2m-1}^{(n)}(u))(1+Y_{2m+1}^{(n)}(u))\\ \times(1+Y_{2m}^{(n)}(u-1))(1+Y_{2m}^{(n)}(u+1))\end{array}}{(1+Y_{m-1}^{(n-1)}(u)^{-1})(1+Y_{m+1}^{(n-1)}(u)^{-1})},$$

$$Y_{2m}^{(n)}(u-1)\,Y_{2m}^{(n)}(u+1) = \frac{1+Y_m^{(n-1)}(u)}{(1+Y_{2m-1}^{(n)}(u)^{-1})(1+Y_{2m+1}^{(n)}(u)^{-1})},$$

$$Y_{2m+1}^{(n)}(u-1)\,Y_{2m+1}^{(n)}(u+1) = \frac{1}{(1+Y_{2m}^{(n)}(u)^{-1})(1+Y_{2m+2}^{(n)}(u)^{-1})}.$$

★ T-system:

$$T_m^{(a)}(u-2)T_m^{(a)}(u+2) = T_m^{(a-1)}(u)T_m^{(a+1)}(u) + T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)$$
$$(1 \le a \le n-2),$$

$$T_m^{(n-1)}(u-2)T_m^{(n-1)}(u+2) = T_m^{(n-2)}(u)T_{2m}^{(n)}(u) + T_{m-1}^{(n-1)}(u)T_{m+1}^{(n-1)}(u),$$

$$T_{2m}^{(n)}(u-1)\,T_{2m}^{(n)}(u+1) = T_m^{(n-1)}(u-1)\,T_m^{(n-1)}(u+1) + T_{2m-1}^{(n)}(u)T_{2m+1}^{(n)}(u),$$

$$T_{2m+1}^{(n)}(u-1)\,T_{2m+1}^{(n)}(u+1) = T_m^{(n-1)}(u)T_{m+1}^{(n-1)}(u) + T_{2m}^{(n)}(u)T_{2m+2}^{(n)}(u).$$

Outline

Tropical y-variable

★ tropical y-variable (called principal coefficient in [FZ07]):
Recall the tropical semifield $\mathbb{P}_{\mathrm{trop}}(y)$ generated by $y = (y_i)_{i \in I}$ with

$$\prod_i y_i^{a_i} \oplus \prod_i y_i^{b_i} = \prod_i y_i^{\min(a_i, b_i)}$$

and the semifield homomorphism

$$\pi_{\mathbf{T}} : \mathbb{P}_{\mathrm{univ}}(y) \to \mathbb{P}_{\mathrm{trop}}(y)$$

Let us write

$$[y'_i] := \pi_{\mathbf{T}}(y'_i) \quad \text{tropical } y\text{-variable}$$

Recall (Separation formula)

$$y'_i = \left(\prod_{j \in I} y_j^{c'_{ji}} \right) \prod_{j \in I} F'_j(y)_{\oplus}^{b'_{ji}},$$

Theorem [Fomin-Zelevinsky 07]

$$[y'_i] = \prod_{j \in I} y_j^{c'_{ji}}, \quad [F'_i(y)_{\oplus}] = 1.$$

tropical y-variables = C-matrix

Categorification

★ Genelized cluster category (Buan et al, ... , Amiot, Plamondon)
B: any skew symmetric matrix
$\mathcal{C}$: generalized cluster category for B
To each seed (B', x', y') of $\mathcal{A}(B, x, y)$, one can canonically assign some rigid object $T' = \bigoplus_{i \in I} T'_i$ in $\mathcal{C}$.

Theorem (Plamondon10)

T: the rigid object assigned to the initial seed (B, x, y).
T': the rigid object assigned to a given seed (B', x', y').
Then,

$$\tilde{Q}' = \textit{the quiver of } \mathrm{End}_{\mathcal{C}}(T'), \quad (\tilde{Q}' \textit{: principal extension of } Q') \tag{2}$$

$$c'_{ij} = -\mathrm{ind}_{T'}(T_i[1])_j = \mathrm{ind}^{\mathrm{op}}_{T'}(T_i)_j, \tag{3}$$

$$g'_{ij} = \mathrm{ind}_T(T'_j)_i, \tag{4}$$

$$F'_i(y) = \sum_{e \in \mathbb{Z}^{\tilde{I}}_{\geq 0}} \chi(\mathrm{Gr}_e(\mathrm{Hom}_{\mathcal{C}}(T, T'_i[1]))) \prod_{j \in I} y_j^{e_j}. \tag{5}$$

Here, $\mathrm{Gr}_e(\,\cdot\,)$ is the quiver Grassmannian with dimension vector e, and $\chi(\,\cdot\,)$ is the Euler characteristic.

Criterion of periodicity

Definition

Let $\nu : I \to I$: be a bijection and let $(k_1, \ldots, k_L)$ be an I-sequence. Let (B', x', y') and (B'', x'', y'') be seeds such that for $(B'', x'', y'') = \mu_{k_L} \cdots \mu_{k_1}(B', x', y')$. We say that $(k_1, \ldots, k_L)$ is a ν-period of (B', x', y') if

$$b''_{\nu(i)\nu(j)} = b'_{ij}, \quad x''_{\nu(i)} = x'_i, \quad y''_{\nu(i)} = y'_i, \quad (i, j \in I).$$

Corollary of categorification [Plamondon 10, IIKKN 10]

$(k_1, \ldots, k_L)$ is a ν-period of (B', x', y') if and only if

$$[y''_{\nu(i)}] = [y'_i] \quad (i \in I).$$

Proof. $[y'] \Longrightarrow C' \Longrightarrow G' \Longrightarrow$ index of $T' \Longrightarrow T' \Longrightarrow (B', x', y')$ $\square$

Our slogan: Tropical y-variables know everything.

Outline

Exchange relation of tropical y-variables

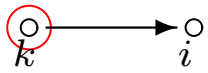
★ ε-expression of exchange relation

$$y_i'' = \begin{cases} {y_k'}^{-1} & i = k, \\ y_i' {y_k'}^{[\varepsilon b_{ki}']_+} (1 \oplus {y_k'}^{\varepsilon})^{-b_{ki}'} & i \neq k, \end{cases}$$

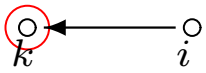
This expression is independent of $\varepsilon \in \{1, -1\}$.
tropical sign ε_k': sign of exponents of $[y_k']$
Set $\varepsilon = \varepsilon_k'$. Then, by definition $[1 \oplus {y_k'}^{\varepsilon_k'}] = 1$, and we have the exchange relation of tropical y-variables

$$[y_i''] = \begin{cases} [y_k']^{-1} & i = k, \\ [y_i'][y_k']^{[\varepsilon_k' b_{ki}']_+} & i \neq k, \end{cases}$$

Sign-arrow coordination: A nontrivial tropical mutation occurs only when $\varepsilon_k' b_{ki}' > 0$, i.e.,



$\varepsilon_k' > 0$ or



$\varepsilon_k' < 0$

Example 1. $(X, \ell) = (A_2, 2)$ [Fomin-Zelevinsky 03]

$$\begin{tikzcd} \overset{+}{\circ}_{1} & \overset{-}{\circ}_{2} \arrow[l] \end{tikzcd}$$

half period: $h + \ell = 3 + 2 = 5$

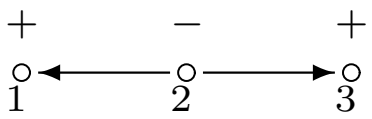
Sequence of mutations $(\dots, +, -, +, -, +, \dots)$:

$$\cdots \overset{+}{\longleftrightarrow} (B(-1), y(-1)) \overset{-}{\longleftrightarrow} (B(0), y(0)) \overset{+}{\longleftrightarrow} (B(1), y(1)) \overset{-}{\longleftrightarrow} \cdots$$

$$\begin{aligned}
y_1(0) &= y_1, & y_2(0) &= y_2,\\
y_1(1) &= y_1^{-1}, & y_2(1) &= y_2(1 \oplus y_1),\\
y_1(2) &= y_1^{-1}(1 \oplus y_1 \oplus y_1 y_2), & y_2(2) &= y_2^{-1}(1 \oplus y_1)^{-1},\\
y_1(3) &= y_1(1 \oplus y_1 \oplus y_1 y_2)^{-1}, & y_2(3) &= y_1^{-1} y_2^{-1}(1 \oplus y_2),\\
y_1(4) &= y_2^{-1}, & y_2(4) &= y_1 y_2 (1 \oplus y_2)^{-1},\\
y_1(5) &= y_2, & y_2(5) &= y_1.
\end{aligned}$$

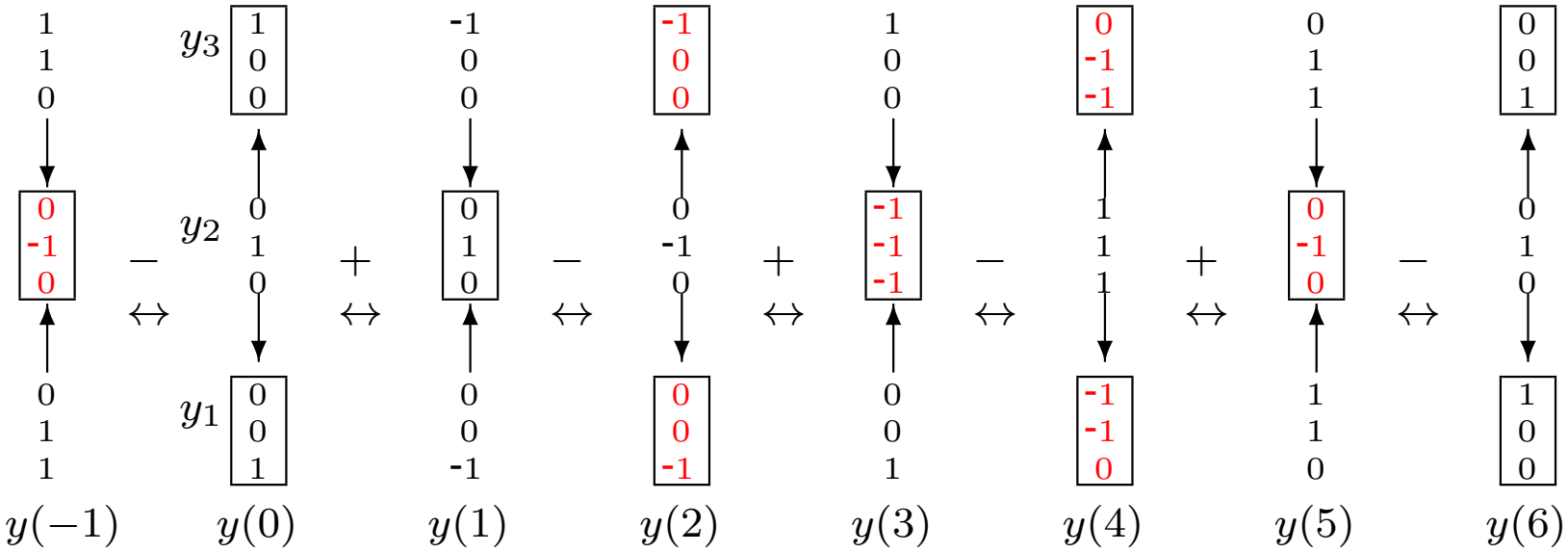
$$\begin{tikzcd}
\boxed{\begin{matrix}-1\\0\end{matrix}} \arrow[u] & y_2 & \begin{matrix}1\\0\end{matrix} \arrow[d] & \boxed{\begin{matrix}1\\0\end{matrix}} \arrow[u] & \begin{matrix}-1\\0\end{matrix} \arrow[d] & \boxed{\begin{matrix}-1\\-1\end{matrix}} \arrow[u] & \begin{matrix}1\\1\end{matrix} \arrow[d] & \boxed{\begin{matrix}0\\1\end{matrix}} \arrow[u] & \begin{matrix}0\\-1\end{matrix} \arrow[d] \\
\begin{matrix}1\\1\end{matrix} & y_1 & \boxed{\begin{matrix}0\\1\end{matrix}} & \begin{matrix}0\\-1\end{matrix} & \boxed{\begin{matrix}0\\-1\end{matrix}} & \begin{matrix}0\\1\end{matrix} & \boxed{\begin{matrix}-1\\0\end{matrix}} & \begin{matrix}1\\0\end{matrix} & \boxed{\begin{matrix}1\\0\end{matrix}} \\
y(-1) \arrow[r, leftrightarrow, "-"] & y(0) \arrow[r, leftrightarrow, "+"] & y(1) \arrow[r, leftrightarrow, "-"] & y(2) \arrow[r, leftrightarrow, "+"] & y(3) \arrow[r, leftrightarrow, "-"] & y(4) \arrow[r, leftrightarrow, "+"] & y(5) \arrow[r, leftrightarrow, "-"] & y(6) &
\end{tikzcd}$$

Example 2. $(X, \ell) = (A_3, 2)$ [Fomin-Zelevinsky 03]



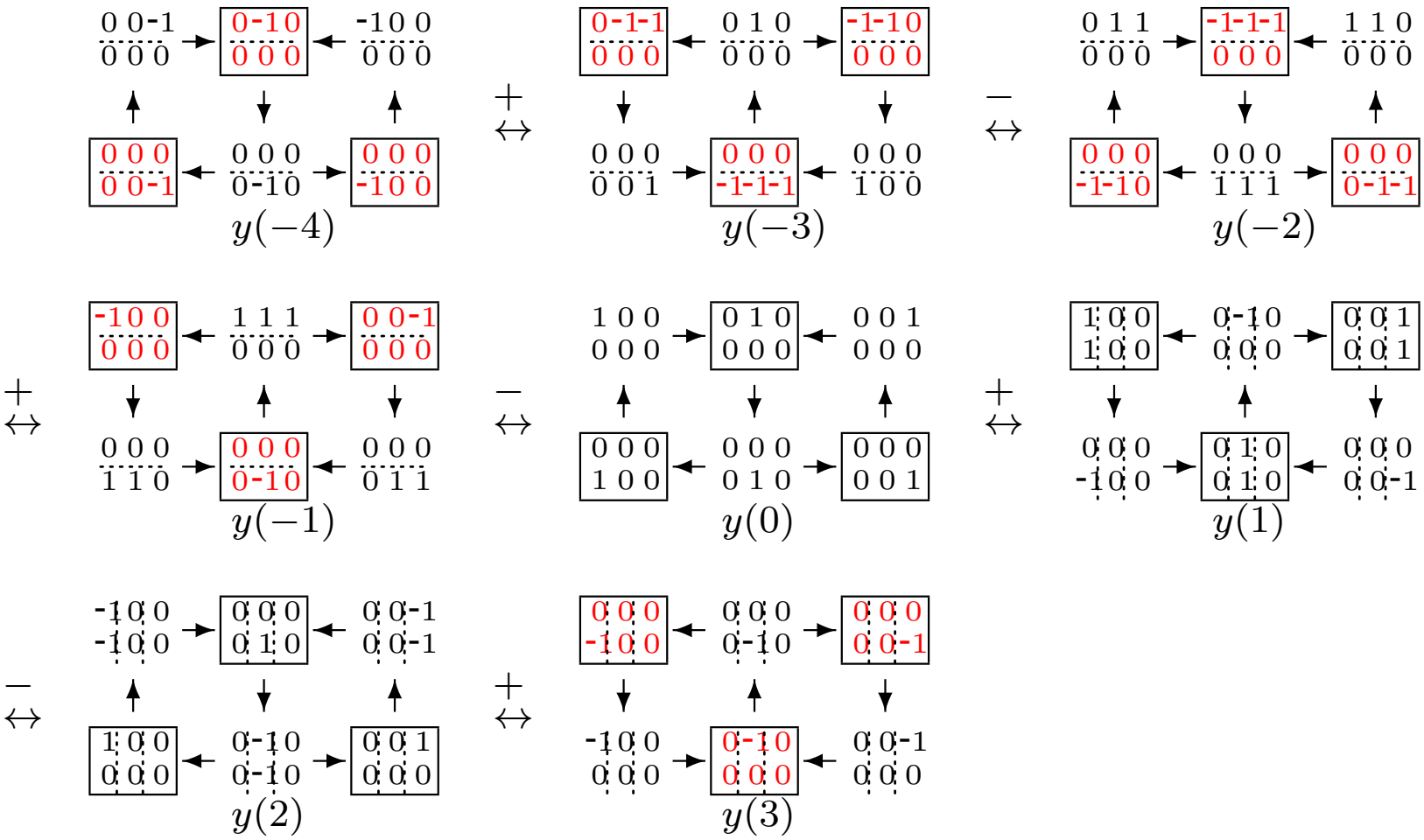
half period: $h + \ell = 4 + 2 = 6$

Sequence of mutations $(\ldots, +, -, +, -, +, \ldots)$:



Example 3. $(X, \ell) = (A_3, 3)$ [N09,IIKKN10]

half period: $h + \ell = 4 + 3 = 7$



Factorization property (due to sign-arrow coordination) [N09]

Outline

Pentagon relation for classical dilogarithm

Euler dilogarithm

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad (|x| < 1)$$
$$= -\int_0^x \frac{\log(1-y)}{y} dy \quad (x \leq 1).$$

Rogers dilogarithm

$$L(x) = -\frac{1}{2}\int_0^x \left\{ \frac{\log(1-y)}{y} + \frac{\log y}{1-y} \right\} dy \quad (0 \leq x \leq 1).$$

$$L(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log x \log(1-x), \quad (0 \leq x \leq 1),$$
$$-L\left(\frac{x}{1+x}\right) = \mathrm{Li}_2(-x) + \frac{1}{2}\log x \log(1+x), \quad (0 \leq x)$$

Properties:

$$L(0) = 0, \quad L(1) = \frac{\pi^2}{6},$$

(Euler) $\quad L(x) + L(1-x) = \dfrac{\pi^2}{6},$

(Abel, pentagon identity) $\quad L(x) + L(y) = L(xy) + L\left(\dfrac{x(1-y)}{1-xy}\right) + L\left(\dfrac{y(1-x)}{1-xy}\right).$

Central charge identity in conformal field theory

★ Fix (X, ℓ). X: simply laced.

The constant Y-system: For variables $\{Y_m^{(a)} \mid a \in \mathcal{I}, m = 1, \ldots, \ell - 1\}$,

$$(Y_m^{(a)})^2 = \frac{\prod_{b \in \mathcal{I}, b \sim a}(1 + Y_m^{(b)})}{(1 + Y_{m-1}^{(a)\,-1})(1 + Y_{m+1}^{(a)\,-1})}.$$

There exists a unique positive real solution [Nahm-Keegan 09].

Dilogarithm identities conjecture in CFT [Kirillov, Bazhanov, Reshetikhin, 90]

$$\frac{6}{\pi^2} \sum_{a \in \mathcal{I}} \sum_{m=1}^{\ell-1} L\left(\frac{Y_m^{(a)}}{1 + Y_m^{(a)}}\right) = \frac{\ell \dim \mathfrak{g}}{h + \ell} - n,$$

where $\mathfrak{g}$ is the simply Lie algebra of type X and $n = |\mathcal{I}| = \operatorname{rank} X$.

Functional generalization [Gliozzi-Tateo 95]

$$\frac{6}{\pi^2} \sum_{a \in \mathcal{I}} \sum_{m=1}^{\ell-1} \sum_{u=0}^{2(h+\ell)-1} L\left(\frac{Y_m^{(a)}(u)}{1 + Y_m^{(a)}(u)}\right) = 2hn(\ell - 1).$$

The former follows from the latter.

Dilogarithm identitities associated with periods of seeds

$(k_0, \ldots, k_{L-1})$: a ν-period of the initial seed (B, x, y).

$$(B(0), y(0)) = (B, y) \overset{\mu_{k_0}}{\leftrightarrow} (B(1), y(1)) \overset{\mu_{k_1}}{\leftrightarrow} \cdots \overset{\mu_{k_{L-1}}}{\leftrightarrow} (B(L), y(L))$$

ε_t: the tropical sign of $y(t)$ at k_t.
$(\varepsilon_0, \ldots, \varepsilon_{L-1})$: the *tropical sign-sequence*

Theorem (Dilogarithm identities [N 10])

The following equalities hold for any evaluation of the initial y-variables y_i in $\mathbb{R}_{>0}$.

$$\sum_{t=0}^{L-1} \varepsilon_t L\left(\frac{y_{k_t}(t)^{\varepsilon_t}}{1 + y_{k_t}(t)^{\varepsilon_t}}\right) = 0,$$

$$\frac{6}{\pi^2} \sum_{t=0}^{L-1} L\left(\frac{y_{k_t}(t)}{1 + y_{k_t}(t)}\right) = N_-,$$

$$\frac{6}{\pi^2} \sum_{t=0}^{L-1} L\left(\frac{1}{1 + y_{k_t}(t)}\right) = N_+.$$

N_+ and N_-: the total numbers of 1 and -1 among $\varepsilon_0, \ldots, \varepsilon_{L-1}$, respectively.

Apply it for the periods corresponding to the Y-systems and count the number N_-
$\Longrightarrow$ dilogarithm identities in CFT and their functional generalizations

Example: Dilogarithm identity for $(X, \ell) = (A_2, 2)$

Simplest case:

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \underset{1}{\circ} \longleftarrow \underset{2}{\circ}$$

$$(B(0), y(0)) \overset{\mu_1}{\leftrightarrow} (B(1), y(1)) \overset{\mu_2}{\leftrightarrow} (B(2), y(2)) \overset{\mu_1}{\leftrightarrow} (B(3), y(3)) \overset{\mu_2}{\leftrightarrow} \cdots$$

$$\begin{cases} y_1(0) = y_1 \\ y_2(0) = y_2, \end{cases} \quad \begin{cases} y_1(1) = y_1^{-1} \\ y_2(1) = y_2(1 + y_1), \end{cases} \quad \begin{cases} y_1(2) = y_1^{-1}(1 + y_2 + y_1 y_2) \\ y_2(2) = y_2^{-1}(1 + y_1)^{-1}, \end{cases}$$

$$\begin{cases} y_1(3) = y_1(1 + y_2 + y_1 y_2)^{-1} \\ y_2(3) = y_1^{-1} y_2^{-1}(1 + y_2), \end{cases} \quad \begin{cases} y_1(4) = y_2^{-1} \\ y_2(4) = y_1 y_2 (1 + y_2)^{-1}, \end{cases} \quad \begin{cases} y_1(5) = y_2 \\ y_2(5) = y_1. \end{cases}$$

$(+, +, -, -, -)$: tropical sign-sequence.
Putting them into

$$\sum_{t=0}^{L-1} \varepsilon_t L\left(\frac{y_{k_t}(t)^{\varepsilon_t}}{1 + y_{k_t}(t)^{\varepsilon_t}}\right) = 0.$$

we obtain

$$L\left(\frac{y_1}{1 + y_1}\right) + L\left(\frac{y_2(1 + y_1)}{1 + y_2 + y_1 y_2}\right)$$
$$- L\left(\frac{y_1}{(1 + y_1)(1 + y_2)}\right) - L\left(\frac{y_1 y_2}{1 + y_2 + y_1 y_2}\right) - L\left(\frac{y_2}{1 + y_2}\right) = 0.$$

It coincides with the pentagon relation.

Pentagon relation for quantum dilogarithm

quantum dilogarithm $\mathbf{\Psi}_q(x)$ ($|q| < 1$)

$$\mathbf{\Psi}_q(x) = \prod_{k=0}^{\infty} (1 + q^{2k+1})^{-1}, \quad x \in \mathbb{C}.$$

Theorem [Faddeev-Kashaev 94]

(a). Asymptotic behavior: In the limit $q \to 1^-$,

$$\mathbf{\Psi}_q(x) \sim \exp\left(-\frac{\mathrm{Li}_2(-x)}{2\log q}\right).$$

(b). Quantum pentagon identity: If $UV = q^2 VU$, then

$$\mathbf{\Psi}_q(U)\mathbf{\Psi}_q(V) = \mathbf{\Psi}_q(V)\mathbf{\Psi}_q(q^{-1}UV)\mathbf{\Psi}_q(U).$$

Moreover, in the limit $q \to 1^-$, it reduces to the classical pentagon identity.

cf. classical pentagon identity

$$L(x) + L(y) = L(xy) + L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right).$$

It is not a smooth termwise limit.

Classical and quantum dilogarithm identities

$$\begin{tikzcd}
\boxed{\text{period of quantum cluster algebra}} \arrow[r, "\text{(III)}"] \arrow[d, leftrightarrow, "\text{(II)}"'] & \boxed{\text{quantum dilogarithm identity}} \arrow[d, "\text{(IV) semiclassical limit}"] \\
\boxed{\text{period of classical cluster algebra}} \arrow[r, "\text{(I)}"] & \boxed{\text{classical dilogarithm identity}}
\end{tikzcd}$$

specialization to A_2 case: pentagon identity

(I) [N 10]
(II) Essentially [Berenstein-Zelevinsky 05], [Fock-Goncharov 09]
(III) [Fock-Goncharov 09], [Keller 11], [Kashaev-N 11], [Nagao 11]
(IV) [Kashaev-N 11]

Announcement

Infinite Analysis 11 Winter School
Quantum cluster algebras and related topics
Date: December 20 (tue) – 23 (fri), 2011
Place: Graduate Shool of Science, Osaka Univeristy

Three day lecture and one day miniworkshop
Lectures: A. Zelevinsky, B. Leclerc, K. Nagao, T. Nakanishi
Mini-workshop: F. Qin, P. Lampe, Y. Kimura, K. Hasegawa, T. Nakashima

workshop website: https://sites.google.com/site/ia11qca/