Dilogarithm identities and cluster algebras

Tomoki Nakanishi

Nagoya University

Talk presented at JMS meeting at Keio University on March 26, 2010

The pdf file of this presentation will be available at my Web site.

Based on the paper:
[N09] T. Nakanishi, Dilogarithm identities for conformal field theories and cluster algebras: simply laced case, arXiv:0909.5480

Rogers dilogarithm

Rogers dilogarithm function $L(x)$

$$L(x) = -\frac{1}{2}\int_0^x \left\{\frac{\log(1-y)}{y} + \frac{\log y}{1-y}\right\} dy \quad (0 \le x \le 1).$$

Basic properties of $L(x)$

$$L(0) = 0, \quad L(1) = \zeta(2) = \frac{\pi^2}{6},$$

$$L(x) + L(1-x) = \frac{\pi^2}{6} \quad \text{(Euler)},$$

$$L(x) + L(y) + L(1-xy) + L\left(\frac{1-x}{1-xy}\right) + L\left(\frac{1-y}{1-xy}\right) = \frac{\pi^2}{2} \quad \text{(Abel, 5-term relation)}.$$

Only few special values are known, e.g.,

$$\frac{6}{\pi^2}L\left(\frac{1}{2}\right) = \frac{1}{2}, \quad \frac{6}{\pi^2}L\left(\frac{-\sqrt{5}+3}{2}\right) = \frac{2}{5}, \quad \frac{6}{\pi^2}L\left(\frac{\sqrt{5}-1}{2}\right) = \frac{3}{5}.$$

Dilogarithm identities in conformal fi eld theories

X: any simply laced Dynkin diagram of finite type with index set I
$\ell \geq 2$: any integer
constant Y-system: $\{Y_m^{(a)} \mid a \in I; 1 \leq m \leq \ell - 1\}$: a family of positive real numbers

$$(Y_m^{(a)})^2 = \frac{\prod_{b:b\sim a} (1 + Y_m^{(b)})}{(1 + Y_{m-1}^{(a)\,-1})(1 + Y_{m+1}^{(a)\,-1})}, \tag{cY}$$

$b \sim a$: b is adjacent to a in X, $Y_0^{(a)-1} = Y_\ell^{(a)-1} = 0$.
There exists a unique positive real solution of (cY). [Nahm-Keegan 09]

Conjecture 1 (Dilogarithm identities) [Bazhanov, Kirillov, Reshetikhin, 86–90]

For the unique positive real solution $\{Y_m^{(a)} \mid a \in I; 1 \leq m \leq \ell - 1\}$ of (cY),

$$\frac{6}{\pi^2} \sum_{a \in I} \sum_{m=1}^{\ell-1} L\left(\frac{Y_m^{(a)}}{1 + Y_m^{(a)}}\right) = \frac{\ell \dim \mathfrak{g}}{h + \ell} - r,$$

h: Coxeter number of X, $\mathfrak{g}$: simple Lie algebra of type X.

(asymptotics of entropy of spin chains/S-matrix models) = (central charge of CFT)
Proved for $X = A_r$ [Kirillov 90].
Related to Rogers-Ramanujan-type identities, KR modules, hyperbolic 3-folds, etc.
Only partially proved in B.C. (=Before Cluster algebra [2000])

Functional dilogarithm identities (1)

Y-system: [Zamolodchikov 91, Kuniba-Nakanishi 92, Ravanini-Tateo-Valleriani 93]
(X, X'): a pair of simply laced Dynkin diagrams of finite type.
$\{Y_{ii'}(u) \mid i \in I, i' \in I', u \in \mathbb{Z}\}$: a family of variables

$$Y_{ii'}(u-1)Y_{ii'}(u+1) = \frac{\prod_{j:j\sim i}(1+Y_{ji'}(u))}{\prod_{j':j'\sim i'}(1+Y_{ij'}(u)^{-1})}, \tag{Y}$$

where $j \sim i$: j is adjacent to i in X, $j' \sim i'$: j' is adjacent to i' in X'.

Conjecture 2 (Periodicity) [Ravanini-Tateo-Valleriani 93]

For $\{Y_{ii'}(u) \mid i \in I, i' \in I', u \in \mathbb{Z}\}$ satisfying (Y),

$$Y_{ii'}(u + 2(h+h')) = Y_{ii'}(u), \quad h : \text{Coxeter number of } X.$$

Conjecture 3 (Functional dilogarithm identities) [Gliozzi-Tateo 95]

For a family of positive real numbers $\{Y_{aa'}(u) \mid a \in I, a' \in I', u \in \mathbb{Z}\}$ satisfying (Y),

$$\frac{6}{\pi^2} \sum_{(i,i')\in I\times I'} \sum_{0\le u<2(h+h')} L\left(\frac{Y_{ii'}(u)}{1+Y_{ii'}(u)}\right) = 2hrr', \quad r = \operatorname{rank} X.$$

Conjecture 3 $\Longrightarrow$ Conjecture 1; set $X' = A_{\ell-1}$, take a *constant solution* $Y_{ii'} = Y_{ii'}(u)$.

Functional dilogarithm identities (2)

Conjecture 2 (Periodicity) [Ravanini-Tateo-Valleriani 93]

$$Y_{ii'}(u + 2(h + h')) = Y_{ii'}(u), \quad h : \text{Coxeter number of } X.$$

Conjecture 3 (Functional dilogarithm identities) [Gliozzi-Tateo 95]

$$\frac{6}{\pi^2} \sum_{(i,i') \in I \times I'} \sum_{0 \le u < 2(h+h')} L\left(\frac{Y_{ii'}(u)}{1 + Y_{ii'}(u)}\right) = 2hrr', \quad r = \operatorname{rank} X. \qquad \text{(FDI)}$$

(1) [Frenkel-Szenes 95] proved Conjs. 2 & 3 for $(X, X') = (A_r, A_1)$ ($X = A_r$ and $\ell = 2$ case) by explicit solution of (Y).
Conj. 3: (Constancy of LHS in (FDI)) + (evaluation at constant solution).

(2) [Fomin-Zelevinsky 03] proved Conj. 2 for $(X, X') = (\text{any}, A_1)$ by 'cluster algebra-like' formulation of (Y) + Coxeter transformation of X.

(3) [Chapoton 05] proved Conj. 3 for $(X, X') = (\text{any}, A_1)$ by (1) + (2).
Conj. 3: (Constancy of LHS in (FDI)) + (evaluation in some limit).

(4) [Fomin-Zelevinsky 07] fully integrated (Y) into the cluster algebra with coefficients.
also introduced F-polynomials

(5) [Keller 08] proved Conj. 2 for any X and X' by (4) + 'cluster category'.

Theorem [N 09]

Conjecture 3 is true for any X and X'.

Essence of proof

We want to show the identity.

$$\frac{6}{\pi^2}\sum_{(i,i')\in I\times I'}\ \sum_{0\le u<2(h+h')} L\left(\frac{Y_{ii'}(u)}{1+Y_{ii'}(u)}\right)=2hrr',\quad r=\operatorname{rank}X. \qquad \text{(FDI)}$$

Step 1. Formulate the Y-system (Y) by cluster algebra with coefficients $\mathcal{A}(Q,x,y)$, where Q is some quiver. [Keller 08]

Step 2. Show the consistency condition of LHS of (FDI). [Frenkel-Szenes 95]
In $\mathbb{Q}_{\mathrm{sf}}(y)\wedge\mathbb{Q}_{\mathrm{sf}}(y)$

$$\sum_{\substack{(i,i')\in I\times I'\\ 0\le u<2(h+h')}} y_{ii'}(u)\wedge(1+y_{ii'}(u))=0.$$

This is almost automatic by cluster algebra machinery (F-polynomials, etc.).

Step 3. Evaluate the LHS of (FDI) in the '$0/\infty$ limit'. [Chapoton 05]
Recall that

$$\frac{6}{\pi^2}L\left(\frac{Y}{1+Y}\right)=\begin{cases}0 & Y\to 0\\ 1 & Y\to+\infty.\end{cases}$$

Suppose there is a limit s.t. each $Y_{ii'}(u)$ goes either 0 or $+\infty$. Then,

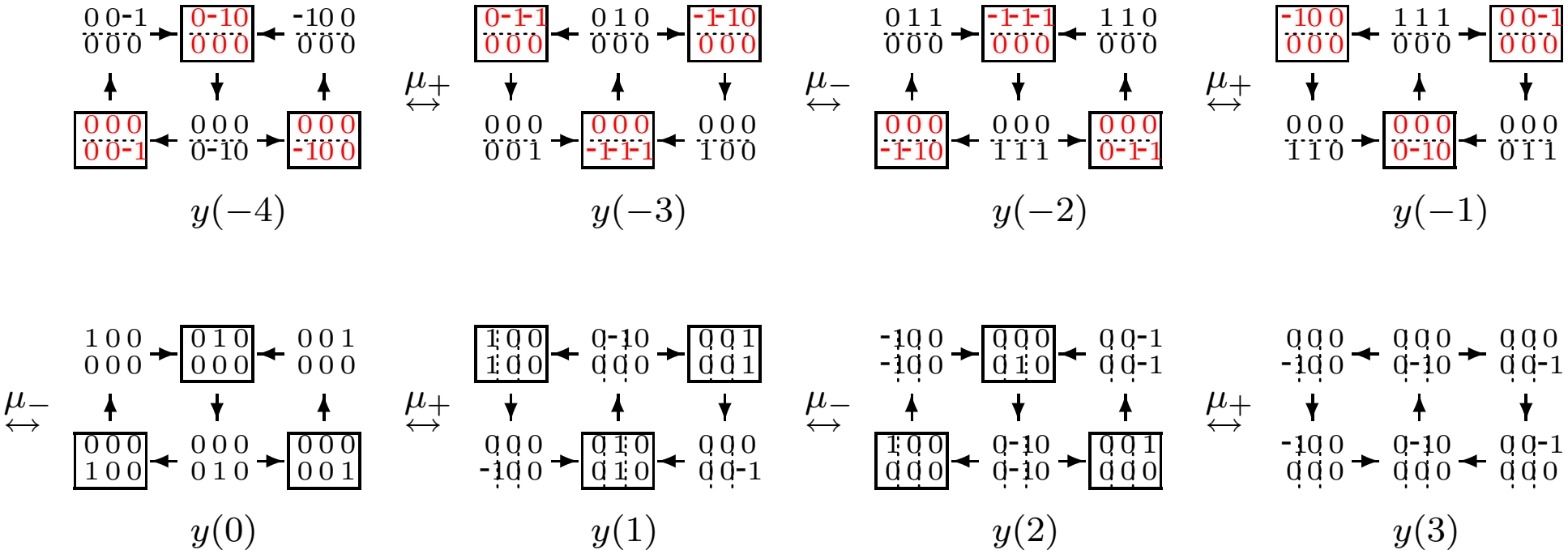
(the LHS of (FDI)) $=$ the total number of $Y_{ii'}(u)$'s going to $+\infty$ for $0\le u<2(h+h')$.

Such a limit can be systematically studied by the tropical Y-system.

Example. Tropical Y-system for $X = A_3$, $X' = A_2$

$X = A_3$, $X' = A_2$. $h = 4$, $h' = 3$, $r = 3$, $r' = 2$.
We want to show the identity.

$$\frac{6}{\pi^2} \sum_{(i,i')\in I\times I'} \sum_{\substack{0\le u<h+h' \\ i+i'+u:\text{even}}} L\left(\frac{Y_{ii'}(u)}{1+Y_{ii'}(u)}\right) = \frac{1}{4}2hrr' = \frac{1}{4}2\cdot 4\cdot 3\cdot 2 = 12. \qquad \text{(FDI)}$$



negative in $-h \le u < 0$, positive in $0 \le u < h'$ 'factorization property'

T and Y-systems, dilogarithm identities and cluster algebras: nonsimply laced case

Tomoki Nakanishi

Nagoya University

Talk presented at JMS meeting at Keio University on March 26, 2010

Based on the paper:
[IIKKN 10] R. Inoue, O. Iyama, B. Keller, A. Kuniba, T. Nakanishi,
Periodicities of T and Y-systems, dilogarithm identities, and cluster algebras I: Type B_r, arXiv:1001.1880
Periodicities of T and Y-systems, dilogarithm identities, and cluster algebras I: Type C_r, F_4, and G_2, arXiv:1001.1881

Review: Simply laced case (1)

X: simply laced Dynkin diagram A_r, D_r, E_6, E_7, E_8 with index set I
$\ell \geq 2$: integer

Y-system [Zamolodchikov 91, Kuniba-Nakanishi 92, Ravanini-Tateo-Valleriani 93]

$\{Y_m^{(a)}(u) \mid \in I; m = 1, \ldots, \ell - 1; u \in \mathbb{Z}\}$: a family of variables

$$Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{\prod_{b:b\sim a}(1 + Y_m^{(b)}(u))}{(1 + Y_{m-1}^{(a)}(u)^{-1})(1 + Y_{m+1}^{(a)}(u)^{-1})}, \tag{Y}$$

where $b \sim a$: b is adjacent to a in X, $Y_0^{(a)}(u)^{-1} = Y_\ell^{(a)}(u)^{-1} = 0$.

They arise from the thermodynamic Bethe ansatz (TBA) equation of integrable lattice/S-matrix models.

T-system [Kuniba-Nakanishi-Suzuki 94]

$\{T_m^{(a)}(u) \mid \in I; m = 1, \ldots, \ell - 1; u \in \mathbb{Z}\}$: a family of variables

$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = \prod_{b:b\sim a} T_m^{(b)}(u) + T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u), \tag{T}$$

where $T_0^{(a)}(u) = T_\ell^{(a)}(u) = 1$.

They are relations among the transfer matrices of integrable lattice models and also q-characters of Kirillov-Reshetikhin modules.

Review: Simply laced case (2)

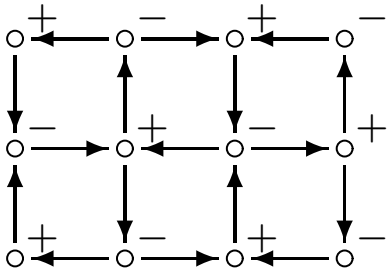
Theorem ([Keller 08], [Inoue-Iyama-Kuniba-Nakanishi-Suzuki 08])

(1) $Y_m^{(a)}(u + 2(h+\ell)) = Y_m^{(a)}(u)$. *($h$: the Coxeter number of X)*
(2) $T_m^{(a)}(u + 2(h+\ell)) = T_m^{(a)}(u)$.

The outline of proof. (after Keller)
Step 1. Formulation by cluster algebra with coefficient [Fomin-Zelevinsky 07]

	cluster algebra	coefficient semifield
Y-system	coefficients y	universal semifield
T-system	cluster variables x	trivial semifield

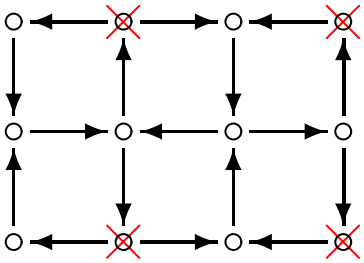
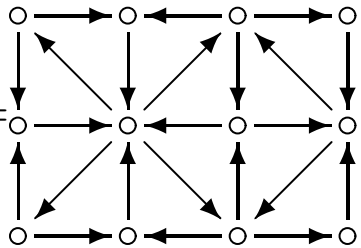
Example. $X = A_4$, $\ell = 4$

$Q =$ 

$\mu_-\mu_+(Q) = Q$

(Q, x, y): initial seed

The theorem is reformulated as $(\mu_-\mu_+)^{h+\ell}(Q, x, y) = (Q, x, y)$.
Step 2. Show periodicity by (generalized) cluster category

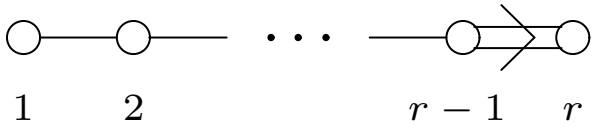
$Q =$  $\overset{\mu_\times}{\longleftrightarrow}$ $Q' =$ 

categorification by cluster category $\mathcal{C}(KX \otimes KA_{\ell-1})$ [Amiot 08]

factorization $\mu_\times\mu_-\mu_+\mu_\times = \tau^{-1} \otimes \mathrm{id} = \mathrm{id} \otimes \tau'$ (τ, τ': AR-translation), $\tau^h(T) = T[2]$.

Nonsimply laced case (1)

Example: $X = B_r$, $I = \{1, \dots, r\}$



$$t_1 = \cdots = t_{r-1} = 1, \quad t_r = 2.$$

$\ell \geq 2$: an integer

Y-system of type B_r [Kuniba-Nakanishi 92]

$\{Y_m^{(a)}(u) \mid a \in I; m = 1, \dots, t_a\ell - 1; u \in \frac{1}{2}\mathbb{Z}\}$: a family of variables

$$Y_m^{(a)}(u-1)Y_m^{(a)}(u+1) = \frac{(1+Y_m^{(a-1)}(u))(1+Y_m^{(a+1)}(u))}{(1+Y_{m-1}^{(a)}(u)^{-1})(1+Y_{m+1}^{(a)}(u)^{-1})} \ (1 \leq a \leq r-2),$$

$$Y_m^{(r-1)}(u-1)Y_m^{(r-1)}(u+1) = \frac{\begin{array}{c}(1+Y_m^{(r-2)}(u))(1+Y_{2m-1}^{(r)}(u))(1+Y_{2m+1}^{(r)}(u)) \\ \times(1+Y_{2m}^{(r)}(u-\frac{1}{2}))(1+Y_{2m}^{(r)}(u+\frac{1}{2}))\end{array}}{(1+Y_{m-1}^{(r-1)}(u)^{-1})(1+Y_{m+1}^{(r-1)}(u)^{-1})},$$

$$Y_{2m}^{(r)}(u-\tfrac{1}{2})\,Y_{2m}^{(r)}(u+\tfrac{1}{2}) = \frac{1+Y_m^{(r-1)}(u)}{(1+Y_{2m-1}^{(r)}(u)^{-1})(1+Y_{2m+1}^{(r)}(u)^{-1})},$$

$$Y_{2m+1}^{(r)}(u-\tfrac{1}{2})\,Y_{2m+1}^{(r)}(u+\tfrac{1}{2}) = \frac{1}{(1+Y_{2m}^{(r)}(u)^{-1})(1+Y_{2m+2}^{(r)}(u)^{-1})},$$

where $Y_m^{(0)}(u) = Y_0^{(a)}(u)^{-1} = Y_{t_a\ell}^{(a)}(u)^{-1} = 0$.

Nonsimply laced case (2)

T-system of type B_r [Kuniba-Nakanishi-Suzuki 94]

$\{T_m^{(a)}(u) \mid a \in I; m = 1, \ldots, t_a\ell - 1; u \in \frac{1}{2}\mathbb{Z}\}$: a family of variables

$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_m^{(a-1)}(u)T_m^{(a+1)}(u) + T_{m-1}^{(a)}(u)T_{m+1}^{(a)}(u)$$
$$(1 \le a \le r-2),$$
$$T_m^{(r-1)}(u-1)T_m^{(r-1)}(u+1) = T_m^{(r-2)}(u)T_{2m}^{(r)}(u) + T_{m-1}^{(r-1)}(u)T_{m+1}^{(r-1)}(u),$$
$$T_{2m}^{(r)}\left(u-\tfrac{1}{2}\right)T_{2m}^{(r)}\left(u+\tfrac{1}{2}\right) = T_m^{(r-1)}\left(u-\tfrac{1}{2}\right)T_m^{(r-1)}\left(u+\tfrac{1}{2}\right)$$
$$+ T_{2m-1}^{(r)}(u)T_{2m+1}^{(r)}(u),$$
$$T_{2m+1}^{(r)}\left(u-\tfrac{1}{2}\right)T_{2m+1}^{(r)}\left(u+\tfrac{1}{2}\right) = T_m^{(r-1)}(u)T_{m+1}^{(r-1)}(u) + T_{2m}^{(r)}(u)T_{2m+2}^{(r)}(u),$$

where $T_m^{(0)}(u) = T_0^{(a)}(u) = T_{t_a\ell}^{(a)}(u) = 1$.

Main results

Theorem 1 [IIKKN 10], conjectured by [Kuniba-Nakanishi-Suzuki 94]

(1) $Y_m^{(a)}(u + 2(h^\vee + \ell)) = Y_m^{(a)}(u)$. ($h^\vee$: the dual Coxeter number of X)
(2) $T_m^{(a)}(u + 2(h^\vee + \ell)) = T_m^{(a)}(u)$.

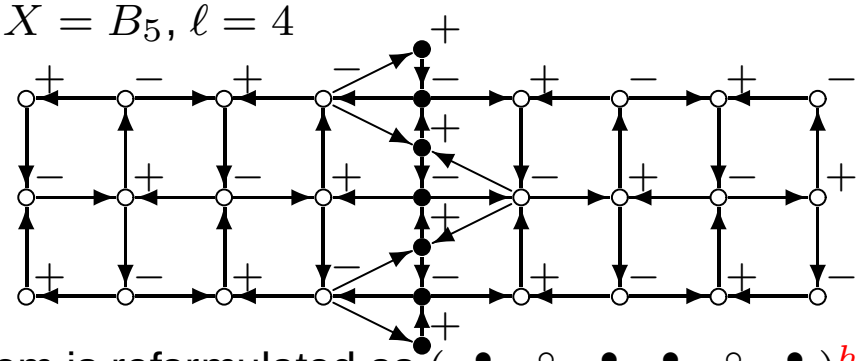
Theorem 2 [IIKKN 10], conjectured by [Kirillov 90], [Kuniba 93]

For the unique positive real solution $\{Y_m^{(a)} \mid a \in I; 1 \le m \le t_a\ell - 1\}$ of the constant Y-system,

$$\frac{6}{\pi^2} \sum_{a \in I} \sum_{m=1}^{t_a\ell - 1} L\left(\frac{Y_m^{(a)}}{1 + Y_m^{(a)}}\right) = \frac{\ell \dim \mathfrak{g}}{h^\vee + \ell} - r,$$

$h^\vee$: dual Coxeter number of X, $\mathfrak{g}$: simple Lie algebra of type X.

Outline of the proof of Theorem 1.
Step 1. Formulation by cluster algebra with coefficient
Example. $X = B_5$, $\ell = 4$

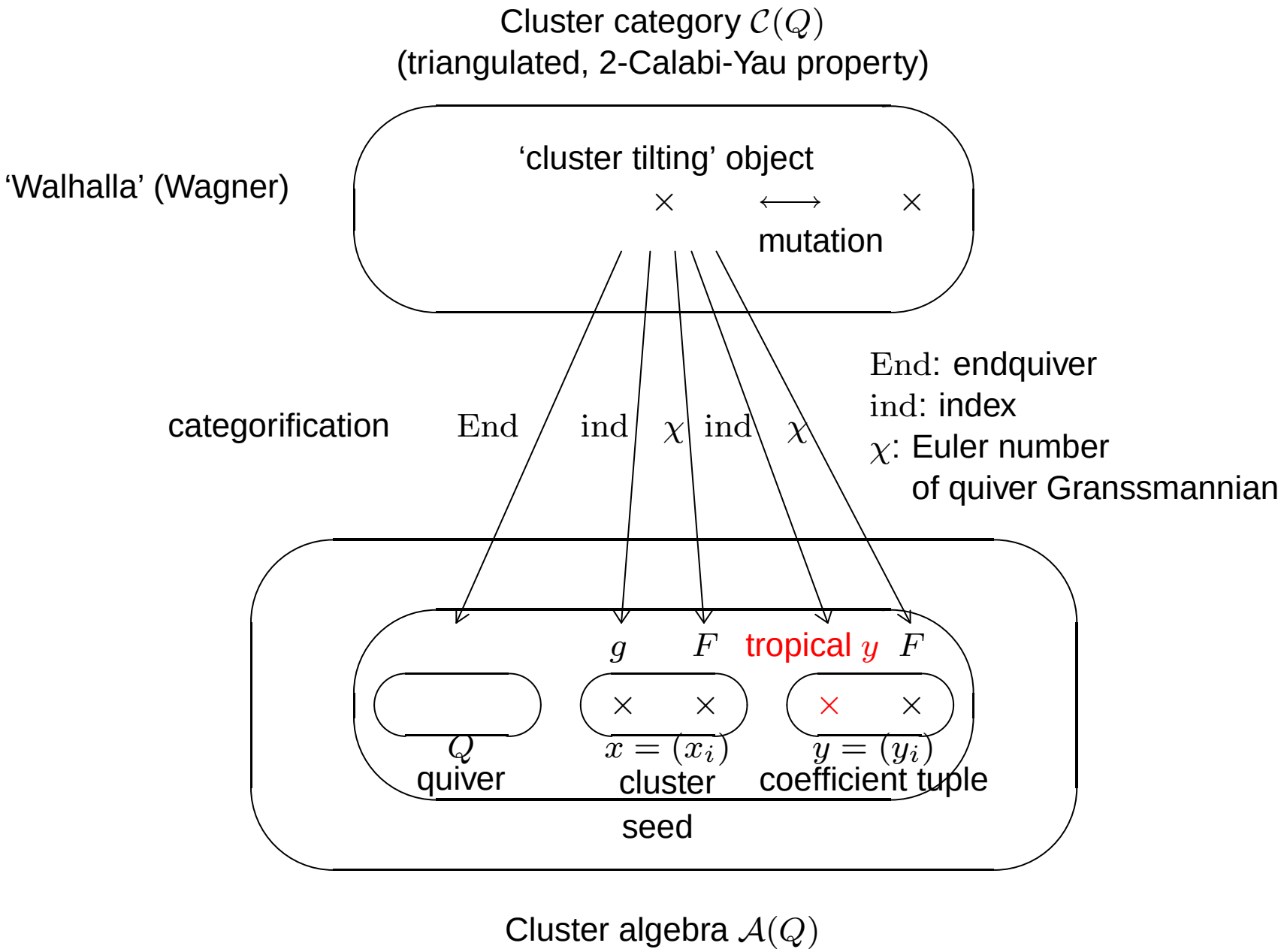
$Q =$ 

$\mu_-^\bullet \mu_-^\circ \mu_+^\bullet \mu_-^\bullet \mu_+^\circ \mu_+^\bullet (Q) = Q$

(Q, x, y): initial seed

The theorem is reformulated as $(\mu_-^\bullet \mu_-^\circ \mu_+^\bullet \mu_-^\bullet \mu_+^\circ \mu_+^\bullet)^{h^\vee + \ell}(Q, x, y) = (Q, x, y)$.
Step 2. Keller's method does not work.
We need a new idea. tropical Y-system + cluster category

Illustration of main idea



Our slogan: "The tropical Y-system knows everything!"