Note on almost sure central limit theorem for branching Brownian motion in random environment

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This is just a note on almost sure central limit theorem on branching Brownian motion in random environment considered by Shiozawa, so there is no abstract and introductions. If you would like to know backgrounds or find references, you had better read other papers. This contains only minimum definition of our model, main result, and its proof.

Let us consider branching Brownian motion in random environment introduced by Shiozawa.

$$\eta : \text{Poisson random measure on } \mathbb{R}_+ \times \mathbb{R}^d \text{ with intensity measure } dtdx. \ (\textbf{environment})$$
$$V_t = \{(s,x) \in \mathbb{R}_; \times \mathbb{R}^d : s \in (0,s], x \in U(B_s(\omega))\}, (\textbf{tube around } (s, B_s)\ )$$

where $U(x)$ is a closed ball in $\mathbb{R}^d$ centered at $x \in \mathbb{R}^d$ with unit volume and $\{B_t : t \in \mathbb{R}_+\}$ is a Brownian motion on $\mathbb{R}^d$.

Let τ be a non-negative random variables of exponential distribution with the mean 1, independent of η, B_t, etc... Fix a parameter $\alpha > 0$, and set

$$S = \inf\{t > 0 : \alpha\eta(V_t) > \tau\}. \quad (\textbf{branching time})$$

Let $\{p_n : n \in \mathbb{N}\}$ be an offspring distribution ($\sum_{n=1}^{\infty} p_n = 1$). Also, we denote by $m^{(p)}$ the p-th moment for $\{p_n : n \in \mathbb{N}\}$;

$$m^{(p)} = \sum_{n=1}^{\infty} n^p \, p_n.$$

Let P^η and $\mathbb{P}$ be probability measure of BBM under fixed environment η and environment. Also, let $P(\cdot) = \int P^\eta(\cdot) d\mathbb{P}$. We denote their expectations by E^η, $\mathbb{E}$, and E.

We define notations as follows:

$$M_t(A) = \sharp\,\{\text{particles locates in } A \text{ at time } t\} \quad \text{for } A \in \mathcal{B}(\mathbb{R}^d).$$
$$M_t = M_t(\mathbb{R}^d)$$
$$N_t(\cdot) = \frac{M_t(\cdot)}{E[M_t]}$$

Here, we remark that $N_t = N_t(\mathbb{R}^d)$ is an $\mathcal{F}_t \otimes \mathcal{G}_t$-martingale, where $\mathcal{F}_t$ and $\mathcal{G}_t$ are filtrations generated by branching Brownian motion under fixed environment and by environment up to time t, respectively. Also, $E[M_t] = e^{\lambda(\beta)t}$ for $\lambda = \lambda(\beta) = e^\beta - 1$ and for $\beta = \log(m^{(1)} - e^{-\alpha}(m^{(1)} - 1))$.

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Theorem 0.1. *Suppose $d \geq 3$. We assume that $\sup_{t\geq 0} E[N_t^2] < \infty$. Then, we have that for any $f \in C_b(\mathbb{R}^d)$*

$$\int_{\mathbb{R}^d} f\left(\frac{x}{\sqrt{t}}\right) N_t(dx) \to N_\infty \int_{\mathbb{R}^d} f(x) d\nu(x),\ P\text{-a.s.}$$

where ν is a Gaussian measure with mean 0 and covariance matrix I_d.

For $\mathbf{a} \in \mathbb{N}^d$, $t \in \mathbb{R}_+$, and $x \in \mathbb{R}^d$, we define $\Phi_{\mathbf{a}}(t,x)$ by

$$\Phi_{\mathbf{a}}(t,x) = \left(\frac{\partial}{\partial\theta}\right)^{\mathbf{a}} \exp\left(\theta \cdot x - \frac{t|\theta|^2}{2}\right)\Bigg|_{\theta=0}.$$

Actually, $\Phi_{\mathbf{a}}(t,x)$ satisfies that

(P1) There exists C_1, C_2, and C_3 such that

$$|\Phi_{\mathbf{a}}| \leq C_1 + C_2|x|^{|\mathbf{a}|} + C_3 t^{|\mathbf{a}|/2}$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

(P2) The process $\Phi_{\mathbf{a}}(t,B_t)$ is a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{B_s : s \leq t\}$.

When we define $M(\Phi_{\mathbf{a}})$ by $M_t(\Phi_{\mathbf{a}}) = M_t(\Phi_{\mathbf{a}}(t,\cdot))$, $M_t(\Phi_{\mathbf{a}})$ is also an $\mathcal{F}_t \otimes \mathcal{G}_t$-martingale [2, Lemma 3.2] and [1, p8 Proof of Theorem 2.1] .

To prove almost sure central limit theorem for BBMRE, it is enough to show the following:

Lemma 0.2. *[1, Lemma 3.1.5] Suppose that $d \geq 3$. We assume that $\sup_{t\geq 0} E[N_t^2] < \infty$. Then, we have*

$$E[M_t(\Phi_{\mathbf{a}})^2] = \mathcal{O}(b_t),\ \text{as } t \nearrow \infty, P\text{-a.s.},$$

where $b_t = 1$ for $|\mathbf{a}| < \frac{d}{2} - 1$, $b_t = \log t$ for $|\mathbf{a}| = \frac{d}{2} - 1$, and $b_t = t^{|\mathbf{a}| - \frac{d}{2} + 1}$ for $|\mathbf{a}| > \frac{d}{2} - 1$.

Moreover, there exists $\kappa \in [0, |\mathbf{a}|)$ such that

$$\sup_{0\leq s\leq t} |M_s(\Phi_{\mathbf{a}})| = \mathcal{O}(t^{\kappa/2}),\quad \text{as } t \nearrow \infty,\ Q\text{-a.s.}$$

Proof of Theorem 0.1. It is well-known that $\Phi_{\mathbf{a}}(t,x) = x^{\mathbf{a}} + \psi_{\mathbf{a}}(t,x)$, where

$$\psi_{\mathbf{a}}(t,x) = \sum_{|\mathbf{b}|+2j=|\mathbf{a}|, j\geq 1} A_{\mathbf{a}}(\mathbf{b},j) x^{\mathbf{b}} t^j,$$

for some $A_{\mathbf{a}}(\mathbf{b},j) \in \mathbb{R}$. Since $t^{-|\mathbf{a}|}\psi_{\mathbf{a}}(t,x) = \psi_{\mathbf{a}}(1, x/\sqrt{t})$, we write

$$\int_{\mathbb{R}^d} \left(\frac{x}{\sqrt{t}}\right)^{\mathbf{a}} N_t(dx) = M_t(\Phi_{\mathbf{a}}) t^{-|\mathbf{a}|/2} - \int_{\mathbb{R}^d} \psi_{\mathbf{a}}\left(1, \frac{x}{\sqrt{t}}\right) N_t(dx).$$

The second term converges almost surely as $t \to \infty$ to

$$-(2\pi)^{-d/2} \int_{\mathbb{R}^d} \psi(1,x) e^{-|x|^2/2} dx = (2\pi)^{-d/2} \int_{\mathbb{R}^d} x^{\mathbf{a}} e^{-|x|^2/2} dx,$$

by induction on $\mathbf{a}$ and $\int_{\mathbb{R}^d} \Phi_{\mathbf{a}}(1,x) e^{-|x|^2/2} dx = 0$. Also, the first term converges to 0 by Lemma 0.2 almost surely. □

Proof of Lemma 0.2. From [2, Lemma 3.4], we have that

$$E\left[M_t(\Phi_{\mathbf{a}})^2\right] = e^{-\lambda t}\mathbb{E}\left[(\Phi_{\mathbf{a}}(t,B_t))^2\right] \tag{0.1}$$

$$+ c\tilde{\nu}\mathbb{E}\left[\int_0^t e^{-\lambda s}\mathbb{E}_{B_s}^{\otimes 2}\left[\Phi_{\mathbf{a}}(t,B_{t-s}^1)\Phi_{\mathbf{a}}(t,B_{t-s}^2)\exp\left(\lambda^2\int_0^{t-s}\left|U(B_u^1)\cap U(B_u^2)\right|du\right)\right]ds\right], \tag{0.2}$$

where $\mathbb{E}$ and $\mathbb{E}_{B_s}^{\otimes 2}$ represent the expectation with respect to Brownian motion starting from the origin and two independent Brownian motion starting from B_s respectively, and

$$c = m^{(2)} - m^{(1)}, \text{ and } \tilde{\nu} = 1 - e^{-\alpha}.$$

We remark that the original result of this was given for $E[N_t(f)^2]$ for bounded Borel function f, but it is easily modified for polynomial growth Borel function f.

We abbreviate $|V_{t-s}(B_1)\cap V_{t-s}(B_2)|$ for $\int_0^{t-s}\left|U(B_u^1)\cap U(B_u^2)\right|du$. Also, we remark that the first term in (0.2) is $\mathcal{O}(1)$. As the proof of [1, Lemma 3.1.5],

$$\exp\left(\lambda^2\left|V_{t-s}(B^1)\cap V_{t-s}(U^2)\right|\right) = 1 + \int_0^{t-s}\lambda^2\left|U(B_u^1)\cap U(B_u^2)\right|\exp\left(\lambda^2\left|V_{t-s}(B^1)\cap V_{t-s}(B^2)\right|\right)$$

so that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t e^{-\lambda s}\mathbb{E}_{B_s}^{\otimes 2}\left[\Phi_{\mathbf{a}}(t,B_{t-s}^1)\Phi_{\mathbf{a}}(t,B_{t-s}^2)\exp\left(\lambda^2\int_0^{t-s}\left|U(B_u^1)\cap U(B_u^2)\right|du\right)\right]ds\right]\\
&= \mathbb{E}\left[\int_0^t e^{-\lambda s}\mathbb{E}_{B_s}^{\otimes 2}\left[\Phi_{\mathbf{a}}(t,B_{t-s}^1)\Phi_{\mathbf{a}}(t,B_{t-s}^2)\right]ds\right]\\
&+ \lambda^2\mathbb{E}\left[\int_0^t e^{-\lambda s}\mathbb{E}_{B_s}^{\otimes 2}\left[\Phi_{\mathbf{a}}(t,B_{t-s}^1)\Phi_{\mathbf{a}}(t,B_{t-s}^2)\int_0^{t-s}\left|U(B_u^1)\cap U(B_u^2)\right|\exp\left(\lambda^2\left|V_u(B^1)\cap V_u(B^2)\right|\right)du\right]ds\right].
\end{aligned} \tag{0.3}$$

From Markov property and martingale property, we have that

$$\mathbb{E}\left[\int_0^t e^{-\lambda s}\mathbb{E}_{B_s}^{\otimes 2}\left[\Phi_{\mathbf{a}}(t,B_{t-s}^1)\Phi_{\mathbf{a}}(t,B_{t-s}^2)\right]ds\right] = \mathbb{E}_0\left[\int_0^t e^{-\lambda s}\Phi_{\mathbf{a}}(s.B_s)^2 ds\right],$$

and this term is finite. The second term in (0.3) can be rewritten as

$$\lambda^2\mathbb{E}\left[\int_0^t e^{-\lambda s}\int_0^{t-s}\mathbb{E}_{B_s}^{\otimes 2}\left[\Phi_{\mathbf{a}}(t,B_{t-s}^1)\Phi_{\mathbf{a}}(t,B_{t-s}^2)\left|U(B_u^1)\cap U(B_u^2)\right|\exp\left(\lambda^2\left|V_u(B^1)\cap V_u(B^2)\right|\right)\right]duds\right] \tag{0.4}$$

Since B_u^1 and B_u^2 are independent, we get from (P2) that

$$\begin{aligned}
&\mathbb{E}_{B_s}^{\otimes 2}\left[\Phi_{\mathbf{a}}(t,B_{t-s}^1)\Phi_{\mathbf{a}}(t,B_{t-s}^2)\left|U(B_u^1)\cap U(B_u^2)\right|\exp\left(\lambda^2\left|V_u(B^1)\cap V_u(B^2)\right|\right)\right]\\
&= \mathbb{E}_{B_s}^{\otimes 2}\left[\Phi_{\mathbf{a}}(s+u,B_u^1)\Phi_{\mathbf{a}}(s+u,B_u^2)\left|U(B_u^1)\cap U(B_u^2)\right|\exp\left(\lambda^2\left|V_u(B^1)\cap V_u(B^2)\right|\right)\right], \ Q\text{-a.s.}
\end{aligned}$$

When we introduce independent Brownian motions $\hat{B}_t$ and $\check{B}_t$ by

$$\hat{B}_t = \frac{B_t^1 - B_t^2}{\sqrt{2}}, \text{ and } \check{B}_t = \frac{B_t^1 + B_t^2}{\sqrt{2}},$$

$U(B_u^1) \cap U(B_u^2) \neq \emptyset$ if and only if $\hat{B}_u \in \sqrt{2}U(0)$ and if $B_0^1 = B_0^2 = B_s$, then $\hat{B}_0 = 0$ and $\check{B}_0 = \sqrt{2}B_s$. Also, it is easy to check from (P1) that

$$\left|\Phi_{\mathbf{a}}(u, B_u^1)\Phi_{\mathbf{a}}(u, B_u^2)\right| \left|U(B_u^1) \cap U(B_u^2)\right| \leq C_1 \left(1 + (s+u)^{|\mathbf{a}|} + |\check{B}_u|^{2|\mathbf{a}|}\right) \mathbf{1}\left\{\hat{B}_u \in \sqrt{2}U(0)\right\}$$

Therefore, we have

$$\begin{aligned}
&\mathbb{E}_{B_s}^{\otimes 2}\left[\Phi_{\mathbf{a}}(t, B_{t-s}^1)\Phi_{\mathbf{a}}(t, B_{t-s}^2) \left|U(B_u^1) \cap U(B_u^2)\right| \exp\left(\lambda^2 \left|V_u(B^1) \cap V_u(B^2)\right|\right)\right] \\
&\leq C\mathbb{E}_{\sqrt{2}B_s}^{\check{B}}\left[\left(1 + (s+u)^{|\mathbf{a}|} + |\check{B}_u|^{2|\mathbf{a}|}\right)\right] \mathbb{E}_0^{\hat{B}}\left[\mathbf{1}\{\sqrt{2}\hat{B}_u \in 2U(0)\} \exp\left(\lambda^2 \int_0^u \left|U(0) \cap U(\sqrt{2}\hat{B}_r)\right| dr\right)\right] \\
&\leq C\mathbb{E}_{\sqrt{2}B_s}^{\check{B}}\left[\left(1 + (s+u)^{|\mathbf{a}|} + |\check{B}_u|^{2|\mathbf{a}|}\right)\right] (1 \wedge u^{-d/2}),
\end{aligned}$$

where we have used Lemma 3.1.4 in [1] in the last line. Also, it is easy to check that $\mathbb{E}_{\sqrt{2}B_s}\left[\left(1 + (s+u)^{|\mathbf{a}|} + |\check{B}_u|^{2|\mathbf{a}|}\right)\right] \leq C(1 + |B_s|^{2|\mathbf{a}|} + s^{|\mathbf{a}|} + u^{|\mathbf{a}|})$. Thus, we have that

$$\begin{aligned}
(0.4) &\leq C \int_0^t ds e^{-\lambda s} \int_0^{t-s} du (1 + s^{|\mathbf{a}|} + u^{|\mathbf{a}|})(1 \wedge u^{-d/2}) \\
&\leq \mathcal{O}(b_t).
\end{aligned}$$

The proof of the last part is the same as Proposition 3.1.2 in [1].

□

References

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[2] Y.Shiozawa.: Central limit theorem for branching Brownian motions in random environment. Journal of Statistical Physics, vol. 136, no. 1, pp145-163, 2009.