Hochschild homology and microlocal Euler classes

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This is a joint work with Masaki Kashiwara announced in [3].

On a complex manifold $(X, \mathcal{O}_X)$, the Hochschild homology $\mathcal{HH}(\mathcal{O}_X)$ may be defined as the object $R\mathcal{H}om_{\mathcal{O}_{X\times X}}(\omega_\Delta^{\otimes -1}, \mathcal{O}_\Delta)$ (see [1]), where ω_X is the dualizing complex and Δ is the diagonal. It is a powerful tool to construct characteristic classes of coherent modules and to get index theorems.

Here, I will show how to adapt this formalism to a wide class of sheaves on a real manifold M. For that purpose, we have to work "microlocally", that is, on the cotangent bundle T^*M and to replace the functor $R\mathcal{H}om$ by the functor μhom, a variant of Sato's microlocalization functor. Then $\mathcal{HH}(\mathcal{O}_X)$ is replaced by $\mathcal{HH}(k_M) = \mu hom(\omega_\Delta^{\otimes -1}, k_\Delta)$, where now k is a ring and ω_M stands for the topological dualizing complex. To a so-called Hochschild kernel we associate its microlocal Euler class in $\mathcal{HH}(k_M)$, a class on T^*M supported by the microsupport of the kernel. The main theorem asserts that this class is functorial with respect to the composition of kernels.

This construction applies in particular to constructible sheaves on real manifolds and $\mathcal{D}$-modules (or more generally, elliptic pairs) on complex manifolds giving a new approach to the Riemann-Roch or Atiyah-Singer theorems.

[1] A. Caldararu, *The Mukai pairing II: the Hochschild-Kostant-Rosenberg isomorphism,* Adv. Math. **194** p. 34–66 (2005).

[2] M. Kashiwara and P. Schapira, *Sheaves on Manifolds,* Grundlehren der Math. Wiss. **292** Springer-Verlag (1990).

[3] ———, *Hochschild homology and microlocal Euler classes,* `math.arXiv:1203.4869`