Simple Lie algebra and Legendre variety *

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A Cartan subalgebra decomposes a simple Lie algebra into its eigenspaces. This reduces many problems on simpel Lie algebras to a discrete geometry, called *root system*. Here we study them from a different viewpoint.

In [4] E. Cartan found a 14-dimensional vector space of polynomials of 5 variables which is a simple Lie algebra with respect to the Legendre bracket (see Table 1). The root system is of G_2-type. In this article, we shall show that all (finite dimensional) simple Lie algebras have similar description.

Theorem *A simple Lie algebra $\mathfrak{g}$ has a canonical embedding into the polynomial ring $\mathbf{C}[x_1, \ldots, x_n, p_1, \ldots, p_n, z]$ of $(2n+1)$ variables endowed with a Lie algebra structure by the Legendre bracket.* [1]

More explicitly there is a homogeneous polynomial $F(x,p)$ of degree 4 such that (the image of) $\mathfrak{g}$ is generated by

$$(2z - \sum_{i=1}^{n} x_i p_i)^2 - F(x,p)$$

and variables $x_1, \ldots, x_n, p_1, \ldots, p_n$ as Lie algebra. Therefore, the construction of $\mathfrak{g}$ is equivalent to finding a good quartic form $F(x,p)$.

Proposition *If $\mathfrak{g} \not\simeq sp(n)$ or $sl(n)$, then the quartic form $F(x,p)$ is the discriminant of a symmetric Legendre subvariety $X^{n-1} \subset \mathbf{P}^{2n-1}$ in the odd dimensional projective space (see Definition 3.4 and 3.5).*

In Cartan's case, this projective variety X is a rational cubic curve in $\mathbf{P}^3$ and $F(x,p)$ is the discriminant of a cubic equation. The general case is similar. We just replace the complex number field $\mathbf{C}$ by a *cubic Jordan algebra* $\mathfrak{J}$ over it. Then $X^{n-1} \subset \mathbf{P}^{2n-1}$ in the proposition is the *cubic curve over* $\mathfrak{J}$ and $F(x,p)$ is the discriminant of the *cubic equation over* $\mathfrak{J}$. Jordan algebras may

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[1] This number n is equal to $h^\vee - 3$, where $h^\vee$ is the dual Coxeter number of the extended affine root system (see [14] for the definition).

not be very familiar but as one sees in Table 2, those with cubic norm all appear in elementary Linear Algebra. The key of the classification of semi-simple Jordan algebras is the following Hurwitz theorem (see [3], [10], [9, Appendix]). Therefore, so is the classification of simple Lie algebras.

Theorem *An involutive alternative algebra*[2] *over the real number field* $\mathbf{R}$ *is isomorphic to either* $\mathbf{R}$, $\mathbf{C}$, *the quaternion algebra or the Cayley octanion algebra* $\mathbf{O}$.

Table 1 Fourteen polynomials of 5 variables in Cartan[4]

$\deg f$	number	polynomial $f(x,y,p,q,z)$
0	1	1
1	4	x, y, p, q
2	3 + 1	y^2+4xq, $3xp+yq$, $4q^2-3yp$ $2z-xp-yq$ (*cf.* (1.7))
3	4	$12xz-12x^2p-12xyq-y^3$ $3yz-3xyp-y^2q+4xq^2$ $9pz-4q^3$ $12qz+3y^2p-8yq^2$
4	1	$36z(z-xp-yq)-3y^3p+12y^2q^2+16xq^3$

Table 2 Semi-simple Jordan algebras $\mathfrak{J}$ over $\mathbf{C}$ with cubic norm

	$\mathfrak{J}$	Cubic norm $N_m : \mathfrak{J} \to \mathbf{C}$	Symm. Legendre var.	Lie alg.
(1)	$\mathbf{C}$	$N_m(x) = x^3$	Twisted cubic in $\mathbf{P}^3$	G_2
(2)	$\mathbf{C} \oplus \mathbf{C}^n$	$N_m(x,y) = xq(y)$ with a non-degenerate qudratic form q on $\mathbf{C}^n$	$\mathbf{P}^1 \times Q \subset \mathbf{P}^{2n+3}$	$o(n+6)$
(3)	$\mathrm{Sym}_3\mathbf{C}$	Determinant of symmetric matrices of size 3	6-dim. symplectic Grassmannian in $\mathbf{P}^{13}$	F_4
(4)	$\mathrm{M}_3\mathbf{C}$	Determinant of square matrices of size 3	9-dim. Grassmannian $G(3,6) \subset \mathbf{P}^{19}$	E_6
(5)	$\mathrm{Alt}_6\mathbf{C}$	Pfaffian of skew-symmetric matrices of size 6	15-dim. orthogonal Grassmannian in $\mathbf{P}^{31}$	E_7
(6)	$\mathrm{H}_3\mathbf{O}$	Reduced norm of Hermitian matrices of size 3 over the octanion algebra	27-dim. E_7-variety in $\mathbf{P}^{55}$	E_8

[2]See Theorem 3.25 of [10]. An algebra $\mathfrak{A}$ with a unit 1 is *alternative* if $x^2y = x(xy)$ and $yx^2 = (yx)x$ hold for every $x, y \in \mathfrak{A}$. $\mathfrak{A}$ is *involutive* if it has an anti-automorphism $x \mapsto \bar{x}$ such that both $x + \bar{x}$ and $x\bar{x}$ are constant multiples of the unit 1 for every $x \in \mathfrak{A}$.

1 Polynomial Lie algebra[3]

We review how Lie algebras usually appear in Geometry.

(1.1) All vector fields on a manifold form a Lie algebra by the Lie bracket. In the one variable case, the bracket is

$$\Big[f(z)\frac{d}{dz},\ g(z)\frac{d}{dz}\Big] = \Big(f(z)\frac{dg(z)}{dz} - g(z)\frac{df(z)}{dz}\Big)\frac{d}{dz}$$

(1.2) All functions on a symplectic manifold form a Lie algebra by Poisson bracket. If the symplectic form is $\sum_{i=1}^{n} dx_i \wedge dp_i$, then the Poisson bracket of $f(x,p)$ and $g(x,p)$ is given by

$$(f,g) = \sum_{i=1}^{n}\Big(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial p_i}\Big)$$

The Lie algebra of infinitesimal contact transformations is a certain mixture of the above two examples.

(1.3) All functions on a contact manifold form a Lie algebra by Legendre bracket. For example, if the contact form is

$$\alpha = dz - \sum_{i=1}^{n} p_i dx_i,$$

then the Legendre bracket of two functions $f(x,p,z)$ and $g(x,p,z)$ is given by

$$\{f,g\} = f\frac{\partial g}{\partial z} - g\frac{\partial f}{\partial z} + \sum_{i=1}^{n}\Big(\frac{df}{dx_i}\frac{dg}{dp_i} - \frac{dg}{dx_i}\frac{df}{dp_i}\Big),$$

where

$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + p_i\frac{\partial}{\partial z}$$

for every i. [4] Equivalently we have

$$(1.4) \qquad \{f,g\} = (f - E_p f)\frac{\partial g}{\partial z} - (g - E_p g)\frac{\partial f}{\partial z} + (f,g)$$

in terms of the Poisson bracket (1.2) and the Euler operator

$$E := \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i}.$$

As a class of functions we choose a miminal one,[5]that is, all polynomials. The polynomial ring $\mathbf{C}[x,p]$ of even number of variables and $\mathbf{C}[x,p,z]$ of odd

[3]These are usually called infinite dimensional Lie algebras of Cartan type.
[4]In the theory of partial differential equations, z is the unknown function $z(x)$ of variables $x = (x_1, \dots, x_n)$ and $p_1(x), \dots, p_n(x)$ are its partials. Hence this definition of d/dx is natural.
[5]Another minimal choice is the space of functions with compact supports.

number of variables are both Lie algebras by Poisson bracket (1.2) and Legendre bracket (1.3), respectively. For our purpose the latter is crucial. We endow the polynomial ring $\mathbf{C}[x,p,z]$ with a ring grading by

$$\deg x_i = \deg p_i = 1 \quad \text{and} \quad \deg z = 2.$$

If $f(x,p,z)$ and $g(x,p,z)$ are (quasi-) homogeneous with respect to this grading, then so is the Legendre bracket $\{f,g\}$. Moreover, we have

$$\deg\{f,g\} = \deg f + \deg g - 2$$

(if it is not zero). Hence $\deg f - 2$ is a Lie algebra grading of $\mathbf{C}[x,p,z]$. We denote this graded Lie algebra by $L = \bigoplus_{i\in\mathbf{Z}} L_i$. Its negative part $L_{-1} \oplus L_{-2}$ is generated by the variables $x_1,\ldots,x_n$, $p_1,\ldots,p_n$ which satisfy the relations

$$\{x_i,x_j\} = \{p_i,p_j\} = 0 \quad \text{and} \quad \{x_i,p_i\} = \delta_{ij}$$

for all $1 \le i,j, \le n$. Hence we have

(1.5) the polynomial Lie algebra $\mathbf{C}[x,p,z]$ is zero in degree less than -2 and the negative part is the Heisenberg Lie algebra with (1-dimensional) center L_{-2}.

Moreover, L is universal among all graded Lie algebras with this property. In other words, L is the *algebraic prolongation* of the Heisenberg Lie algebra $L_{<0}$ in the sense of [11].

Proposition 1.6 *Any isomorphism from the negative subalgebra of a graded Lie algebra $\mathfrak{g}$ to that of L extends a homomorphism $\Phi : \mathfrak{g} \longrightarrow L$ uniquely.*

An element δ of a graded Lie algebra $\mathfrak{g} = \bigoplus_{i\in\mathbf{Z}} \mathfrak{g}_i$ is a *scaling element* if $[\delta,x] = (\deg x)x$ holds for every homogeneous element x of $\mathfrak{g}$. For $L = \mathbf{C}[x,p,z]$,

$$(1.7) \qquad \tilde{z} = 2z - \sum_{i=1}^{n} x_i p_i$$

is the unique scaling element. Since the homomorphism Φ preserves degree, we have $\Phi(\delta) = \tilde{z}$ if $\mathfrak{g}$ has a scaling element δ.

2 Embedding of simple Lie algebras into polynomial rings

Let $\mathfrak{g}$ be a finite simple Lie algebra over the complex number field $\mathbf{C}$ and $\mathbf{P}(\mathfrak{g})$ the associated projective space. Then $\mathfrak{g}$ acts on the nilpotent subvariety $\mathcal{N} \subset \mathbf{P}(\mathfrak{g})$[6]and it has a unique closed orbit. We denote it by $\mathcal{Z}$. Take an $sl(2)$-triple $\{x_\theta, h, y_\theta\}$ (cf. [5]) with $[x_\theta] \in \mathcal{Z}$ and decompose $\mathfrak{g}$ by the action of h.

[6] $\mathcal{N}$ is the complete intersection of ℓ invariant hypersurfaces ([15]), where ℓ is the rank of $\mathfrak{g}$.

h is semi-simple and the eigenvalues are only $0, \pm 1$ or ± 2. So we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2. \tag{2.1}$$

Moreover, the negative part is the Heisenberg algebra with center $\mathfrak{g}_{-2} = \mathbf{C} \cdot y_\theta$. Hence, by Proposition 1.6, we have,

Proposition 2.2 *There exists an embedding Φ of graded Lie algebras from $\mathfrak{g}$ to $L = \mathbf{C}[x, p, z]$ which is an isomorphism on negative parts.*

The element h of the $sl(2)$-triple is the scaling element of (2.1). Hence we have $\Phi(h) = \tilde{z}$. Since $\Phi(y_\theta)$ is a nonzero constant, we normalize so that $\Phi(y_\theta) = 1$. $\Phi(x_\theta)$ is a homogeneous quartic polynomial and is equal to $\tilde{z}^2 - F_{\mathfrak{g}}(x, p)$ for a quartic polynomial $F_{\mathfrak{g}}$ of x and p. Let $L(F)$ be the subalgebra in $L = \mathbf{C}[x, p, z]$ generated by the variables and $\tilde{z}^2 - F(x, p)$ for a polynomial $F(x, p)$.

Corollary 2.3 *$\mathfrak{g}$ is isomorphic to $L(F_{\mathfrak{g}})$ for the quartic $F_{\mathfrak{g}}(x, p)$ determined from the decomposition (2.1).*

Remark 2.4 Let R be $\mathbf{C}[x_1, \ldots, x_n]$ and

$$V = R\frac{\partial}{\partial x_1} \oplus \cdots \oplus R\frac{\partial}{\partial x_n}$$

the Lie algebra of the vector fields with polynomial coefficient. V has a natural grading $\bigoplus_{i \geq -1} V_i$, whose negative part V_{-1} is abelian. Moreover, V is the most universal one among such graded Lie algebras. For example, every symmetric Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is embedded into V (see *e.g.*, [8, §7]). Our Proposition 2.2 is the Heisenberg analogue of this fact.

It is very rare that $L(F)$ is of finite dimension. In Cartan's case (Table 1), we have

$$9F = 9(xp+yq)^2+3y^3p-12y^2q^2-16xq^3 = (3xp+yq)^2-(y^2+4xq)(4q^2-3yp). \tag{2.5}$$

We give two simpler examples (but exceptional as we will see later).

Example 2.6 If the quartic $F(x, p)$ is identically zero, then $L(F)$ consists of all quadratic polynomials in the usual sense. Simple computation shows that the Lie algebra $L(F)$ is the symplectic Lie algebra $sp(n + 2)$.

Example 2.7 If $F(x, p)$ is the square of a non-degenerate quadratic form, say, $(\sum_{i=1}^n x_i p_i)^2$, then $L(F)$ is the special linear Lie algebra $sl(n + 2)$.

Remark 2.8 The latter example becomes more interesting if we consider over the real $\mathbf{R}$. Regard the n-dimensional complex vector space $\mathbf{C}^n$ as a $2n$-dimensional real vector space $\mathbf{R}^{2n}$ by the usual coordinate $z_i = x_i + \sqrt{-1}y_i$, $1 \le i \le n$. This space has a skew inner product $\sum_{i=1}^n x_i^* \wedge y_i^*$ and an inner product $q(x,y) = \sum_{i=1}^n (x_i^2 + y_i^2)$, as the imaginary and real part of Hermitian inner product $H(z,w) = \sum_{i=1}^n z_i \bar{w}_i$, respectively. Then the Lie algebra $L(q^2)$ is isomorphic to the special unitary Lie algebra $su(1, n+1)$. This Lie algebra acts on the unit ball $B^n \subset \mathbf{C}^n$ and its boundary sphere S^{2n-1}. The latter has a real contact structure as CR-manifold. [7]

It is well-known that the homogeneous projective variety $Z_{\mathfrak{g}} \subset \mathbf{P}(\mathfrak{g})$ is a disjoint union of finite cells $\mathbf{C}^k$'s. Let $f_1(x,p,z), \dots, f_N(x,p,z)$, $N = \dim \mathfrak{g}$, be a basis of the image of Φ in Proposition 2.2 and consider the morphism

$$\mathbf{C}^{2n-1} \longrightarrow \mathbf{P}(\mathfrak{g}), \quad (x,p,z) \longmapsto (f_1(x,p,z) : \cdots : f_N(x,p,z)).$$

Then the image of this morphism is the (open) top dimensional cell.

3 Legendre projective variety

Let $\mathbf{P}^{2n-1}$ be the projective space associated with an even dimensional vector space $\mathbf{C}^{2n}$ with a non-degenerate skew inner product

$$\langle\ ,\ \rangle : \mathbf{C}^{2n} \times \mathbf{C}^{2n} \longrightarrow \mathbf{C}.$$

For every nonzero vector v, the symplectic complement $v^\perp = \{u \in \mathbf{C}^{2n} \mid \langle u, v\rangle = 0\}$ is a vector subspace of codimension one which contains v. In the language of projective geometry, a hyperplane H_p passing through p is assigned linearly for every point $p \in \mathbf{P}^{2n-1}$. This is referred as *nil-correlation.* [8]

Definition 3.1 An $(n-1)$-dimensional smooth closed subvariety X of $\mathbf{P}^{2n-1}$ is *Legendrian* if the tangent space of X at p is contained in H_p for every $p \in X$.

The symplectic group $Sp(n)$ of the skew inner product $\langle\ ,\ \rangle$ acts on $\mathbf{P}^{2n-1}$ preserving the nil-correlation $p \mapsto H_p$.

Definition 3.2 A Legendre subvariety $X^{n-1} \subset \mathbf{P}^{2n-1}$ is *symmetric* if an algebraic subgroup of $Sp(n)$ acts on X transitively and if X is symmetric with respect to the action.

[7] $(\mathbf{R}^{2n}, \sum dx_i \wedge y_i)$ is the phase space of $\mathbf{R}^n$ and $E = \frac{1}{2}q : \mathbf{R}^{2n} \longrightarrow \mathbf{R}$ is the energy of harmonic oscillators. The complex space $\mathbf{P}^{n-1}$ is obtained as the reduced phase space $E^{-1}(1)/S^1$.

[8] This is an example of (holomorphic) contact structure. An odd dimensional projective space is a contact Fano manifold.

Example 3.3 The image R of the third Veronese embedding

$$\mathbf{P}^1 \longrightarrow \mathbf{P}^3, \quad z \longmapsto (1 : z : z^2 : z^3)$$

of the projective line $\mathbf{P}^1$ is a symmetric Legendre subvariety. Let $(x : y : q : p)$ be the homogeneous coordinate of $\mathbf{P}^3$. Then we have

1. The tangent vector of R at the point $p = (1 : a : a^2 : a^3)$ is contained in the plane
$$a^3x - 3a^2y + 3aq - p = 0.$$
Hence R is Legendrian with respect to the skew inner product
$$x^* \wedge p^* - 3y^* \wedge q^*.$$

2. An algebraic subgroup of $Sp(2)$ isomorphic to $SL(2)$ acts on R transitively and R is symmetric with respect to this action.

Special quartics are obtained as discriminant of symmetric Legendre subvarieties.

Definition 3.4 The locus $\check{X}$ of p such that the intersection $H_p \cap X$ is not transversal is a closed subvariety of $\mathbf{P}^{2n-1}$. This is called the *(projective) dual* of X.

For a Legendre subvariety X, the dual $\check{X}$ coincides with the union of all tangent spaces of $X \subset \mathbf{P}^{2n-1}$.

Proposition-Definition 3.5 *For a symmetric Legendre subvariety, its dual is a quartic hypersurface. We call its defining equation the* discriminant *of X.*

It is easy to find the discriminant for Example 3.3. The plane

$$ap - 3bq + 3cy - dx = 0$$

corresponds to a point $(a : b : c : d) \in \mathbf{P}^3$ by the nil-correlation. The plane does not cut R transversally if and only if

$$f(z) := az^3 - 3bz^2 + 3cz - d = 0$$

has a multiple root. Therefore, the discriminant of $R \subset \mathbf{P}^3$ is that of the cubic equation $f(z) = 0$, which is equal to

$$\begin{aligned} D(a,b,c,d) &= a^2d^2 - 6abcd - 3b^2c^2 + 4ac^3 + 4b^3d \\ &= (ad - bc)^2 - 4(b^2 - ac)(c^2 - bd) \end{aligned} \tag{3.6}$$

and essentially the quartic (2.5). Other simple Lie algebras are obtained from similar discriminants.

Theorem 3.7 *The Lie subalgebra $L(D)$ is of finite dimension if $D = D(x,p)$ is the discriminant of a symmetric Legendre subvariety, where $(x:p)$ is a symplectic homogeneous coordinate of $\mathbf{P}^{2n-1}$. Moreover, every simple Lie algebra $\mathfrak{g} \not\simeq sp(n)$ or $sl(n)$ is obtained in this way.*

We have already seen the quartics F which yield $sp(n)$ and $sl(n)$ as $L(F)$ in Examples 2.7 and 2.8.

Example 3.8 Let S be a regular symmetric matrix of size n and define the inner product on $\mathbf{C}^n$ by

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} S^t \vec{y}.$$

Furthermore, endow the $2n$-dimensional vector space $\mathbf{C}^n \otimes \mathbf{C}^2 = \mathbf{C}^n \oplus \mathbf{C}^n$ with the skew inner product by $\begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix}$. Let Q be the quadratic hypersurface defined by $\langle \vec{x}, \vec{x} \rangle = 0$ in $\mathbf{P}^{n-1}$. Then the image of $\mathbf{P}^1 \times Q$ by the Segre embedding

$$\mathbf{P}^1 \times \mathbf{P}^{n-1} \hookrightarrow \mathbf{P}^{2n-1}$$

is a symmetric Legendre subvariety. The discriminant is equal to the difference of the two sides of the Cauchy-Schwarz inequality, that is,

$$D(\vec{x}, \vec{p}) = \langle \vec{x}, \vec{p} \rangle^2 - \langle \vec{x}, \vec{x} \rangle \langle \vec{p}, \vec{p} \rangle$$

for the coordinate $(\vec{x} : \vec{p}) \in \mathbf{P}(\mathbf{C}^n \oplus \mathbf{C}^n)$. The Lie algebra $L(D)$ is the orthogonal Lie algebra $o(n+4)$.

4 Jordan algebra and Legendre subvariety

Definition 4.1 A vector space $\mathfrak{J}$ with a multiplication $\mathfrak{J} \times \mathfrak{J} \longrightarrow \mathfrak{J}$ is a *Jordan algebra* if

$$xy = yx \qquad \text{and} \qquad x^2(yx) = (x^2 y)x$$

hold for every $x, y \in \mathfrak{J}$.

Example 4.2 If $\mathfrak{J}$ is an associative algebra, then $\mathfrak{J}$ with the new multiplication $x \cdot y = \frac{1}{2}(xy + yx)$ is a Jordan algebra.

The notion of center and (semi-)simplicity is defined as usual. We restrict ourselves to central semi-simple Jordan algebras over $\mathbf{C}$ (cf. [3], [9, Appendix 1]).

For an element x of a finite dimensional Jordan algebra $\mathfrak{J}$, let $\Delta(x)$ be the determinant of the multiplication by x:

$$\Delta(x) := \det[\mathfrak{J} \xrightarrow{x} \mathfrak{J}] \in \mathbf{C}.$$

Then $\Delta(x)$ is a homogeneous polynomial of x. If $\mathfrak{J}$ is simple, then $\Delta(x)$ is a some power of an irreducible polynomial $F(x)$, which is called the *reduced norm* of $\mathfrak{J}$.

An arbitrary power of $F(x)$ is called a *norm*. A semi-simple Jordan algebra is the direct product of simple ones. A polynomial on it is called a norm if it is product of norms of simple factors.

Now we pay our attentions to Jordan algebras $\mathfrak{J}$ with a cubic norm $\mathrm{Nm} : \mathfrak{J} \longrightarrow \mathbf{C}$. Table 2 is the classification of them. Note that they have a quadratic map $\mathfrak{J} \ni x \mapsto x' \in \mathfrak{J}$ such that $xx' = \mathrm{Nm}(x) \cdot 1$. For example, x' is the cofactor matrix in the cases (3) and (4) of the table. Imitating (3.3) we consider the morphism

$$(4.3) \qquad \mathfrak{J} \longrightarrow \mathbf{P}(\mathbf{C} \oplus \mathfrak{J} \oplus \mathfrak{J} \oplus \mathbf{C}), \quad x \longmapsto (1 : x : x' : \mathrm{Nm}(x)).$$

The closure of the image of this morphism is called the (cubic) *Veronese variety* over $\mathfrak{J}$. In the case (1) of the table, this is the cubic rational curve.

Proposition 4.4 *The Veronese variety over a Jordan algebra with a cubic norm is a symmetric Legendre subvariety. Moreover, the converse holds also.*

For example, the projective variety $\mathbf{P}^1 \times Q \subset \mathbf{P}^{2n-1}$ in (3.8) is the cubic Veronese over the decomposable Jordan algebra of (2) in Table 2. These Veronese variety is symmetric since the symmetric Lie algebra $\mathfrak{J} \oplus (\mathfrak{J} \Box \mathfrak{J}) \oplus \mathfrak{J}$ of $\mathfrak{J}$ acts on it transitively. Combining with Theorem 3.7, we have that all simple Lie algebra other than $sp(n)$ and $sl(n)$ are obtained from cubic Jordan algebras. [9]

We explain this connection more directly in the next section.

5 G_2-decomposition of simple Lie algebras

We return to the graded Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

in §2 and consider the subset

$$(5.1) \qquad X_{\mathfrak{g}} = \{[y] \mid [y, [y, x_\theta]] = 0\}.$$

of $\mathbf{P}(\mathfrak{g}_{-1})$.

Proposition 5.2 *$X_{\mathfrak{g}}$ is empty if and only if $\mathfrak{g}$ is isomorphic to the symplectic Lie algebra $sp(n)$.*

In the sequel, we assume that $X_{\mathfrak{g}}$ is non-empty.

[9] The quartic discriminant form $D(a, b, c, d)$ is equal to

$$\mathrm{Trace}\{(ad - b\Box c)^2 - k(b' - ac)\Box(c' - db)\}$$

for $(a, b, c, d) \in \mathbf{C} \oplus \mathfrak{J} \oplus \mathfrak{J} \oplus \mathbf{C}$, where k is a constant and the trace is taken over $\mathfrak{J}$.

Proposition 5.3 *There is a Lie subalgebra $\mathfrak{s} \subset \mathfrak{g}$ which is isomorphic to $sl(3)$ and which contains the $sl(2)$-triple $\{x_\theta, h, y_\theta\}$.*

We decompose $\mathfrak{g}$ by the adjoint action of this subalgebra $\mathfrak{s} \simeq sl(3)$. $\mathfrak{s}$ itself is one irreducible component. An irreducible component other than $\mathfrak{s}$ is either trivial or isomorphic to the standard 3-dimensional representation V or its dual $V^\vee$. Let $\mathfrak{g}_{\bar{0}}$ be the normalizer of $\mathfrak{s}$. Let $\mathfrak{g}_{\bar{1}}$ (resp. $\mathfrak{g}_{\bar{2}}$) be the union of subrepresentations isomorphic to V (resp. $V^\vee$). Then the decomposition

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}_{\bar{2}},$$

is a Lie algebra grading by integers modulo 3. By definition (and the complete reducibility), there exists a vector space $\mathfrak{J}$, which is trivial as representation of $\mathfrak{s}$, such that

$$\mathfrak{g}_{\bar{1}} \cong \mathfrak{J} \otimes_{\mathbf{C}} V, \qquad \mathfrak{g}_{\bar{2}} \cong \mathfrak{J}^\vee \otimes_{\mathbf{C}} V^\vee.$$

The bracket [,] of $\mathfrak{g}$ induces $\mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{2}}$, $\mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}^\vee$ and finally the cubic map $N : \mathfrak{J} \longrightarrow \mathbf{C}$.

Proposition 5.5 *If the cubic map $N : \mathfrak{J} \longrightarrow \mathbf{C}$ is zero, then $\mathfrak{g}$ is isomorphic to the special linear Lie algebra $sl(n)$ with $n \geq 3$.*

Proposition 5.5 *If $N : \mathfrak{J} \longrightarrow \mathbf{C}$ is not zero, then $\mathfrak{J}$ is a Jordan algebra and N is its cubic norm.*

Conversely, if $Nm : \mathfrak{J} \longrightarrow \mathbf{C}$ is the cubic norm of a Jordan algebra, then the multiplication of $\mathfrak{J}$ and quadratic map $\mathfrak{J} \ni x \mapsto x' \in \mathfrak{J}$ defines a bracket [,] on

$$\begin{array}{ccccccc} \mathfrak{g} & = & (sl(3) \oplus \mathfrak{J}\Box\mathfrak{J}) & \oplus & (\mathfrak{J} \otimes \mathbf{C}^3) & \oplus & (\mathfrak{J}^\vee \otimes \mathbf{C}^3), \\ & & \updownarrow & & \updownarrow & & \updownarrow \\ & & \mathfrak{g}_{\bar{0}} & & \mathfrak{g}_{\bar{1}} & & \mathfrak{g}_{\bar{2}} \end{array} \tag{5.7}$$

which satisfies the Jacobi law. This makes $\mathfrak{g}$ a simple Lie algebra.

The six roots of $sl(3)$ and the weights of 3-dimensional representations locate in $\mathbf{R}^2$ as follows

$\circ$: 6 roots of $sl(3)$
v : 3 weights of V
a : 3 weights of $V^\vee$

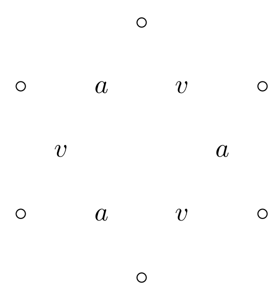

Figure 1

and the Lie algebra (5.7) is visualized as follows:

$$
\begin{array}{ccccccc}
 & & & \mathbf{C} & & & \\
\mathbf{C} & & \mathfrak{J}^\vee & & \mathfrak{J} & & \mathbf{C} \\
 & \mathfrak{J} & & \star & & \mathfrak{J}^\vee & \\
\mathbf{C} & & \mathfrak{J}^\vee & & \mathfrak{J} & & \mathbf{C} \\
 & & & \mathbf{C} & & &
\end{array}
$$

Figure 2 $\star = \mathbf{C}^2 \oplus (\mathfrak{J} \square \mathfrak{J})$

The six $\mathbf{C}$'s on the outermost orbits and $\mathbf{C}^2$ at the center make the Lie algebra $sl(3)$. If the twelve vertices in Figure 1 are all roots, then it is a root system of G_2. Hence it seems natural to call Figure 2 the *decomposition of G_2 type*. [10]

The $X_{\mathfrak{g}}$ in (5.1) is a Legendre subvariety with respect to the skew inner product $[\ ,\ ] : \mathfrak{g}_{-1} \times \mathfrak{g}_{-1} \longrightarrow \mathfrak{g}_{-2} \simeq \mathbf{C}$. This is the Veronese variety of the Jordan algebra in (5.7). Thus simple Lie algebras and symmetric Legendre subvarieties are related with each other as Jordan algebras are their same root.

Postscript

I was led to this consideration in my study of *primary* Fano 3-folds. [11] Though I have not still found any reason, four homogeneous spaces related with

[10] Jordan algebra $\mathfrak{J}$ is the direct sum of $\mathbf{C} \cdot 1$ and the traceless part $\mathfrak{J}_0$ and the Lie algebra $\mathfrak{J} \square \mathfrak{J}$ is isomorphic to $\mathfrak{J} \oplus \mathrm{Der}\ \mathfrak{J}$. Hence, rearranging (5.7) and Figure 2, we have

$$
\mathfrak{g} = \underbrace{\begin{pmatrix}
 & & & \mathbf{C} & & & \\
\mathbf{C} & & \mathbf{C} & & \mathbf{C} & & \mathbf{C} \\
 & \mathbf{C} & & \mathbf{C}^2 & & \mathbf{C} & \\
\mathbf{C} & & \mathbf{C} & & \mathbf{C} & & \mathbf{C} \\
 & & & \mathbf{C} & & &
\end{pmatrix}}_{\updownarrow\ \mathrm{Der}\ \mathbf{O}} \oplus \underbrace{\begin{pmatrix}
 & \mathfrak{J}_0 & & \mathfrak{J}_0 & \\
\mathfrak{J}_0 & & \mathfrak{J}_0 & & \mathfrak{J}_0 \\
 & \mathfrak{J}_0 & & \mathfrak{J}_0 &
\end{pmatrix}}_{\updownarrow\ \mathbf{O}_0 \otimes \mathfrak{J}_0} \oplus \ \mathrm{Der}\ \mathfrak{J},
$$

where $\mathbf{O}$ is the octanion algebra and $\mathbf{O}_0$ is the subspace of pure imaginary ones. This construction of Lie algebras is the last row of the *Freudenthal magic square*. See Theorem 4.13 in [10].

[11] A smooth complete algebraic variety is called *Fano* if the anti-canonical class $-K$ is ample and *primary* if the Picard group is generated by $-K$. Note that a primary Fano variety does not exist in dimension ≤ 2 over algebraically closed fields. There are exactly 10 types of primary Fano 3-folds, whose genera are $2 \leq g \leq 10$ and $g = 12$. The Fano 3-folds V_{22} of genus 12 are still mysterious though we have several nice descriptions of them (see [17] [18]).

cubic Jodan algebras appeared as *key varieties* of Fano 3-folds of genus $g = 8, 9$ and 10 when I classified them by vector bundles (*cf.* [7]). In the case $g = 8$, the homogeneous space is the 8-dimensional Grassmannian, which is the Severi variety[12] associated with the cubic Jordan algebra $\mathrm{Alt}_6\mathbf{C}$. In the case $g = 9$, it is the 6-dimensional symplectic (or Lagrangian) Grassmannian, which is the Veronese variety over the Jordan algebra $\mathrm{Sym}_3\mathbf{C}$. In the case $g = 10$, it is the minimal (co-)adjoint orbit in $\mathbf{P}^{13} = \mathbf{P}(\mathfrak{g})$ of the simple Lie algebra $\mathfrak{g}$ of type G_2. As explained in (2.9), this is the closure of the image of the morphism $\mathbf{C}^5 \longrightarrow \mathbf{P}^{13}$ defined by 14 polynomials of Cartan.

Finally I mention about a connection with the existing literatures in my best knowledge. The condition that the Lie subalgebra $L(F) \subset \mathbf{C}[x, p, z]$ do not contain a polynomial of degree ≥ 5 is equivalent to the system of axioms of the symplectic triple system defined in [2] and [13], and to that of the Freudenthal triple system in [6]. It is proved in [2] that every simple Lie algebra is obtained from such a triple system. [13] The equivalence of such triple systems and cubic Jordan algebras is proved in [6].

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References

[1] Arnold, V.I.: *Mathematical Methods in Classical Mechanics*, Springer Verlag, 1978.

[2] Asano, H.: Symplectic triple system and simple Lie algebras, Res. Inst. Math. Sci. Kōkyūroku, **308**(1977), 41-59.

[3] Braun, H. and Koecher, M.: *Jordan Algebren*, Springer-Verlag, Berlin-Heiderberg-New York, 1996.

[4] Cartan, E.: Sur la structure des groupes simples finis et continus, C. R. Acad. Paris, 1893, pp.784–786.

[5] Kostant, B.: The principle three-dimensional subgroup and the Betti numbers of a complex simple Lie group, Amer. J. Math. **81**(1959), 973–1032.

[6] Meyberg, K.: Eine Theorie der Freudenthalschen Tripelsysteme I, Indag. Math., **30**(1968), 162–174: II, *ibid.* 175–190.

[7] Mukai, S.: New development of Fano 3-folds, Sūgaku, **47**(1995),124–144: English translation, to appear in Sūgaku Exposition, Amer. Math. Soc., Providence.

[12] See [16].

[13] See also [19]. This kind of description of Lie algebras is really needed when we classify them over number fields. The decomposition (2.1) is called the relative root system of type BC_1.

[8] Satake, I.: *Algebraic Structures of Symmetric Domain*, Publ. Math. Soc. Japan, 1980.

[9] —— : Lie kann no hanashi (in japanese), Nihon Hyouron-sya, 1987.

[10] Schafer, R.D.: *An Introduction to Nonassociative Algebras*, Academic Press, New York-London, 1966.

[11] Tanaka, N.: On differential systems, graded Lie algebras and pseudo groups, J. Math. Kyoto Univ. **10**(1970), 1–82.

[12] Yamaguchi, K.: Differential systems associated with simple graded Lie algebras, in '*Progress in Differential Geometry*', Shiohama, K., ed., Adv. Stud. Pure Math. **22**(1993), 413–494.

[13] Yamaguti, K. and Asano, H.: On the Freudenthal's construction of exceptional Lie algebras, Proc. Japan Acad. **51**(1975), 253–258.

References added in translation

[14] Kac, V.: *Infinte Dimensional Lie Algebra*, 3rd. ed., Cambridge Univ. Press, 1990.

[15] Kostant, B.: Lie group representations on polynomial rings, Amer. J. Math. **85**(1963), 327–404.

[16] Lazarsfeld, R. and van de Ven, A.: Topics in geometry of projective space : Recent work of F.L. Zak, DMV Seminar **4**, Birkhäuser, 1984.

[17] Mukai, S.: Biregular classification of Fano threefolds and Fano manifolds of coindex 3, Proc. Nat. Acad. Sci. USA, **86** (1989), 3000–3002.

[18] —— : Fano 3-folds, in '*Complex Projective Geometry: Selected Papers*', London Math. Soc. Lecture Note, **179**, (Ellingsrud G. ed.), 1992, pp. 255–263.

[19] Seligman, G.B.: *Rational Methods in Lie Algebras*, Marcel Dekker, 1976.

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