Equations defining a space curve

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Let $C \subset \mathbf{P}^3$ be a smooth irreducible complete algebraic curve of degree d embedded in a projective 3-space over an algebraically closed field k. It is called *n-regular* if $H^1(\mathcal{I}_C(n-1)) = H^2(\mathcal{I}_C(n-2)) = 0$, where $\mathcal{I}_C$ is the ideal sheaf of C. Among all this implies that the homogeneous ideal $I_C = \bigoplus_l H^0(\mathcal{I}_C(l)) \subset k[x_0, x_1, x_2, x_3]$ is generated in degree $\leq n$ ([7], Lecture 14) and $C \subset \mathbf{P}^3$ is (scheme-theoretically) an intersection of surfaces of degree n. A line $\ell \subset \mathbf{P}^3$ is called an *s-secant line* if $\deg(\ell \cap C) \geq s$. If C is an intersection of surfaces of degree n, then it has no $(n+1)$-secant lines. We say that the number of s-secant lines is *Picard finite* if the number of isomorphism classes of the line bundles $\mathcal{O}_C(\ell \cap C)$, ℓ moving all s-secant lines, is finite. A line bundle ξ on C is called a g^2_s if $\deg \xi = s$ and $h^0(\xi) \geq 3$. In this article, applying a vanishing of Raynaud type to a certain family of vector bundles on C, we shall show the following:

Theorem 1 *Assume that $C \subset \mathbf{P}^3$ has no $(n+1)$-secant lines and that C has no g^2_s of degree $s < d - n$. Then we have*

(a) $C \subset \mathbf{P}^3$ is an intersection of surfaces of degree n if $n \geq d/2$, and

(b) $C \subset \mathbf{P}^3$ is n-regular if $n \geq d/2 + 1$, if the number of n-secant lines is Picard finite and if there are only finitely many (isomorphism classes of) g^2_{d-n} and at most 1-dimensional families of g^2_{d-n+1}.

In the case $n = d-1$, (b) says that every non-planar curve is $(d-1)$-regular, which is Castelnuovo's theorem. (A curve with $(d-1)$-secant line is rational. Hence the number of its $(d-1)$-secant lines is Piacrd finite.) In the cases $n = d-2$ and $d-3$, (b) follows from the results of Gruson-Lazarsfeld-Peskine[4] and D'Almeida[2].

We prove the theorem by constructing an $N \times n$-matrix $R = (f_{ij})$ whose entries f_{ij} are linear forms on $\mathbf{P}^3$ and such that all n-minors vanish on C, where we put $N = 3n - d \geq n$. Such a matrix R is equivalent to a complex

$$N\mathcal{O}_{\mathbf{P}}(-1) \xrightarrow{\phi} n\mathcal{O}_{\mathbf{P}} \longrightarrow i_*\zeta \longrightarrow 0 \tag{1}$$

of coherent sheaves on $\mathbf{P}^3$, where ζ is a line bundle on C and $i : C \hookrightarrow \mathbf{P}^3$ is the natural inclusion. For every point $p \in \mathbf{P}^3$, we find such a complex, with suitable ζ, which is exact at p. Then the n-th exterior product

$$\bigwedge^n \phi : \begin{pmatrix} N \\ n \end{pmatrix} \mathcal{O}_{\mathbf{P}}(-1) \xrightarrow{\epsilon} \mathcal{I}_C \subset \mathcal{O}_{\mathbf{P}}$$

is a surjection onto the ideal sheaf $\mathcal{I}_C$ at p. Thus a global section of $\mathcal{I}_C(n)$ not vanishing at p will be obtained as an n-minor of R, which is our proof of (a).

In order to prove (b), we construct *one* complex (1) which is exact except a finite subset $\bar{Q} \subset \mathbf{P}^3$ and follow the argument of §1 in [4]. Let $K^\bullet$ be the Eagon-Northcott complex

$$\begin{array}{c} 0 \longrightarrow \bigwedge^N E \otimes S^{N-n} F^\vee \longrightarrow \bigwedge^{N-1} E \otimes S^{N-n-1} F^\vee \longrightarrow \cdots \\ \cdots \longrightarrow \bigwedge^{n+2} E \otimes S^2 F^\vee \longrightarrow \bigwedge^{n+1} E \otimes F^\vee \longrightarrow \bigwedge^n E \end{array} \tag{2}$$

of ϕ, where $E = N\mathcal{O}_{\mathbf{P}}(-1)$ and $F = n\mathcal{O}_{\mathbf{P}}$. Then each vector bundle $\bigwedge^j E \otimes S^{j-n} F^\vee$, $j = N, N-1, \ldots, n+1, n$, is a direct sum of a certain number of copies of $\mathcal{O}_{\mathbf{P}}(-j)$. The complex $K^\bullet \stackrel{\epsilon}{\longrightarrow} \mathcal{I}_C \longrightarrow 0$ is exact outside $C \cup \bar{Q}$ and ϵ is surjective outside $\bar{Q}$, which shows the n-regularity of C.

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§1. Let C be a smooth curve in an m-dimensional projective space $\mathbf{P}^m$, $m \geq 3$, and $\pi : Bl_C\mathbf{P}^m \longrightarrow \mathbf{P}^m$ the blowing up with center C. We denote the exceptional divisor by D and the the natural inclusion morphism $D \hookrightarrow Bl_C\mathbf{P}^m$ by j. Then $C \subset \mathbf{P}^m$ is an intersection of hypersurfaces of degree n if and only if the the complete linear system $|nH - D|$ is free (from base points), where H is the pull-back of the hyperplane class of $\mathbf{P}^r$. More precisely the sheaf $\mathcal{I}_C(n)$ is generated by global sections at p if and only if $|nH - D|$ is free on $\pi^{-1}(p)$. In order to show it, we construct a complex

$$N\mathcal{O}_{Bl\mathbf{P}}(-H) \stackrel{\phi}{\longrightarrow} n\mathcal{O}_{Bl\mathbf{P}} \stackrel{\psi}{\longrightarrow} j_*L \longrightarrow 0,$$

for a suitable line bundle L on D, using a family of vector bundles $\{E_x\}$ on C parameterized by the blow-up $Bl_C\mathbf{P}^m$. Let

$$0 \longrightarrow \Omega_{\mathbf{P}}(1) \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}} \stackrel{ev}{\longrightarrow} \mathcal{O}_{\mathbf{P}}(1) \longrightarrow 0. \tag{3}$$

be the universal exact sequence on $\mathbf{P}^m$. Here, $\mathcal{O}_{\mathbf{P}}(1)$, V, $\Omega_{\mathbf{P}}$ and ev are the tautological line bundle, the $(m+1)$-dimensional space $H^0(\mathbf{P}^m, \mathcal{O}_{\mathbf{P}}(1))$ of linear forms, the cotangent bundle and the evaluation homomorphism, respectively. Taking dual, we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow V^* \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow T_{\mathbf{P}}(1) \longrightarrow 0.$$

Hence the space of global sections of $T_{\mathbf{P}}(-1) \boxtimes \mathcal{O}_{\mathbf{P}}(1)$ on $\mathbf{P}^m \times \mathbf{P}^m$, the outer tensor of two bundles $T_{\mathbf{P}}(-1)$ and $\mathcal{O}_{\mathbf{P}}(1)$, is isomorphic to $\mathrm{End}(V)$. Moreover, if s is the global section corresponding to an automorphism $f \in GL(V)$, then its zero locus $(s)_0 \subset \mathbf{P}^m \times \mathbf{P}^m$ is the graph of the automorphism of $\mathbf{P}^m$ induced from f. In particular, taking f to be the identity, we have the exact sequence

$$\Omega_{\mathbf{P}}(1) \boxtimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow \mathcal{O}_{\mathbf{P} \times \mathbf{P}} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0, \tag{4}$$

where Δ is the diagonal of $\mathbf{P}^m \times \mathbf{P}^m$.

We denote the restriction of (3) to C by

$$0 \longrightarrow M \longrightarrow V \otimes \mathcal{O}_C \xrightarrow{ev} \mathcal{O}_C(1) \longrightarrow 0 \tag{5}$$

and the pull-back of the exact sequence (4) by the morphism $i \times \pi : C \times Bl_C\mathbf{P}^m \longrightarrow \mathbf{P}^m \times \mathbf{P}^m$ by

$$M \boxtimes \mathcal{O}_{Bl\mathbf{P}}(-H) \xrightarrow{\Phi} \mathcal{O}_{C \times Bl\mathbf{P}} \xrightarrow{\Psi} \mathcal{O}_{\tilde{D}} \longrightarrow 0, \tag{6}$$

where the subscheme $\tilde{D} \subset C \times Bl_C\mathbf{P}^m$ is the pull-back of the diagonal Δ and coincides with the image of $(\pi|_D, j) : D \longrightarrow C \times Bl_C\mathbf{P}^m$. Since $\tilde{D}$ is of codimension two in $C \times Bl_C\mathbf{P}^m$, the kernel of Φ is locally free. We denote this rank $m-1$ vector bundle by $\mathcal{E}$ and its restriction to $C \times x$ by E_x for $x \in Bl_C\mathbf{P}^m$. By the exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow M \boxtimes \mathcal{O}_{Bl\mathbf{P}}(-H) \longrightarrow \mathcal{I}_{\tilde{D}} \longrightarrow 0, \tag{7}$$

we have

$$\det \mathcal{E} \simeq \mathcal{O}_C(-1) \boxtimes \mathcal{O}_{Bl\mathbf{P}}(-H) \quad \text{and} \quad \det E_x \simeq \mathcal{O}_C(-1) \tag{8}$$

for every x.

Let ζ be a line bundle on C. Take the tensor product of (6) with the pull-back of ζ, and take the direct image by the projection p_2 onto $Bl_C\mathbf{P}^m$. Then we obtain the complex

$$p_{2*}((6) \otimes p_1^*\zeta) : H^0(C, M \otimes \zeta) \otimes \mathcal{O}_{Bl\mathbf{P}}(-H) \xrightarrow{\phi} H^0(C, \zeta) \otimes \mathcal{O}_{Bl\mathbf{P}} \xrightarrow{\psi} j_*(\pi|_D)^*\zeta. \tag{9}$$

Lemma 1 *This complex is exact at $x \in Bl_C\mathbf{P}^n$ if $H^1(C, E_x \otimes \zeta) = 0$.*

Proof. The kernel of ψ coincides with the direct image $p_{2*}(\mathcal{I}_{\tilde{D}} \otimes p_1^*\zeta)$ by virtue of the exact sequence

$$[0 \longrightarrow \mathcal{I}_{\tilde{D}} \longrightarrow \mathcal{O}_{C \times Bl\mathbf{P}} \longrightarrow \mathcal{O}_{\tilde{D}} \longrightarrow 0] \otimes p_1^*\zeta.$$

Taking the direct image of (7) $\otimes p_1^*\zeta$ by p_2, we have the exact sequence

$$H^0(C, M \otimes \zeta) \otimes \mathcal{O}_{Bl\mathbf{P}}(-H) \xrightarrow{\alpha} p_{2*}(\mathcal{I}_{\tilde{D}} \otimes p_1^*\zeta) \longrightarrow R^1p_{2*}(\mathcal{E} \otimes p_1^*\zeta).$$

By assumption and the base change theorem, the first direct image $R^1p_{2*}(\mathcal{E} \otimes p_1^*\zeta)$ is zero at x. Hence, α is surjective there. □

Definition For a point $p \in \mathbf{P}^n$, V_p is the space of linear forms vanishing at p,

$$ev_p : V_p \otimes \mathcal{O}_C \longrightarrow \mathcal{O}_C(1)$$

is the restriction of the homomorphism ev in (5) to V_p and M_p is its kernel.

The homomorphism ev_p is surjective if $p \notin C$ and a surjection onto $\mathfrak{m}_p(1)$ if $p \in C$, where $\mathfrak{m}_p = \mathcal{O}_C(-p)$ is the maximal ideal at p. Hence $\det M_p$ is isomorphic to $\mathcal{O}_C(-1)$ or $\mathcal{O}_C(-1) \otimes \mathcal{O}_C(p)$ according or $p \notin C$ or $p \in C$.

Let x be a point of $Bl_C\mathbf{P}^n$ and $\phi_x : M \longrightarrow \mathcal{O}_C$ the restriction of the homomorphism Φ in (6) to $C \times x$. We put $p = \pi(x)$. Then the kernel of ϕ_x is isomorphic to M_p and there is a natural injection $E_x \hookrightarrow \operatorname{Ker} \phi_x$, which is an isomorphism if $x \notin D$. If $x \in D$, two vector bundles E_x and M_p of the same rank differ but only at p, the unique intersection of $\tilde{D}$ and $C \times x$. Since $\det E_x \simeq (\det M_p)(-p)$ by (8), we have the following:

Lemma 2 *The vector bundle $E_x = \mathcal{E}|_{C\times x}$ is isomorphic to M_p if $x \notin D$ and to the kernel of a nonzero homomorphism $M_p \longrightarrow k(p)$ to the sky-scraper sheaf if $x \in D$.*

Assume that $x \in D$ and let

$$0 \longrightarrow M_p \longrightarrow V_p \otimes \mathcal{O}_C \longrightarrow \mathfrak{m}_p(1) \longrightarrow 0$$

be the defining exact sequence of the vector bundle M_p. By the tensor product

$$0 \longrightarrow M_p \otimes k(p) \longrightarrow V_p \longrightarrow \mathfrak{m}_p/\mathfrak{m}_p^2 \longrightarrow 0.$$

with the skyscraper sheaf $k(p)$, we identify the fiber of M_p at p with the conormal space of $C \subset \mathbf{P}^n$ at p. Thus x corresponds a surjective homomorphism $\gamma_x : M_p \longrightarrow k(p)$.

Proposition 1 *If $x \in D$, the vector bundle E_x is isomorphic to the kernel of this homomorphism $\gamma_x : M_p \longrightarrow k(p)$.*

We omit the proof since we don't need this for the proof of Theorem 1.

§2. We return to the case $m = 3$ and prove Theorem 1. The following is a variant of Raynaud's vanishing [9] for rank two bundles.

Proposition 2 *Let F be a rank two vector bundle on a smooth curve C. If $\chi(F) \leq 0$ and $\chi(\xi) \leq 0$ for every line subbundle ξ of F, then $H^0(F \otimes \mu)$ vanishes for a general line bundle μ of degree* 0.

For the proof and the later use, we introduce the subset

$$S(F) = \{\mu | \deg \mu = 0, H^0(F \otimes \mu) \neq 0\} \subset \operatorname{Pic}^0(C)$$

for a vector bundle F on C. $S(F) = \operatorname{Pic}^0(C)$ if $\chi(F) > 0$. $S(\xi)$ is birationally equivalent to the e-th symmetric product of C if ξ is a line bundle of degree $e \leq g$. In particular, we have $\dim S(\xi) \leq \deg \xi$. If ξ is a line subbundle of F, then $S(\xi)$ is a subset of $S(F)$. We denote the union of $S(\xi)$ for all line subbundles $\xi \subset F$ of degree e by $S_e(F)$. Obviously $S(F)$ is the union of $S_e(F)$ for all (nonnegative) integers e.

Let $\mu \in S_e(F)$. Then there exists an effective divisor D of degree e such that $\mu^{-1}(D)$ is a line subbundle of F. Consider the deformation of the exact sequence

$$0 \longrightarrow \mu^{-1}(D) \longrightarrow F \longrightarrow \alpha \longrightarrow 0 \tag{10}$$

in the quot scheme of F or the Hilbert scheme of the $\mathbf{P}^1$ bundle P over C associated with F. We denote the line bundle $\det(F \otimes \mu(-D))$ by β. This is the normal bundle of the section of P corresponding to (10). Therefore, the space of the first order infinitesimal deformations of (10) is canonically isomorphic to $H^0(\beta) \simeq \mathrm{Hom}\,(\mu^{-1}(D), \alpha)$. Hence we have $\dim_\mu S_e(F) \leq h^0(\beta) + e$. Since $\deg \beta = \deg F - 2e$, we have

$$\dim_\mu S_e(F) \begin{cases} \leq e & \text{if } H^0(\beta) = 0, \\ = \deg F + 1 - g - e & \text{if } H^1(\beta) = 0, \text{ and} \\ = \frac{1}{2}(\deg F - \mathrm{Cliff}\,\beta) + 1 & \text{if } \beta \text{ is special} \end{cases} \tag{11}$$

by the Riemann-Roch theorem, where $\mathrm{Cliff}\,\beta$ is the Clifford index $\deg \beta - 2 \dim |\beta|$.

Proof of Proposition 2. If C is rational, then $\chi(\mathcal{O}_C)$ is positive. Hence the assumption $\chi(\xi) \leq 0$ directly implies $\mathrm{Hom}(\mathcal{O}_C, F) = 0$. Therefore, we assume that the genus g of C is positive and prove that $\dim_\mu S_e(F) \leq g - 1$ for every $0 \leq e \leq g - 1$ and $\mu \in S_e(F)$. Put $\beta = \det(F \otimes \mu(-D))$ as above. By our assumption $\deg F \leq 2g - 2$, (11) becomes

$$\dim_\mu S_e(F) \leq \begin{cases} g - 1 & \text{if } H^0(\beta) = 0, \\ g - 1 - e & \text{if } H^1(\beta) = 0, \text{ and} \\ g - \frac{1}{2}\mathrm{Cliff}\,\beta & \text{if } \beta \text{ is special.} \end{cases}$$

Hence the assertion is obvious if β is special. Assume that β is special. Then, by Clifford's theorem, $\mathrm{Cliff}\,\beta$ is nonnegative. Moreover, it is zero if and only if $\beta \simeq \mathcal{O}_C, K_C$ or C is hyperelliptic and β is a multiple of the unique g^1_2 (*e.g.*, see [5] Chap.IV §5). Here K_C is the canonical line bundle of C. In particular, there are only finitely many special line bundles of Clifford index zero. Hence we have $\dim_\mu S(F) \leq g - 1$ except at a finite number of μ's. Since $g \geq 1$, we have $\dim S(F) \leq g - 1$ everywhere. □

Let ζ_0 be a fixed line bundle with $\chi(\zeta_0) = n$ and apply the proposition to $F = K_C \otimes E^\vee \otimes \zeta_0$. Then, by the Serre duality, we have

Corollary 1 *Let E be a rank two vector bundle on a smooth curve C. If $\deg E \geq -2n$ and $\deg \alpha \geq -n$ for every quotient line bundle α of E, then $H^1(E \otimes \zeta)$ vanishes for a general line bundle ζ with $\chi(\zeta) = n$.*

Note that the assumption on quotient line bundles α is equivalent to the vanishing of $H^0(E \otimes \xi)$ for every line bundle ξ of degree $< -\deg E - n$. We apply the corollary to $E = E_x$. In this case, the vanishing of $H^0(E_x \otimes \xi)$ follows from that of

$$H^0(\xi \otimes M_p) = \mathrm{Ker}\,[H^0(\xi \otimes ev_p) : H^0(\xi) \otimes V_p \longrightarrow H^0(\xi \otimes \mathcal{O}_C(1)] \tag{12}$$

by Lemma 2. Here we give two examples of line bundles ξ for which $H^0(\xi \otimes M_p) \neq 0$.

Example i) If ℓ is a line passing through p and $F = C \cap \ell$, then $H^0(\xi \otimes M_p) \neq 0$ for $\xi = \mathcal{O}_C(-F) \otimes \mathcal{O}_C(1)$. In fact, the skew-symmetric part $\bigwedge^2 \bar{V}_\ell$ of $\bar{V}_\ell \otimes \bar{V}_\ell \subset H^0(\xi) \otimes V_p$ is contained in $H^0(\xi \otimes M_p)$, where $\bar{V}_\ell$ is the space of linear forms vanishing along ℓ regarded as subspaces of $H^0(\xi)$ and $H^0(\mathcal{O}_C(1))$.

ii) Let $R_3 \subset \mathbf{P}^3$ be a twisted cubic passing through p and F the intersection $C \cap R_3$. Then $H^0(\xi \otimes M_p) \neq 0$ for $\xi = \mathcal{O}_C(-F) \otimes O_C(2)$. In fact, choose a system of homogeneous coordinates $(x_0 : x_1 : x_2 : x_3)$ such that $p = (1000)$ and R_3 is defined by three minors $q_{12}(x), q_{13}(x)$, and $q_{23}(x)$ of the matrix $\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$. Then ξ is generated by three global sections $\bar{q}_{12}$, $\bar{q}_{13}$ and $\bar{q}_{23}$ corresponding to the minors and the tensor $\bar{q}_{23} \otimes \bar{x}_1 - \bar{q}_{13} \otimes \bar{x}_2 + \bar{q}_{23} \otimes \bar{x}_3$ is contained in $H^0(\xi \otimes M_p)$.

Note that in both examples $H^0(\xi \otimes M_p) \neq 0$ for infinitely many p.

Let w be a tensor in the kernel $H^0(\xi \otimes M)$ of the multiplication map

$$H^0(\xi \otimes ev) : H^0(\xi) \otimes V \longrightarrow H^0(\xi \otimes \mathcal{O}_C(1)).$$

Then w is expressed as $\sum_{i=1}^r s_i \otimes \bar{f}_i$ for an integer $0 \leq r \leq 4$, where $\{f_1, f_2, f_3, f_4\}$ is a basis of V and s_i's are linearly independent global sections of ξ. This number r is independent of the choice of basis. In fact, it is equal to the rank of the homomorphism $V^\vee \longrightarrow H^0(\xi)$ induced from w. So we call it the rank of w. Since C is irreducible, r is not equal to 1. If w belongs to $H^0(\xi \otimes M_p)$, then its rank r is equal to 0, 1 or 2.

Lemma 3 *Assume that a tensor $w \in H^0(\xi \otimes M)$ is of rank 2. Then we have*

(1) *there exists a line $\ell \subset \mathbf{P}^3$ such that* $\mathrm{Hom}(\mathcal{O}_C(1) \otimes \mathcal{O}_C(-F), \xi) \neq 0$, *where $F = \ell \cap C$ is the intersection divisor,*

(2) *w belongs to $H^0(\xi \otimes M_p)$ if and only if the line ℓ passes through p, and*

(3) *$w \in H^0(\xi \otimes E_x)$ if and only if the strict transform of ℓ passes through x.*

Proof. The tensor w is equal to $s_1 \otimes \bar{f}_1 + s_2 \otimes \bar{f}_2$. Let ℓ be the line defined by $f_1(x) = f_2(x) = 0$. Then F is the common zero locus of $\bar{f}_1$ and $\bar{f}_2 \in H^0(\mathcal{O}_C(1))$ and we have the exact sequence

$$0 \longrightarrow \mathcal{O}_C(-1) \otimes \mathcal{O}_C(F) \overset{(f_2, -f_1)}{\longrightarrow} \mathcal{O}_C \oplus \mathcal{O}_C \overset{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}}{\longrightarrow} \mathcal{O}_C(1) \otimes \mathcal{O}_C(-F) \longrightarrow 0.$$

Tensor with ξ and take the global sections. Then we have $H^0(\xi \otimes \mathcal{O}_C(1) \otimes \mathcal{O}_C(F)) \neq 0$, which shows (1). (2) is obvious since $w \in H^0(\xi \otimes M_p)$ is equivalent to f_1 and $f_2 \in V_p$. (3) follows from Proposition 1. □

Now we prove Theorem 1(a). Let x be an arbitrary point of $Bl_C\mathbf{P}^3$ and α a quotient line bundle of E_x. Then, by (8), we have

$$0 \neq \mathrm{Hom}(E_x, \alpha) \simeq H^0(E_x \otimes \mathcal{O}_C(1) \otimes \alpha) \subset H^0(M \otimes \mathcal{O}_C(1) \otimes \alpha). \qquad (13)$$

Let w be a nonzero tensor in it. If w is of rank 2, then $\mathrm{Hom}(\mathcal{O}_C(-\ell \cap C), \alpha) \neq 0$ for some line ℓ by the above lemma. Hence we have $\deg \alpha \geq -\deg(\ell \cap C) \geq -n$ by our assumption. If w is of rank 3, then we have $\dim H^0(\mathcal{O}_C(1) \otimes \alpha) \geq 3$ and $\deg \mathcal{O}_C(1) \otimes \alpha \geq d-n$ by our assumption. Hence we have proved $\deg \alpha \geq -n$. Since $\deg E_x = -d \geq -2n$, $H^1(E_x \otimes \zeta)$ vanishes for a general line bundle ζ with $\chi(\zeta) = n$ by Corollary 1. So the complex (9) is exact at x by Lemma 1. Since $\chi(\zeta) = n > 0$ and ζ is general, ψ is surjective at $\pi(x)$. Since ζ is general, we also have $H^1(\zeta) = 0$ and $\dim H^0(\zeta) = n$. Thus we obtain the complex

$$H^0(M \otimes \zeta) \otimes \mathcal{O}_{Bl\mathbf{P}}(-H) \xrightarrow{\phi} n\mathcal{O}_{Bl\mathbf{P}} \xrightarrow{\psi} j_*(\pi|_D)^*\zeta \longrightarrow 0,$$

which is exact at x. Hence, as we saw in the introduction, x is not a base point of the linear system $|nH - D|$. $\square$

Let $\Sigma_{n+1} \subset Bl_C\mathbf{P}^3$ be the union of the strict transforms of all $(n+1)$-secant lines and the total transforms of all $(n+2)$-secant lines. Proposition 1 improve Theorem 1 as follows:

Theorem 2 *Assume that $2n \geq d$ and that C has no g_s^2 of degree $s < d-n$. Then the linear system $|nH - D|$ is free outside Σ_{n+1}.*

Proof. Assume $x \notin \Sigma_{n+1}$ and let α be a quotient line bundle of E_x. Then by (3) of Lemma 3 and by the non-existence of g_s^2, we have $\deg \alpha \geq -n$. Hence by the same argument as above the linear system $|nH - D|$ is free at x. $\square$

We give two remarks on the special case $d = 2n$ of Theorem 1(a). Firstly the proof can be restated in terms of moduli and the *determinant line bundle* $\mathcal{L}$ on it if $g(C) \geq 2$. In fact, the vector bundle E_x is semi-stable for every x and we have the classification morphism

$$f : Bl_C\mathbf{P}^3 \longrightarrow \bar{M}_C(2, \mathcal{O}_C(-1)), \quad x \mapsto [E_x]$$

of $\mathcal{E}$ to the moduli space of semi-stable rank two vector bundles of determinant $\mathcal{O}_C(-1)$ on C. For every line bundle ζ with $\chi(\zeta) = n$, the subset $\{E | H^0(E \otimes \zeta) \neq 0\}$ with suitable multiplicity is the zero locus of a global section of $\mathcal{L}$. Hence $\mathcal{L}$ is free by Raynaud's vanishing (cf. [1]). The line bundle $\mathcal{O}_{Bl\mathbf{P}}(nH - D)$ is free since it is isomorphic to the pull-back of $\mathcal{L}$ by f. In the case d odd, the determinant line bundle of $M_C(2, \mathcal{O}_C(-1))$, which is a positive generator of the Picard group, is also free and pulled back to $\mathcal{O}_{Bl\mathbf{P}}(dH - 2D)$.

The second remark is concerned with the condition on g_s^2. An irreducible curve $Z \subset \mathbf{P}^3$ of degree a with $\deg(Z \cap C) \geq an + 1$, *e.g.*, a $(3n+1)$-secant twisted cubic, is a potential obstruction for $C \subset \mathbf{P}^3$ to be an intersection of surfaces of degree n , or more precisely, for the divisor $nH - D$ to be nef. As we saw in Example ii), if $R_3 \subset \mathbf{P}^3$ is an s-secant twisted cubic, then $\xi = \mathcal{O}_C(-F) \otimes \mathcal{O}_C(2)$ is a linear net of degree $2d - s$. Hence the non-existence of g_{n-1}^2, assumed in Theorem 1, directly forbids a $(3n+1)$-secant twisted cubic.

§**3**. We prove Theorem 1(b) using the following generalization of Proposition 2.

Proposition 3 *Let F be a rank two vector bundle on a curve C of genus g and c a nonnegative integer. If $\chi(F) \le -c$, then $\dim S(F) \le g-c-1$ outside the following two subvarieties.*

a) the union of $S_e(F)$ for $e \ge g-c$ and

b) a closed proper subvariety B which depends on C and $\det F$ but not on F.

Proof. There is nothing to prove if $g-c \le 0$. So we assume that $g \ge c+1$. Let $V_{\det F} \subset \operatorname{Pic} C$ be the locus of line bundles ξ of degree $< g-c$ such that $(\det F) \otimes \xi^{-2}$ is a special line bundle of Clifford index $\le c$. By the theorem of Martens [6], the dimension of the locus of such special line bundles in $\operatorname{Pic} C$ is at most c. Hence so is $\dim V_{\det F}$. We take as B the union of $S(\xi)$ for all $\xi \in V_{\det F}$. Then we have

$$\dim B = \dim V_{\det F} + \max \dim S(\xi) \le c + (g-c-1) = g-1$$

and B is a proper closed subvariety. It suffices to show that $\dim_\mu S_e(F) \le g-c-1$ for every $0 \le e \le g-c-1$ assuming $\mu \notin B$. By our assumption $\deg F \le 2g-2-c$, (11) becomes

$$\dim_\mu S_e(F) \le \begin{cases} g-c-1 & \text{if } H^0(\beta)=0, \\ g-c-1-e & \text{if } H^1(\beta)=0, \text{ and} \\ g-\frac{1}{2}(c+\operatorname{Cliff}\beta) & \text{if } \beta \text{ is special.} \end{cases}$$

So the assertion follows from our assumption $\beta \notin B$ if β is special and is obvious otherwise. □

For a vector bundle E on C, we set

$$T(E) = \{\zeta | \chi(\zeta) = n \text{ and } H^1(E \otimes \zeta) \ne 0\} \subset \operatorname{Pic}^{n+g-1}(C).$$

If α is a line bundle, then $\dim T(\alpha) = \min\{g-n-\deg\alpha-1, g\}$. In particular $T(\alpha)$ is empty if $\deg \alpha \ge g-n$. If α is a quotient bundle of F, then $T(\alpha)$ is a subset of $T(F)$. We denote the union of $T(\alpha)$ for all quotient line bundles α of F of degree e by $T_e(F)$. $T(F)$ is the union of $T_e(F)$ for all integers e.

We apply the following, putting $c=2$, to the family $\{E_x\}_{x \in Bl\mathbf{P}}$ constructed in §1.

Corollary 2 *Let E be a rank two vector bundle on a curve C of genus g and c a nonnegative integer. If $\deg E \ge -2n+c$, then $\dim T(E) \le g-c-1$ outside the following two subvarieties of $\operatorname{Pic}^{n+g-1}(C)$.*

a) the union $A(E)$ of $T_e(E)$ for all $e \le -n+c-1$ and

b) a closed proper subvariety B which depends only on the curve C and $\det E$.

Since $\det E_x$ is the same for every x, the subset B in the corollary does not depend on x. For integers i and r, we set

$$Y_i^{(r)} = \{\alpha | \deg \alpha = -n+i \text{ and } H^0(M \otimes \mathcal{O}_C(1) \otimes \alpha) \text{ contains a tensor of rank } r\}.$$

and $Y_i = Y_i^{(2)} \cup Y_i^{(3)}$ in $\mathrm{Pic}^{-n+i}C$.

Claim: $\dim Y_i \leq i$ for every $i \leq 1 (= c-1)$.

By Lemma 3, $Y_i^{(2)}$ coincides with the set of isomorphism classes of $\mathcal{O}_C(-\ell \cap C)$ for all lines ℓ with $\deg \ell \cap C = n - i$. Hence, by our assumption, $Y_i^{(2)}$ is empty for negative i and $\dim Y_0^{(2)} \leq 0$. Our claim for $i = 1$ is trivial if $g \leq 1$. Hence we assume $g \geq 2$. This implies $d \geq 5$ and $n \geq 4$. Since a general secant line is not a 3-secant line ([5], Chap. IV §3), we have $\dim Y_1^{(2)} \leq 1$.If α belongs to $Y_i^{(3)}$, then $\mathcal{O}_C(1) \otimes \alpha$ is a g^2_{d-n+i}. Hence, by our assumption, $Y_i^{(3)}$ is empty for negative i and $\dim Y_i^{(3)} \leq i$ for $i = 0$ and 1.

By (13), every quotient line bundle α of E_x of degree $-n+i$ belongs to Y_i. Hence the subvariety $A(E_x)$ in Corollary 2 is a subset of the union $\mathcal{A}$ of $T(\alpha)$ for all $\alpha \in \bigcup_{i \leq 1} Y_i$. Since $\dim T(\alpha) \leq g - 1 - i$ for every $\alpha \in Y_i$, we have $\dim \mathcal{A} \leq g - 1$ by the claim, and $\mathcal{A}$ is a proper closed subvariety of $\mathrm{Pic}^{n+g-1}C$. Since $\deg E_x = -d \geq -2n + 2$ by our assumption, we have $\dim T(E_x) \leq g - 3$ outside $\mathcal{A} \cup B$ for every x by Corollary 2. Hence the dimension of the subvariety

$$\mathcal{T} := \{(x, \zeta) | H^1(E_x \otimes \zeta) \neq 0, \zeta \notin \mathcal{A} \cup B\} \subset \coprod_{x \in Bl\mathbf{P}} T(E_x) \subset Bl_C\mathbf{P}^3 \times \mathrm{Pic}^{n+g-1}C$$

is at most g and the projection of $\mathcal{T}$ onto $\mathrm{Pic}^{n+g-1}C$ is generically finite. It follows that the fiber

$$Q_\zeta = \{x \in Bl_C\mathbf{P}^3 | H^1(E_x \otimes \zeta) \neq 0\}$$

is finite for a general line bundle $\zeta$with $\chi(\zeta) = n$. Thus the complex (9) is exact outside Q_ζ by Lemma 1. This means that we have a complex (1) on $\mathbf{P}^3$ which is exact off the image $\bar{Q}_\zeta$. Since $\chi(\zeta) \geq 2$, ζ is free and ψ is surjective. Hence the Eagon-Northcott complex (2)

$$\begin{array}{c} 0 \longrightarrow a_N\mathcal{O}_{\mathbf{P}}(-N) \longrightarrow a_{N-1}\mathcal{O}_{\mathbf{P}}(-N+1) \longrightarrow \cdots \\ \cdots \longrightarrow a_{n+2}\mathcal{O}_{\mathbf{P}}(-n-2) \longrightarrow a_{n+1}\mathcal{O}_{\mathbf{P}}(-n-1) \longrightarrow a_n\mathcal{O}_{\mathbf{P}}(-n) \stackrel{\epsilon}{\longrightarrow} \mathcal{I}_C \longrightarrow 0 \end{array} \qquad (14)$$

is exact outside $C \cup \bar{Q}_\zeta$ (cf. [3] or Appendix C of [8]). Hence the image $\mathcal{J}$ of ϵ is n-regular. Since the quotient $\mathcal{J}/\mathcal{I}_C$ is supported by the finite set $\bar{Q}$, so is $\mathcal{I}_C$. So we have completed the proof of Theorem 1(b). □

Assume that $C \subset \mathbf{P}^3$ is irreducible and reduced and let $i : \tilde{C} \longrightarrow \mathbf{P}^3$ be its normalization. We define an s-secant line ℓ of $C \subset \mathbf{P}^3$ by $\deg F \geq s$, where $F \subset \tilde{C}$ is the common zero locus of the pull-backs $\bar{f}_1$ and $\bar{f}_2 \in H^0(\tilde{C}, i^*\mathcal{O}_{\mathbf{P}}(1))$ of the defining linear forms f_1 and f_2 of ℓ. Then Theorem 1(b) holds true for a singular curve if we understand a g^2_s to be a line bundle with $h^0 \geq 3$ on the nomalization $\tilde{C}$. The proof is almost the same: Let $Bl^0_C\mathbf{P}^3$ be the complement of $\pi^{-1}\mathrm{Sing}\, C$ in $Bl_C\mathbf{P}^3$. Then we can construct the rank two vector bundle $\mathcal{E}$ on the product $\tilde{C} \times Bl^0_C\mathbf{P}^3$, or the family $\{E_x\}$ of vector bundles on $\tilde{C}$ parameterized by $Bl^0_C\mathbf{P}^3$, similarly. Let ζ be a general member of $\mathrm{Pic}^{n+g-1}\tilde{C}$. Then the complex (9) is exact on $Bl^0_C\mathbf{P}^3 \setminus Q_\zeta$. The Eagon-Northcott complex (14) is exact outside $\bar{Q}_\zeta \cup C$ and ϵ is surjective outside $\bar{Q}_\zeta \cup \mathrm{Sing}\, C$, which is still a finite set. Hence $C \subset \mathbf{P}^3$ is n-regular.

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