Abelian variety and spin representation*

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Abelian varieties and K3 surfaces[1] have bigger symmetry than their automorphisms. We have learned this from the study of vector bundles on them. This phenomenon is similar to the action of metaplectic group on the function space $L^2(\mathbf{R}^n)$ and to sphere geometry of Lie. In this article, extending the Fourier functor defined in [M2], we shall show that a *unitary* group[2] $U(X \times \hat{X})$ acts on the derived category $\mathbf{D}^b_c(X)$ of an abelian variety modulo shift of complex (Theorem 1.14). Moreover, using the spin representations (§2), we shall show that a double covering group $USpin(X \times \hat{X})$ has a finer action. The Chern character map and Riemann-Roch theorem are equivariant on this group action (§3). The action will be constructed by semi-homogeneous vector bundles (or their universal family), in place of the Poincaré line bundle (§4). In §5, we show that the Lie group $U(X \times \hat{X})_{\mathbf{R}}$ is of Hermitian type and that the group $U(X \times \hat{X})$ of autoequivalences acts on the tube domain

$$D_X = \mathrm{NS(X)}_{\mathbf{R}} + \sqrt{-1}(\text{ample cone})$$

associated with the (formally real) Jordan algebra $\mathrm{NS(X)}_{\mathbf{R}}$ of Néron-Severi group. (This might be a sign of a *mirror symmetry* for Abelian varieties if there's any.)

Notation

- g denotes the dimension of an abelian variety X.
- $\hat{X}$ is the dual abelian variety of X, that is, the neutral component $\mathrm{Pic}^0 X$ of the Picard group $\mathrm{Pic}\, X$. The double dual $\hat{\hat{X}}$ is canonically isomorphic to X.
- For a homomorphism of abelian varieties $\phi : X \longrightarrow Y$, $\hat{\phi} : \hat{Y} \longrightarrow \hat{X}$ is its transpose, which is the pull-back $\phi^* : \mathrm{Pic}\, Y \longrightarrow \mathrm{Pic}\, X$ by definition.
- $\mathcal{P}$ is a *normalized* Poincaré bundle, that is, universal bundle on $X \times \hat{X}$ such that both $\mathcal{P}|_{X\times 0}$ and $\mathcal{P}|_{0\times\hat{X}}$ are trivial.

[0] This is a translation of the author's article in Proceedings of the symposium "Hodge theory and Algebraic Geometry", Sapporo, 1994, pp. 110-135. He is grateful to the University of Warwick for various supports where this translation was made in stimulating atomosphere during his stay in 1998. All footnotes were added in this occasion.

[1] See [6], [7] and [8].

[2] In the original article this group was called orthogonal and denoted by $O(X \times \hat{X})$.

- $\mathrm{Aut}\, X$ denotes the group $\{\phi : X \longrightarrow X | \phi(0) = 0\}$ of automorphisms of X as Abelian variety.
- For a point $a \in X$, $T_a : X \longrightarrow X$ is the translation $x \mapsto x + a$, which is an automorphism of X as variety.
- For a morphism $\pi : X \longrightarrow Y$ of schemes, $\mathbf{R}\pi_* : \mathbf{D}^b_c(X) \longrightarrow \mathbf{D}^b_c(Y)$ is the derived functor of the functor $\pi_* : (\mathcal{O}_X - mod) \longrightarrow (\mathcal{O}_Y - mod)$ taking the direct image of sheaves by π.
- $\chi(E, F)$ is the alternating sum $\sum_i (-1)^i \dim \mathrm{Ext}\,^i_{\mathcal{O}}(E, F)$ for a pair of coherent sheaves E and F on X.

1 Fourier transformation and Fourier functor

We recall the basic back-ground. Let G be a finite abelian group and G^* the group of characters χ. A function on G is expanded to a linear combination $\sum c_\chi \chi$ of characters in the unique way and we get a function

$$(1.1) \qquad \hat{f} : G^* \longrightarrow \mathbf{C}, \quad \chi \mapsto c_\chi = \frac{1}{|G|} \sum_{g \in G} f(g)\chi(g)$$

on G^*. This is the simplest example of the Fourier transformation. Note that this is an isometry of two inner product spaces $\mathrm{Map}(G, \mathbf{C})$ and $\mathrm{Map}(G^*, \mathbf{C})$.

The most famous one is obtained, at least formally, by replacing the pair (G, G^*) of groups with the pair $(V, V^\vee)$ of a real vector space V and its dual. The Fourier transformation is defined by

$$(1.2) \qquad \hat{f}(\alpha) = \int_V f(x) \exp(2\pi\sqrt{-1} < x, \alpha >) dx$$

for $f \in L^2(V)$. As is well-known, this gives an isometry of two Hilbert spaces $L^2(V)$ and $L^2(V^\vee)$.

These are special cases of the expansion of a function by special functions (characters in the above case). We make an analogy of this in Algebraic Geometry. Namely we consider the *expansion* of a sheaf by a family of special sheaves. As *coefficients*, we obtain a sheaf on another variety, the moduli space. The best sample[3]is the Fourier functor, which expands a sheaf on an abelian variety by the family of line bundles.

We denote the line bundle (or more precisely its isomorphism class) corresponding to a point $\alpha \in \hat{X} = \mathrm{Pic}\,^0 X$ by P_α. The Fourier transform $\hat{F}$ of a sheaf F on X is the sheaf, or precisely speaking a complex of sheaves, on $\hat{X}$ obtained in the following way regarding the cohomology $H^\bullet(X, F \otimes P_\alpha)$ as the coefficient or multiplicity of $P_{-\alpha}$.

[3]Another sample is a K3 surface. See references in the footnote of the first page. Beilinson's spectral sequence is also an expansion of a sheaf. His functor gives an equivalence of the derived categories of sheaves on $\mathbf{P}^n$ and modules over the exterior algebra.

Definition 1.3 Let $\mathcal{P}$ be the normalized Poincaré bundle on $X \times \hat{X}$. For a coherent sheaf F on X, its Fourier transform $\hat{F}$ is the element

$$\mathbf{R}\pi_{\hat{X},*}(\pi_X^* F \otimes \mathcal{P})$$

of the derived category $\mathbf{D}_c^b(\hat{X})$, where π_X and $\pi_{\hat{X}}$ are the projections of the direct product $X \times \hat{X}$ onto the first and second factors, respectively.

Remark 1.4 $\mathbf{D}_c^b(\hat{X})$ is the triangulated category consisting of certain equivalence classes, called quasi-isomorphism, of complexes

$$K^\bullet : \quad \cdots \longrightarrow K^{n-1} \longrightarrow K^n \longrightarrow K^{n+1} \longrightarrow \cdots$$

of quasi-coherent sheaves K^n of $\mathcal{O}_X$-modules such that the cohomology group $H^i(K^\bullet)$ is coherent for every i and zero except a finite number of i ([H]). This category is considered as a (cohomological) completion of the category of coherent sheaves.[4]

The following is the main result of [M2].

Theorem 1.5

$$\hat{\hat{}} = (-1_X)^*[g],$$

where -1_X is the isomorphism $x \mapsto -x$ of X and $[g]$ is the g-shift operator of a complex. In particular, $\hat{}$ is an equivalence between the derived categories $\mathbf{D}_c^b(X)$ and $\mathbf{D}_c^b(\hat{X})$.

This theorem has a several variants. Recall that the Grothendiek K-group $K(X)$ is the abelian group whose generators are the isomorphism classes $[F]$ of coherent sheaves F on X and relations are $[F_1] - [F_2] + [F_3]$ for all exact sequences

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

on X. We obtain the well-defined homomorphism $\hat{}: K(X) \longrightarrow K(\hat{X})$ by

$$(1.6) \qquad [F] \mapsto \sum_i (-1)^i [R^i \pi_{\hat{X},*}(\pi_X^* F \otimes \mathcal{P})]$$

and the duality $\hat{\hat{}} = (-1)^g(-1_X)^*$. We have similar duality $\hat{}$ for the Chow group $CH^\bullet(X) \otimes \mathbf{Q}$ and singular cohomology group $H^*(X, \mathbf{Z})$ by using the Chern characters of of the Poincaré bundle $\mathcal{P}$. We have the commutative diagram:

$$(1.7) \qquad \begin{array}{ccccccc} \mathbf{D}_c^b(X) & \longrightarrow & K(X) & \stackrel{Chern\ char.}{\longrightarrow} & CH^\bullet(X)_{\mathbf{Q}} & \longrightarrow & H^*(X, \mathbf{Q}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{D}_c^b(\hat{X}) & \longrightarrow & K(\hat{X}) & \longrightarrow & CH^\bullet(\hat{X})_{\mathbf{Q}} & \longrightarrow & H^*(\hat{X}, \mathbf{Q}) \end{array}$$

[4]Compare with the fact that $L^2(\mathbf{R}^n)$ is a completion of the space of usual functions.

where the downward arrows are all Fourier isomorphisms. The most right one is simple. Its degree i part is just the isomorphism

$$(1.8) \qquad \bigwedge^{i} H^1(X, \mathbf{Q}) \longrightarrow \bigwedge^{2g-i} H^1(\hat{X}, \mathbf{Q})$$

of the complementary exterior products for the pair of mutually dual vector spaces $H^1(X, \mathbf{Q})$ and $H^1(\hat{X}, \mathbf{Q})$.

The group action on the derived category $\mathbf{D}^b_c(X)$, which we are going to discuss, has grown out from the following:

Observation 1.9 ([M2], p.163) Let (X, L) be a principally polarized abelian variety. We identify X and its dual $\hat{X}$ by the isomorphism $\phi_L : X \longrightarrow \hat{X}$ (see (4.3)). If we neglect the shift of complexes, then the relation between the two equivalences

$$F \mapsto \hat{F}, \quad \text{(Fourier transformation)}$$

and

$$F \mapsto F \otimes L, \quad \text{(basic twist)}$$

of $\mathbf{D}^b_c(X)$ is the same as the two matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. In other words, the modular group $SL(2, \mathbf{Z})$ acts on $\mathbf{D}^b_c(X)$ (modulo shift).

If we look for group actions on the category $\mathbf{D}^b_c(X)$ for general abelian varieties, then we immediately find the following two actions:

(1.10) If ϕ is an automorphism of X as variety, then $\phi^* : \mathbf{D}^b_c(X) \longrightarrow \mathbf{D}^b_c(X)$ is an autoequivalence. Therefore the automorphism group of X acts on $\mathbf{D}^b_c(X)$.

(1.11) If L is a line bundle on X, then $F \mapsto F \otimes L$ is an autoequivalence. Therefore the Picard group of X acts on $\mathbf{D}^b_c(X)$.

The automorphism group of X as variety is generated by all the translations $T_a, a \in X$, and the automorphisms as Abelian variety. Hence both X and the dual abelian variety $\hat{X}$ act on $\mathbf{D}^b_c(X)$. In the sequel we consider actions and autoequivalences modulo these actions. Two discrete groups, that is, the automorphism group $\mathrm{Aut}\, X$ of X and the Néron-Severi group $\mathrm{NS(X)} = \mathrm{Pic\, X/Pic^0 X} \subset \mathrm{H^2(X, \mathbf{Z})}$ still act on $\mathbf{D}^b_c(X)$ by (1.10) and (1.11), respectively.

Problem 1.12 Does $\mathbf{D}^b_c(X)$ has autoequuivalences other than the semi-direct product $(\mathrm{Aut}\, X) \cdot \mathrm{NS(X)}$?

The answer is yes by virtue of the Fourier equivalence $\mathbf{D}^b_c(X) \xrightarrow{\simeq} \mathbf{D}^b_c(\hat{X})$. In fact, the Picard group $\mathrm{Pic}\, \hat{X}$ of the dual abelian variety $\hat{X}$ acts on $\mathbf{D}^b_c(\hat{X})$. Though the action of $\mathrm{Pic}^0 \hat{X} \simeq X$ is translation T_a^*, the action of the Néron-Severi group $\mathrm{NS(\hat{X})}$ does not belong to $(\mathrm{Aut}\, X) \cdot \mathrm{NS(X)}$. A better answer to the question is this:

Definition 1.13 We denote an endomorphism ϕ of the product abelian variety $X \times \hat{X}$ in the matrix form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathrm{End}\,(X) & \mathrm{Hom}\,(X, \hat{X}) \\ \mathrm{Hom}\,(\hat{X}, X) & \mathrm{End}\,(\hat{X}) \end{pmatrix} = \mathrm{End}\,(X \times \hat{X}).$$

Then the *unitary group* [5] $U(X \times \hat{X})$ associated with an abelian variety X is the group consisting of all automorphisms ϕ which satisfy

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \hat{d} & -\hat{b} \\ -\hat{c} & \hat{a} \end{pmatrix} = \begin{pmatrix} 1_X & 0 \\ 0 & 1_{\hat{X}} \end{pmatrix}.$$

Theorem 1.14 *The group $U(X \times \hat{X})$ acts on the derived category $\mathbf{D}^b_c(X)$ of an abelian variety X modulo shift of complex (and modulo the actions of X and $\hat{X}$).*

The functors induced from $L \in \mathrm{NS(X)}$, $\varphi \in \mathrm{Aut}\,X$ and $M \in \mathrm{NS}(\hat{\mathrm{X}})$ are included in this action and the following matrices correspond to them:

$$\begin{pmatrix} 1 & \phi_L \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \varphi & 0 \\ 0 & \hat{\varphi}^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \phi_M & 1 \end{pmatrix}.$$

Example 1.15 (1) If (X, L) is a principally polarized abelian variety, then the group $U(X \times \hat{X})$ contains

$$\left\{ \begin{pmatrix} a_X & b\phi_L \\ c\phi_L^{-1} & d_{\hat{X}} \end{pmatrix} \middle| a, b, c, d \in \mathbf{Z}, \quad ad - bc = 1 \right\} \simeq SL(2, \mathbf{Z})$$

as its subgroup. They coincide when $\mathrm{End}\,(X) \simeq \mathbf{Z}$.

(2) When $\mathrm{End}\,(X) = \mathbf{Z}$, $U(X \times \hat{X})$ is isomorphic to the Hecke group

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| c \equiv \bmod N \right\} \subset SL(2, \mathbf{Z}),$$

where N is the smallest natural number which annihilates the kernel of the generator $\phi : X \longrightarrow \hat{X}$ of $\mathrm{Hom}\,(X, \hat{X}) \simeq \mathbf{Z}$.

(3) If X is the product of g copies of the same elliptic curve E (or the same abelian variety), then $U(X \times \hat{X})$ contains a subgroup isomorphic to the symplectic group $Sp(2g, \mathbf{Z})$. They coincide when $\mathrm{End}\,(X) \simeq \mathbf{Z}$.

We endow $H^1(X \times \hat{X}) = H^1(X) \oplus H^1(X)^\vee$ with the inner product $\begin{pmatrix} 0 & I_{2g} \\ I_{2g} & 0 \end{pmatrix}$. Then the group $U(X \times \hat{X})$ coincides with the group of Hodge isometries preserving this inner product.

We improve the theorem using the spin representation in §3. ($U(X \times \hat{X})$ has no natural action on $K(X)$ or $CH(X)_{\mathbf{Q}}$.)

[5] If we tensor $\mathbf{Q}$, then this group is the $SL(2)$ over the algebra $\mathrm{End}_{\mathbf{Q}}(X)$ with the Rosati involution $'$. See §5.

2 Metaplectic representation and spin representation

First we review these representations a little bit. We consider the endomorphism algebra of the polynomial ring $k[x_1, \ldots, x_n]$ as vector space. The multiplications by x_i and partial derivations $\partial/\partial x_i$, $i = 1, 2, \ldots, n$, form a $2n$-dimensional subspace, which we denote by V. The subalgebra W generated by them is called the Weyl algebra. The commutator $[A, B] = AB - BA$ induces a non-degenerate skew-inner product on V and makes W a Lie algebra.[6] Moreover, the subspace spanned by

$$x_i x_j, \quad x_i \frac{\partial}{\partial x_j} + \frac{1}{2}\delta_{ij} \quad \text{and} \quad \frac{\partial^2}{\partial x_i \partial x_j}$$

is a Lie subalgebra of W and isomorphic to the symplectic Lie algebra $sp(V) \simeq sp(2n)$ of V. Hence the polynomial ring $k[x_1, \ldots, x_n]$ is a representation of $sp(2n)$. The *metaplectic representation* is the lift of this Lie algebra representation to a unitary representation of a Lie group in the case $k = \mathbf{R}$.[7] This action does not lift to $Sp(2n, \mathbf{R})$ but to its double covering denoted by $Mp(2n, \mathbf{R})$. Moreover, the space is no more the polynomial ring but itscompletion $L^2(\mathbf{R}^n)$.

This story goes similarly when the variables $x_1, \cdots, x_n$ anti-commutes. We replace the polynomial ring with the exterior algebra $\bigwedge^\bullet(x_1, \ldots, x_n)$ and consider its endomorphism algebra

$$\mathrm{End}_k \bigwedge^\bullet(x_1, \ldots, x_n)$$

as vector space. This algebra is generated by n multiplications x_i and n partial derivations[8] $\partial/\partial x_i$. Let V be the subspace sppened by these $2n$ generators. The anti-commutator $[A, B]_+ = A \circ B + B \circ A$ induces an inner product on this vector space V.[9] The *Clifford algebra* is the pair of this algebra $\mathrm{End}_k \bigwedge^\bullet(x_1, \ldots, x_n)$ and this generating vector space V. We denote it by $\mathcal{C}$. The vector subspace spanned by

$$x_i \wedge x_j, \quad x_i \wedge \frac{\partial}{\partial x_j} + \frac{1}{2}\delta_{ij} \quad \text{and} \quad \frac{\partial^2}{\partial x_i \partial x_j}$$

is a Lie subalgebra [10] isomorphic to the orthogonal Lie algebra $so(V) \simeq so(2n)$ of V. Hence $\bigwedge^\bullet(x_1, \ldots, x_n)$ is a representation of $so(2n)$. The (group-theoretic)

[6]The $(2n+1)$-dimensional vector space $V \oplus \mathbf{C} \cdot 1$ is also a Lie subalgebra. This is called a Heisenberg algebra. $sp(V)$ is the normalizer of this in W.

[7]Metaplectic representation is called the Weil representation in arithmetic context. This plays a crucial role in the theory of modular functions and theta functions. See [11], [12], [5] and [10]. Metaplectic representation appears in Frenel optics also. See[3].

[8]In literatures, the symbol $x_i ? u$ is used in place of $\partial u/\partial x_i$.

[9]The $(2n+1)$-dimensional space $V \oplus \mathbf{C} \cdot 1$ is closed under anti-commutator and normalized by $so(V)$. This is called a Heisenberg superalgebra.

[10]The anti-commutator $[\ ,\ ]_+$ and the commutator $[\ ,\ ]$ are the same on the even part of algebras.

spin representation is a lift of this Lie algebra representation to a representation of an algebraic group. The even part and odd parts are both irreducible representations and called half spin representations. Similar to the symplectic case, the Lie algebra representation does not lift to $SO(2n)$ but its double cover, $Spin(2n)$. The construction is as follows:

The even invertible elements $g \in \mathcal{C}$ such that $gVg^{-1} = V$ form a group under multiplication. This is called the *special Clifford group* and denoted by $CSpin(2n)$. This acts on V preserving the inner product and we obtain the exact sequence

$$1 \longrightarrow \mathbf{G}_m \longrightarrow CSpin(2n) \longrightarrow SO(V) \longrightarrow 1. \tag{2.1}$$

The spinor group is the subgroup of this Clifford group consisting of those g with spinor norm[11] 1. Hence we have the exact sequence

$$1 \longrightarrow Spin(2n) \longrightarrow CSpin(2n) \overset{sp.\ norm}{\longrightarrow} \mathbf{G}_m \longrightarrow 1. \tag{2.2}$$

Combining the two exact sequences, we have

$$1 \longrightarrow \{\pm 1\} \longrightarrow Spin(2n) \longrightarrow SO(V) \longrightarrow 1, \tag{2.3}$$

which shows $Spin(2n)$ is a double over of $SO(2n)$.

3 Action of $USpin(X \times \hat{X})$ on the derived category

As Theorem 1.14 shows the derived category $\mathbf{D}^b_c(X)$ of an abelian variety X has a bigger symmetry than its automorphism and Picard group. This is similar to the Hamiltonian formalism in classical and quantum mechanics and also to the contact transformation of Lie in the classical theory of partial differential equations. We recall the former a little bit (cf. [A]). Let

$$m\frac{d^2\vec{x}}{dt^2} = \vec{F} \quad \text{in} \quad \mathbf{R}^n$$

be the equation of motion of the location $\vec{x} = (x_1, \ldots, x_n)$ of particles. This is transformed to the canonical equations

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} \tag{3.1}$$

on the phase space $\mathbf{R}^n \oplus \mathbf{R}^n$ by introducing the new variables $p_i = m\frac{dx_i}{dt}$, where $H = H(\vec{x}, \vec{p})$ is the Hamiltonian function. The symmetry of the symplectic group $Sp(2n)$ thus obtained is a great advantage of this formalism.

[11] There is a unique anti-automorphism * of $\mathcal{C}$ which is identity on V. If $g \in CSpin(2n)$, then $gg^* = N(g) \cdot id$. This $N(g)$ is called the spinor norm.

When we pass to the quantum mechanics, the problem changes from the motion of particles in $\mathbf{R}^n$ to the (wave) functions on the phase space. But the symmetry of $Sp(2n)$ is still vital. It survives as the action of metaplectic group $Mp(2n)$ reviewed in the previous section. We are now making an analogy of this for Abelian varieties. The dictionary is this:

(3.2)

$\mathbf{R}^n$	Abelian variety X
function $f(x)$	sheaf F
multiplication $f(x)g(x)$	tensor product $F \otimes G$
integral transformation $\int K(x,y)f(y)dy$	integral functor $\pi_*(\mathcal{K} \otimes \tau^* F)$
Hilbert space $L^2(\mathbf{R}^n)$	derived category $\mathbf{D}_c^b(X)$
polynomial ring $\mathbf{C}[x_1, \dots, x_n]$	exterior algebra $\bigwedge^\bullet (x_1, \dots, x_n)$
Heisenberg group $1 \longrightarrow \mathbf{C}^* \longrightarrow H \longrightarrow \mathbf{R}^n \oplus \mathbf{R}^n \longrightarrow 0$	odd Heisenberg group[12]
$Mp(2n) \xrightarrow{2:1} Sp(n, \mathbf{R})$	$USpin(X \times \hat{X}) \xrightarrow{2:1} U(X \times \hat{X})$

Restricting the exact sequence (2.3), we have the exact sequence

$$\begin{array}{ccccccc} 1 \longrightarrow \{\pm 1\} \longrightarrow Spin(4g, \mathbf{Z}) \longrightarrow & O(H_1(X \times \hat{X}), \mathbf{Z}) & \longrightarrow 1 \\ & \cup & \\ & U(X \times \hat{X}). & \end{array}$$

Definition 3.3 $USpin(X \times \hat{X})$ is the inverse image in $Spin(4g, \mathbf{Z})$ of $U(X \times \hat{X})$.

Theorem 3.4 *The group $USpin(X \times \hat{X})$ acts on the derived category $\mathbf{D}_c^b(X)$ modulo even shift of complexes. The nontrivial element z of the kernel of $USpin(X \times \hat{X}) \xrightarrow{2:1} U(X \times \hat{X})$ shifts complexes by one under this action,* i.e., $K^\bullet \overset{z}{\mapsto} K^\bullet[1]$.

The group $USpin(X \times \hat{X})$ also acts on $K(X), CH(X)$ and $H^\bullet(X)$ and the action of z is the multiplication by -1. Note that the even part

$$H^{ev}(X) = \bigwedge^{ev} H^1(X)$$

is the half spin representation of $Spin(4g)$, by definition.

Theorem 3.5 *The Chern character homomorphism $ch : \mathbf{D}_c^b(X) \longrightarrow H^{ev}(X)$ is equivariant with respect to the action of $USpin(X \times \hat{X}) \subset Spin(4g)$. Moreover,*

$$\chi(E, F) = \beta(ch(E), ch(F))$$

[12]The exact sequence $1 \longrightarrow \mathbf{C}^* \longrightarrow \mathcal{P}^* \longrightarrow X \times \hat{X} \longrightarrow 0$ was put here in the original article instead.

holds for a pair of coherent sheaves on X, where β is the invariant bilinear form on the half spin representation.

The second half is the equivariance of the Riemann-Roch formula with respect to the action of $USpin(X \times \hat{X})$. The bilinear form β is called the *fundamental polar form* in [Ca, §102]. This is symmetric or anti-symmetric according as g is even or odd. [13]

4 Semi-homogeneous sheaf

The Fourier functor for sheaves is an integral functor whose kernel (sheaf) is the Poincaré line bundle $\mathcal{P}$. We prove Theorems 1.14 and 3.4 by replacing $\mathcal{P}$ with the universal family of semi-homogeneous sheaves. [14]

Let E be a coherent sheaf on an abelian variety. We define a subgroup of $X \times \hat{X}$ by

$$(4.1) \qquad \Phi(X) = \{(X, \alpha) | T_x^* E \simeq E \otimes P_\alpha\}.$$

This is also a subvariety[15]. We denote its neutral component by $\Phi^0(E)$.

Definition 4.2 A coherent sheaf E on an abelian variety X is *semi-homogeneous* if $\dim \Phi(E) = \dim X$ holds.

If E is a (holomorphic) vector bundle, then the first projection $\Phi(E) \longrightarrow X$ is finite. Hence the above definition is compatible with the one in [M1], that is, a vector bundle E is semi-homogeneous if for every $x \in X$ there exists $P \in \mathrm{Pic}^0 X$ such that $T_x^* E \simeq E \otimes P$. A line bundle L is always semi-homogeneous and the subgroup $\Phi(L)$ is the graph of the homomorphism

$$(4.3) \qquad \phi_L : X \longrightarrow \mathrm{Pic}^0 X = \hat{X}, \quad x \mapsto T_x^* L \otimes L^{-1}$$

associated with L. Another typical example of semi-homogeneous sheaf is a sheaf which has finite support. In this case, the connected component $\Phi^0(E)$ is $0 \times \hat{X}$. The category of semi-homogeneous sheaves is closed under two operations, the pull-backs π_* and the direct images π^* by isogenies π. It is also closed under Fourier transformation. For example, the category of artinian sheaves and that of homogeneous vector bundles are interchanged by the Fourier functor ([M2, (2.9)]).

We look at the Abelian subvariety $\Phi^0(E)$ more closely. The homomorphism (4.3) defined for a line bundle L is characterized by the symmetric property

[13] Note that $\chi(E, F) = (-1)^g \chi(F, E)$ holds by Hirzebruch-Riemann-Roch theorem or by Serre duality.

[14] The basic reference of this section is [M1]. The proof of results is similar to the case of semi-homogeneous vector bundles, or reduces easily to that case. Note that if F is a semi-homogeneous sheaf, then the Fourier transform of $F \otimes L$ is a semi-homogeneous vector bundle for a sufficiently ample line bundle L.

[15] More precisely, a natural scheme structure on $\Phi(E)$ is defined as in [M1] using the technique of [Mum §10].

$\hat{\phi}_L = \phi_L$ among all homomorphisms from X to $\hat{X}$. So the image NS(X) of $\operatorname{Pic} X$ in the exact sequence

$$(4.4) \qquad \begin{array}{ccccc} 0 \longrightarrow \operatorname{Pic}^0 X \longrightarrow & \operatorname{Pic} X & \longrightarrow & \operatorname{Hom}(X, \hat{X}) \\ & L & \mapsto & \phi_L \end{array}$$

coincides with the subgroup $\{\hat{\phi} = \phi\}$ in $\operatorname{Hom}(X, \hat{X})$. When we regard the linear map $H_1(\phi_L) : H_1(X) \longrightarrow H_1(\hat{X})$ as a tensor $w \in H^1(X) \otimes H^1(X)$, $\hat{\phi}_L = \phi_L$ is equivalent to the anti-symmetricity of w. This tensor w is nothing but the Chern class $c_1(L) \in H^2(X) = \bigwedge^2 H^1(X)$ of L. The anti-symmetricity is also equivalent to saying that the graph of $H_1(X) \longrightarrow H_1(\hat{X})$ is totally isotropic with respect to the inner product $\begin{pmatrix} 0 & I_{2g} \\ I_{2g} & 0 \end{pmatrix}$.

Proposition 4.5 *$\Phi^0(E) \subset X \times \hat{X}$, or more precisely, the subspace $H_1(\Phi^0(E))$ is totally isotropic with respect to $\begin{pmatrix} 0 & I_{2g} \\ I_{2g} & 0 \end{pmatrix}$. In particular, we have $\dim \Phi(E) \leq g$.*

Remark 4.6 Let $\psi : Z \longrightarrow \hat{Z}$ be a homomorphism from an abelian variety to its dual. We can define a homomorphism $f : Y \longrightarrow Z$ be *isotropic* with respect to ψ by the property $\hat{f} \circ \phi \circ f = 0$. Then the totally isotropicness in the proposition is equivalent to that of the natural inclusion homomorphism $Y \hookrightarrow X \times \hat{X}$ with respect to the homomorphism

$$J := \begin{pmatrix} 0 & 1_X \\ -1_{\hat{X}} & 0 \end{pmatrix} : X \times \hat{X} \longrightarrow \hat{X} \times X = \text{dual of } X \times \hat{X}.$$

(It will be natural to call this homomorphism J *skew-polarization* since it satisfies $\hat{J} = -J$.)

Recall that a maximally totally isotropic subspace, or a Lagrangian, determines a vector in the spin representation, which is unique up to constant multiplication ([Ch], [Ca]). We call it the spinor coordinate. In our situation, E is semi-homogeneous and $H_1(\Phi^0(E)) \subset H_1(X) \oplus H_1(\hat{X})$ is a Lagrangian. Hence we obtain an element of the cohomology group $H^*(X)$ as spinor coordinate.The following is important:

(4.7) For a semi-homogeneous sheaf E its Chern character $ch(E)$ is the spinor coordinate of $H_1(\Phi^0(E))$.

(4.8) The Chern character of semi-homogeneous vector bundle E is equal to

$$r(E) \exp(\frac{c_1(E)}{r(E)}) \in \bigwedge^{ev} H^1(X),$$

where $r(E)$ is the rank of E. For a general semi-homogeneous sheaf E, the Chern character $ch(E)$ is equal to $[Y] \wedge \exp w$ for suitable $w \in H^{1,1}(X, \mathbf{Q})$, where $[Y]$ is a connected component of the support of E.

A semi-homogeneous sheaf E has a filtration

$$E = E_n \supset E_{n-1} \supset \cdots \supset E_2 \supset E_1 \supset E_0 = 0$$

such that all successive quotients E_i/E_{i-1} are simple, semi-homogeneous and have the same Abelian subvariety, *i.e.* $\Phi^0(E_i/E_{i-1}) = \Phi^0(E)$. (A semi-homogeneous vector bundle E is semi-stable and the above filtration coincides with the JHS-filtration.) Therefore, simple ones are important. They are classified in the following way: (A sheaf E is simple if $\mathrm{End}_{\mathcal{O}}(E) = \mathbf{C}$.)

Theorem 4.9 (1) *If E is simple and semi-homogeneous, then $\Phi(E)$ is connected.*

(2) (Riemann-Roch) *For a pair of simple semi-homogeneous sheaves E_1 and E_2, we have*

$$\chi(E_1, E_2)^2 = \begin{cases} |\Phi(E_1) \cap \Phi(E_2)| \\ 0 \end{cases}$$

according as the intersection $\Phi(E_1) \cap \Phi(E_2)$ is finite or not. In particular, we have

$$\chi(E)^2 = |\Phi(E) \cap X \times 0| \quad \text{and} \quad r(E)^2 = |\Phi(E) \cap 0 \times \hat{X}|$$

for a semi-homogeneous sheaf E.

(3) *Let E_1 and E_2 be as above. Then $\Phi(E_1)$ and $\Phi(E_2)$ coincide if and only if there exist $a \in X$ and $\alpha \in \hat{X}$ such that $E_1 \simeq T_a^* E_2 \otimes P_\alpha$. When both E_1 and E_2 are vector bundles, then these are also equivalent to the condition*

$$\frac{c_1(E_1)}{r(E_1)} = \frac{c_1(E_2)}{r(E_2)} \quad \text{in} \quad H^1(X, \mathbf{Q}).$$

(4) *For a Lagrangian Abelian subvariety Y of $X \times \hat{X}$, there exists a simple semi-homogeneous sheaf E on X such that $\Phi(E) = Y$.*

Remark 4.10 For a semi-homogeneous vector bundle E, the Abelian subvariety $\Phi^0(E)$ coincides with the image of the homomorphism

$$(r_X, \phi_L) : X \longrightarrow X \times \hat{X}$$

where r is the rank of X and L is determinant $\det E$. Hence the semi-homogeneous vector bundles modulo deformation are parameterized by NS(X) $\otimes\, \mathbf{Q}$ by the correspondence $E \mapsto \phi_{\det E}/r(E)$. All semi-homogeneous sheaves are parameterized by the rational points of the projective line over the Jordan algebra NS(X), which is the natural compactification of NS(X).

All deformations of a simple semi-homogeneous sheaf E is isomorphic to $T_a^* E \otimes P_\alpha$ for some a and α and the moduli space $M(E)$ is isomorphic to the quotient Abelian variety $(X \times \hat{X})/\Phi(E)$. Now assume that there exists another semi-homogeneous sheaf F with $\chi(E, F) = \pm 1$. By (2) of the theorem, $X \times \hat{X}$ is the direct product $\Phi(E) \times \Phi(F)$. Moreover on the product $X \times M(E)$, there exists a universal family $\mathcal{E}$. The direct image $\mathbf{R}\pi_{M,*}(\pi_X^* F^\vee \otimes \mathcal{E})$ is a line bundle on $M(E)$ (modulo shift). We normalize $\mathcal{E}$ so that this line bundle is trivial.

Proposition 4.11 *The integral functor*

$$\mathbf{R}\pi_{X,*}(\mathcal{E} \otimes \pi_M^* ?) : \mathbf{D}_c^b(M(E)) \longrightarrow \mathbf{D}_c^b(X)$$

is an equivalence of categories and send the sky-scraper sheaf $k(0)$ *to* E *and the structure sheaf* $\mathcal{O}_X$ *to* F *(modulo shift).*

Using Theorem 4.9, especially (4) of it, we can eliminate E and F from the above statement.

Proposition 4.12 *Assume that an isomorphism*

$$\omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathrm{Hom}\,(Y, X) & \mathrm{Hom}\,(Y, \hat{X}) \\ \mathrm{Hom}\,(\hat{Y}, X) & \mathrm{Hom}\,(\hat{Y}, \hat{X}) \end{pmatrix} = \mathrm{Hom}\,(Y \times \hat{Y}, X \times \hat{X})$$

satisfies

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \hat{d} & -\hat{b} \\ -\hat{c} & \hat{a} \end{pmatrix} = \begin{pmatrix} 1_X & 0 \\ 0 & 1_{\hat{X}} \end{pmatrix}.$$

Then there exists an integral functor

$$\Omega : \mathbf{D}_c^b(Y) \longrightarrow \mathbf{D}_c^b(X)$$

whose kernel is a universal family of semi-homogeneous sheaves and such that

$$\Omega(T_y^* G \otimes P_\alpha) \simeq T_{a(y)+c(\alpha)}^* \Omega(G) \otimes P_{b(y)+d(\alpha)}$$

holds [16] *for every* $G \in \mathbf{D}_c^b(Y)$, $y \in Y$ *and* $\alpha \in \hat{X}$.

Note that the assumption of theorem implies that the images of $Y \times 0$ and $0 \times \hat{Y}$ are both totally isotropic subvarieties of $X \times \hat{X}$. Theorem 1.14 is the special case $Y = X$ of this proposition.

Remark 4.13 In the notation of §2, there exists a standard isomorphism

$$: \; : \; : \bigwedge^{\bullet} V \longrightarrow \mathcal{C}$$

from the exterior algebra $\bigwedge^{\bullet} V$ to the Clifford algebra ([SMJ]). In our case the isomorphism is

$$: \; : \; : H^*(X \times \hat{X}) \longrightarrow \mathcal{C}.$$

The equivalence of categories associated with

$$\tilde{\omega} \in USpin(X \times \hat{X}) \subset \mathcal{C},$$

a lift of ω, is the composite of two functors: one is the integral functor whose kernel is the semi-homogeneous sheaf $\mathcal{F}$ on $X \times \hat{X}$ with $: ch(\mathcal{F} \otimes \mathcal{P}^{\pm 1}) := \tilde{\omega}$ and the other is the Fourier functor. This has a similarity with the formalism of Fourier integral operators in the theory of partial differential equations ([D]).

[16] This functor Ω may be called semi-homogeneous by this property. Another possible name will be a spinor functor.

5 Action on a tube domain

The unitary group $U(X \times \hat{X})$ of an abelian variety X is the set of integral points of an algebraic group. The Lie algebra[17]of this algebraic group is

$$(5.1) \qquad u(X \times \hat{X}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} \hat{d} & -\hat{b} \\ -\hat{c} & \hat{a} \end{pmatrix} = 0 \right\} \subset \mathrm{End}\,(X \times \hat{X})$$

and decomposed into three parts:

$$(5.2) \qquad \begin{array}{ccccc} u(X \times \hat{X}) & \simeq & \mathrm{NS(X)} & \oplus \mathrm{End}\,(X) \oplus & \mathrm{NS(\hat{X})} \\ \begin{pmatrix} \varphi & \phi_L \\ \phi_M & -\hat{\varphi} \end{pmatrix} & \leftrightarrow & (L, & \varphi, & M) \end{array} .$$

In this section we study the structure of this Lie algebra $u(X \times \hat{X})$ over $\mathbf{Q}$.

We fix an ample line bundle L on X and let $\phi_L : X \longrightarrow \hat{X}$ be the homomorphism in (4.3). Since this is an isogeny, ϕ_L^{-1} is defined as element of $\mathrm{Hom}\,(\hat{X}, X)_{\mathbf{Q}}$. The map

$$\mathrm{End}\,(X)_{\mathbf{Q}} \ni a \mapsto a' = \phi_L^{-1} a \phi_L \in \mathrm{End}\,(X)_{\mathbf{Q}}$$

is called the Rosati involution. The positivity $\mathrm{Tr}(aa') > 0$ is famous ([Mum, §21]). By (4.4) the Néron-Severi group $\mathrm{NS(X)}_{\mathbf{Q}}$ is isomorphic to the subspace $\{a' = a\}$. This subspace is closed under the product $a \circ b = (ab + ba)/2$. Hence $\mathrm{NS(X)}_{\mathbf{Q}}$ becomes a Jordan algebra whose unit is ϕ_L. This Jordan algebra is formally real by the above mentioned positivity ([Mum, ibid.]).

Now we consider the element

$$(5.3) \qquad H_L = \frac{1}{2} \begin{pmatrix} 0 & \phi_L \\ -\phi_L^{-1} & 0 \end{pmatrix}$$

of the Lie algebra $u(X \times \hat{X})_{\mathbf{Q}}$. By easy computation, $u(X \times \hat{X})_{\mathbf{Q}}$ decomposes into the direct sum of 0-eigenspace

$$(5.4) \qquad \mathfrak{k} = \left\{ \begin{pmatrix} a & \phi_L b \\ -b\phi_L & -\hat{a} \end{pmatrix} \middle| a, b \in \mathrm{End}\,(X)_{\mathbf{Q}}, \quad a + a' = 0, \quad b = b' \right\}$$

and (-1)-eigenspace

$$(5.5) \qquad \mathfrak{p} = \left\{ \begin{pmatrix} a & \phi_L b \\ b\phi_L^{-1} & -\hat{a} \end{pmatrix} \middle| a, b \in \mathrm{End}\,(X)_{\mathbf{Q}}, \quad a = a', \quad b = b' \right\}$$

of $(ad\ H_L)^2$. By the positivity of Rosati involution $\ '$, the Lie algebra $\mathfrak{k}_{\mathbf{R}}$ is of compact type. Moreover, $ad\ H_L$ induces a complex structure on $\mathfrak{p}$.

[17]a) This Lie algebra acts on the cohomology group $H^*(X)$. The subaction of

$$\left\{ \begin{pmatrix} a & b\phi_L \\ c\phi_{\hat{L}} & -a \end{pmatrix} \middle| a, b, c \in \mathbf{R} \right\}$$

is the $sl(2)$-action of the Lefschetz decomposition (see [2]).

b) This kind of Lie algebras is studied for general varieties in [4].

Proposition 5.6 *The Lie algebra $u(X \times \hat{X})$ is of Hermitian type and H_L is its H-element (see [S, Chap. II] for the terminology.)*

In particular, the Lie group $U(X \times \hat{X})_{\mathbf{R}}$ acts transitively on a bounded symmetric domain. In our case the domain is the tube domain

$$D_X = \mathrm{NS(X)}_{\mathbf{R}} + \sqrt{-1}\mathrm{B} \subset \mathrm{NS(X)} \otimes \mathbf{C},$$

where B is the set of positive elements of $\mathrm{NS(X)}_{\mathbf{R}}$, that is, the ample cone. So the discrete group $U(X \times \hat{X})$, studied in the previous sections, acts this domain D_X discontinuously. It will be interesting to study how the category relates with the quotient variety $U(X \times \hat{X}) \backslash D_X$, Shimura variety, Hodge group, Hodge structures, the Kuga-Satake Abelian varieties of K3 surfaces, etc.. [18] But we do not pursue them here.

Remark 5.7 When L is a principal polarization, then the H-element of (5.3) corresponds to the Fourier transformation $\hat{}$.

Example 5.8 Let X be the product $E \times \cdots \times E$ of elliptic curve as in (3) of (1.15). If $\mathrm{End}\, E = \mathbf{Z}$, the Lie algebra $u(X \times \hat{X})_{\mathbf{R}}$ is the symplectic Lie algebra $sp(2g, \mathbf{R})$ and $\mathfrak{k}$ is the unitary group $u(g)$. Hence the tube domain D_X is the Siegel upper half space of degree g. When E has a complex multiplication, then $u(X \times \hat{X})_{\mathbf{R}}$ is isomorphic to the general linear Lie algebra $gl(2g, \mathbf{R})$ and $\mathfrak{k}$ is a subalgebra isomorphic to $u(g) \oplus u(g)$. The domain D_X is of dimension g^2 and of type $I_{g,g}$.

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References

[A] Arnold, V.I.: *Mathematical Methods in Classical Mechanics*, Springer Verlag, 1978.

[Ca] Cartan, E.: *The Theory of Spinors*, Hermann, 1966.

[Ch] Chevalley, C.: *The Algebraic Theory of Spinors*, Columbia Univ. Press, 1954.

[D] Duistermaat, J.J.: Fourier integral operators, Courant Inst., 1973.

[H] Hartshorne, R.: Residues and duality, Lecture Notes in Math. **20**(1966).

[M1] Mukai, S.: Semi-homogeneous vector bundles on an abelian variety. J. Math. Kyoto. Univ., **18** (1978), 239–272.

[M2] —— : Duality between $\mathbf{D}(X)$ and $\mathbf{D}(\hat{X})$ with its application to Picard sheaves, Nagoya Math. J., **81**(1981), 153–175.

[18] Cf. [9], [1] and [13].

[Mu] Mumford, D.: *Abelian Varieties*, Oxford Univ. Press, 1970.

[S] Satake, I.: *Algebraic Structure of Symmetric Domains*, Iwanami Shoten and Princeton Univ. Press, 1980.

[SMJ] Sato, M. Miwa, T. and Jimbo, M.: Holomorphic quantum field I, Publ. Res. Inst. Math. Sci., Kyoto Univ., **14**(1978), 223-267.

Added in translation

[1] Deligne, P.: La conjecture de Weil pour les surfaces K3, Inv. math. **15**(1972), 206-226.

[2] Griffiths, P.A. and Harris, J.: *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[3] Guillemin, V. and Stenberg, S.: *Symplectic Techniques in Mathematical Physics*, Cambridge Univ. Press.

[4] Looijenga, E. and Lunts, V.A.: A Lie algebra attached to a projective variety, Invent. Math. **129**(1997), 361-412.

[5] Lion, G. and Vergne, M. : *The Weil Representation, Maslov Index and Theta Series*, Birkhäuser, 1978.

[6] Mukai, S.: On classification of vector bundles on an abelian surface, Res. Inst. Math. Sci. Kōkyūroku **409**(1980), 103-127 (in japanese).

[7] —— :On the moduli space of bundles on $K3$ surfaces, I, in 'Vector Bundles on Algebraic Varieties ', Tata Institute of Fundamental Research, Bombay, 1987, pp. 341–413.

[8] —— : Duality of polarized K3 surfaces, to appear in Proceedings of Euroconference on Algebraic Geometry, (K. Hulek and M. Reid eds.), 1998.

[9] Mumford, D.: Families of abelian varieties, Proc. Symp. in Pure Math. **9**(1966), 347-351.

[10] —— , Nori, M. and Norman, P.: *Tata Lectures on Theta III*, Progress in Math. **97**, Birkhäuser, 1991, Boston.

[11] Weil, A.: Sur certain groupes d'operateurs unitaires, Acta Math. **111**(1964), 143 211.

[12] —— : Sul la formula de Siegel dans l théorie de groupes classique, Acta Math. **113**(1965), 1-87.

[13] —— : Abelian varieties and the Hodge ring, Collected works, III, pp. 421-429.

Added after translation[19]

[19]I thank Prof. A. Tyurin who taught the author this reference.

[14] Orlov, D.: On equivalence of derived categories of coherent sheaves on abelian varieties: e-print, alg-geom/9712017.

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