HIDDEN HECKE ALGEBRAS AND KOSZUL DUALITY

JOSEPH CHUANG AND HYOHE MIYACHI

Contents

1. Introduction

Let $\mathbf{k}$ be a field and let $q, Q_1, \ldots, Q_l \in \mathbf{k}$. The Ariki-Koike algebra $\mathcal{H}_m(q; Q_1, \ldots, Q_l)$ is defined to be the $\mathbf{k}$-algebra generated by $T_0, \ldots, T_{m-1}$ subject to the relations

$$\begin{aligned} (T_i - q)(T_i + 1) &= 0 \qquad (i = 1, \ldots, m-1); \\ (T_0 - Q_1)\ldots(T_0 - Q_l) &= 0; \\ T_iT_j &= T_jT_i, \qquad (i, j = 1, \ldots, m-1 \text{ with } |i-j| > 1); \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} \qquad (i = 1, \ldots, m-2); \\ T_0T_1T_0T_1 &= T_1T_0T_1T_0. \end{aligned}$$

If $\mathbf{k}$ contains a primitive l-th root of unity ζ, then specializing $q = 1$ and $Q_i = \zeta^{i-1}$, we recover the group algebra of the complex reflection group of type $G(l, 1, m)$:

$$\mathcal{H}_m(1; 1, \zeta, \ldots, \zeta^{l-1}) \cong \mathbf{k}[\mathbf{Z}/l\mathbf{Z} \wr \mathfrak{S}_m].$$

Now suppose that all the parameters $q, Q_1, \ldots, Q_l$ are nonzero. By a theorem of Dipper and Mathas [DM02], under this restriction any Ariki-Koike algebra is Morita equivalent to a direct sum of tensor products of algebras of the form

$$\mathcal{H}_m[\mathbf{s}] := \mathcal{H}_m(q; q^{s_1}, \ldots, q^{s_l}),$$

where $\mathbf{s} = (s_1, \ldots, s_l) \in \mathbf{Z}^l$. Let e be the multiplicative order of q if $l > 1$, or the quantum characteristic $\inf\{i \in \mathbf{N} \mid 1 + q + q^2 + \cdots + q^{i-1} = 0 \in \mathbf{k}\}$ if $l = 1$. We say that $\mathcal{H}_m[\mathbf{s}]$ has *degree m*, *level l* and *rank e*. When $e = 1$ (equivalently $q = 1$ and $l > 1$), then

$$\mathcal{H}_m[\mathbf{s}] = \mathcal{H}_m(1; 1, \ldots, 1) = \mathbf{k}[x]/(x^l) \wr \mathfrak{S}_m,$$

is an indecomposable algebra, independent of $\mathbf{s}$; we denote it by $A_m(l)$.

Foda, Leclerc, Okado, Thibon and Welsh [FLO+99] established a remarkable relationship between branching coefficients for Ariki-Koike algebras of rank e and level l and those of rank l and level e. The purpose of this paper is to propose a further connection in terms of Koszul dualities between blocks of certain associated Ginzburg-Guay-Opdam-Rouquier module categories $\mathcal{O}[\mathbf{s}]$ over the rational Cherednik algebras [GGOR03], and to explore the consequences of the conjectured dualities. Our approach is inspired by Uglov's construction [Ugl00] of higher level Fock spaces $\mathcal{F}[s]$ inside a quantization of Frenkel's level-rank duality spaces [Fre82], and by Rouquier's conjectured categorification [Rou08b] (see also [Yvo06]) of $\mathcal{F}[s]$ by the GGOR category $\mathcal{O}[\mathbf{s}]$.

While our point of view applies to all levels and ranks, for the rest of this introduction we focus on the case $l = 1$, describing our results and conjectures in this case. The algebra $\mathcal{H}_m[\mathbf{s}]$ is isomorphic to the Hecke algebra $\mathcal{H}_m(q)$ of the symmetric group $\mathfrak{S}_m$ (and in particular does not depend on $\mathbf{s} = (s_1)$). Let $\mathcal{S}_m(q)$ be the q-Schur algebra of degree m. The Schur functor $\mathcal{S}_m(q)$-mod $\to \mathcal{H}_m(q)$-mod is an exact functor, fully faithful on projective modules; it is an example of a *quasihereditary cover*.

Theorem 1. *Suppose $\mathbf{k} = \mathbf{C}$ is the complex field and q is a primitive e-th root of unity.*

(1) *The Schur algebra $\mathcal{S}_m(q)$ is a Koszul algebra and its Koszul dual algebra $\mathcal{S}_m(q)^!$ is quasihereditary.*
(2) *Let B be a block of $\mathcal{S}_m(q)$ of e-weight w (see §4 for a definition of e-weight). Then there is a projective-injective $B^!$-module P_B such that* $\mathrm{End}_{B^!}(P_B) \cong A_w(e)$.
(3) *The functor* $\mathsf{F}_B = \mathrm{Hom}_{B^!}(P_B, -) : B^!\text{-mod} \to A_w(e)\text{-mod}$ *is a quasihereditary cover for at least one block of each e-weight $w \geq 0$.*
(4) *Let B and B' be blocks of $\mathcal{S}_m(q)$ and $\mathcal{S}_{m'}(q)$ of the same e-weight. Then there exists a left exact (or right exact) functor* $\mathsf{F} : B\text{-mod} \to B'\text{-mod}$ *such that the derived functor* $\mathbf{R}\mathsf{F} : \mathcal{D}^b(B\text{-mod}) \xrightarrow{\sim} \mathcal{D}^b(B'\text{-mod})$ *(or* $\mathbf{L}\mathsf{F}$*) is an equivalence.*

Conjecture 2. *Let B be a block of $\mathcal{S}_m(q)$ of e-weight w. Then $B^!$-mod is equivalent to GGOR category $\mathcal{O}$ of the rational Cherednik algebra of $\mathbf{Z}/e\mathbf{Z} \wr \mathfrak{S}_w$ for a certain choice of parameters, and under this equivalence* *the functor* F_B *identifies with the Knizhnik-Zamolodchikov functor* $\mathsf{KZ} : \mathcal{O} \to A_w(e)$-mod*. In particular* F_B *is a quasihereditary cover.*

If F_B were fully faithful on projectives, as the conjecture predicts, then B could be recovered from the $A_w(e)$-module $M_B := \mathsf{F}_B(B^!)$ as follows:

$$B \cong \mathrm{End}_{A_w(e)}(M_B)^!.$$

The derived equivalences of part (4) of Theorem 1 are described explicitly as compositions of perverse equivalences (see [Rou06]). As a consequence the indecomposable summands of $M_{B'}$ may be derived from those of M_B by a prescribed sequence of relative syzygies. On the other hand it is tempting to seek a direct description of M_B (or at least a module with the same isomorphism classes of indecomposable summands) as a 'generalised permutation module' over $A_w(e)$.

The paper is organized as follows. In § 2, we shall recall higher level Fock spaces $\mathcal{F}[\mathbf{s}]$ following Uglov [Ugl00]. The space $\mathcal{F}[\mathbf{s}]$ is expected to be categorified by a direct sum $\mathcal{O}[\mathbf{s}]$ of category $\mathcal{O}$'s of rational Cherednik algebras with a fixed "multicharge" $\mathbf{s}$ (see [Rou08b, Sha09]). So in § 3 we describe the relevant choice of parameters for Ariki-Koike algebras, cyclotomic q-Schur algebras and rational Cherednik algebras. Then we explain Rouquier's point of view: categories $\mathcal{O}$ of rational Cherednik algebras of complex reflection group of type $G(l, 1, m)$ are 'charged q-Schur algebras' generalizing cyclotomic q-Schur algebras. We compare the weight spaces of $\mathcal{F}[\mathbf{s}]$ with the blocks of $\mathcal{O}[\mathbf{s}]$. Then, we formulate the Main Conjecture 6. According to Rouquier's approach in § 3, the Koszul dual $B^!$ of B should have a "Hecke algebra" if we assume the truth of Conjecture 12. So in § 4 we give a concrete realization of this Hecke algebra. To do so, we require the so-called "Rouquier blocks" of q-Schur algebras. In § 5, we make a proposition mirroring Beilinson-Ginzburg-Soergel's numerical Koszul duality criterion [BGS96]. We then check that Ariki's graded Schur algebra has an involutory contravariant autofunctor with a good shift of grading which fixes all simple modules. In § 6, assuming the Lusztig conjecture for quantum groups and a Koszulity criterion (Proposition 21), we shall show that q-Schur algebras in characteristic zero are (standard) Koszul. We also show that the finite dimesional module category of type I over Lusztig's divided power big quantum general linear group is standard Koszul. Finally in § 8, we provide more evidence for the Main Conjecture 6.

Before closing the introduction, we write some notational conventions here : $\lambda \models r$ means λ is a composition of r. $\lambda \vdash r$ means λ is a partition of r. $|\lambda| := \sum_i \lambda_i$. $\langle \cdot, \cdot \rangle$ is used for an appropriate inner product for a suitable space.

2. Higher level Fock spaces

Fix $e, l \geq 1$. Fix also a *charge* $s \in \mathbf{Z}$, and let $\mathbf{Z}^l(s) := \{(s_1, \ldots, s_l) \mid \sum s_i = s\}$ be the set of *multicharges*. Let v be an indeterminate and put $u = -v^{-1}$. Uglov [Ugl00] constructed commuting actions of the quantum affine algebras $U_v(\hat{\mathfrak{sl}}'_e)$ and $U_u(\hat{\mathfrak{sl}}'_l)$, of level l and e respectively, on the $\mathbf{C}(v)$-vector space

$$\Lambda^{s+\frac{\infty}{2}} = \bigoplus \mathbf{C}(v)|\lambda, r\rangle$$

with basis indexed by charged (e, l)-multipartitions $|\lambda, r\rangle$. Here $\lambda = (\lambda^{ij})$ is an $e \times l$ matrix of partitions and $r = (r_{ij})$ a $e \times l$ matrix of integers summing to s. The charged (e, l)-multipartitions may be represented on an $e \times l$ array of 'runners', with runner (i, j) being of charge r_{ij}. So the 'abacus' of $|\lambda, r\rangle$ is a subset of $[e] \times [l] \times \mathbf{Z}$, with 'beads' situated at positions $(i, j, \lambda^{ij}_k + r_{ij} - k + 1)$. By regarding the rows (columns) of the array of runners as e-runner (l-runner) abaci for partitions, we obtain two alternative indexations for the basis of $\Lambda^{s+\frac{\infty}{2}}$ (see [Ugl00] for details):

- charged e-multipartitions $|\boldsymbol{\lambda}, \mathbf{s}\rangle$, consisting of an l-multipartition $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^l)$ and $\mathbf{s} \in \mathbf{Z}^l(s)$
- charged l-multipartitions $|\boldsymbol{\mu}, \mathbf{t}\rangle$ consisting of an e-multipartition $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^e)$ and $\mathbf{t} \in \mathbf{Z}^e(s)$.

The charges in the three different representations are connected by the equations $s_i = \sum_j r_{ij}$ and $t_j = \sum_i r_{ij}$.

Uglov gives an explicit description of commuting actions of $U_v(\hat{\mathfrak{sl}}'_e)$ and $U_u(\hat{\mathfrak{sl}}'_l)$ on $\Lambda^{s+\frac{\infty}{2}}$ in terms of the indexation of the basis by charged l-multipartitions and by charged e-multipartitions, respectively, and of a further commuting action of a Heisenberg algebra H. He also introduces a degree operator d which provides an extension to (noncommuting) actions of $U_v(\hat{\mathfrak{sl}}_e)$ and $U_u(\hat{\mathfrak{sl}}_l)$.

Each basis element of $\Lambda^{s+\frac{\infty}{2}}$ is a weight vector for the action of both $U_v(\hat{\mathfrak{sl}}'_e)$ and $U_u(\hat{\mathfrak{sl}}'_l)$:

$$\mathrm{wt}\,|\boldsymbol{\mu},\mathbf{t}\rangle = (t_e - t_1 - l)\Lambda_0 + (t_1 - t_2)\Lambda_1 + (t_2 - t_3)\Lambda_2 + \dots (t_{e-1} - t_e)\Lambda_{e-1}$$

and

$$\dot{\mathrm{w}}\mathrm{t}|\boldsymbol{\lambda},\mathbf{s}\rangle = (s_l - s_1 - e)\dot{\Lambda}_0 + (s_1 - s_2)\dot{\Lambda}_1 + (s_2 - s_3)\dot{\Lambda}_2 + \dots (s_{e-1} - s_e)\dot{\Lambda}_{l-1}.$$

Here, we are using Kac's notation for the fundamental weights ([Kac90]), and to distinguish the fundamental weights for $U_v(\hat{\mathfrak{sl}}'_e)$ and $U_u(\hat{\mathfrak{sl}}'_l)$ we put 'dots' on the level side. It follows that

$$\Lambda^{s+\frac{\infty}{2}} = \bigoplus_{\mathbf{s}\in\mathbf{Z}^l(s)} \mathcal{F}[\mathbf{s}], \qquad \mathcal{F}[\mathbf{s}] := \bigoplus_{\lambda} \mathbf{C}(v)|\boldsymbol{\lambda},\mathbf{s}\rangle$$

and

$$\Lambda^{s+\frac{\infty}{2}} = \bigoplus_{\mathbf{t}\in\mathbf{Z}^e(s)} \mathcal{F}[\mathbf{t}], \qquad \mathcal{F}[\mathbf{t}] := \bigoplus_{\mu} \mathbf{C}(v)|\boldsymbol{\mu},\mathbf{t}\rangle$$

are weight space decompositions for the actions of $U_v(\hat{\mathfrak{sl}}'_e)$ and $U_u(\hat{\mathfrak{sl}}'_l)$, respectively. By definition $\mathcal{F}[\mathbf{s}]$ carries an action $U_v(\hat{\mathfrak{sl}}'_e)$; it is the level l Fock space associated to the multicharge $\mathbf{s}$. Similary the $U_u(\hat{\mathfrak{sl}}'_l)$-modules $\mathcal{F}[\mathbf{t}]$ are level e Fock spaces. Define the $\mathbf{C}(v)$-vector space

$$\mathcal{F}[\mathbf{s},\mathbf{t}] := \mathcal{F}[\mathbf{s}] \cap \mathcal{F}[\mathbf{t}]$$

as a simultaneous weight space. In each space $\mathcal{F}[\mathbf{s},\mathbf{t}]$, the operator d acts with eigenvalues $v^z, v^{z+1}, \dots$ for some $z \in \mathbf{Z}$, and we define $\mathcal{F}[\mathbf{s},\mathbf{t}]_w$ to be the v^{z+w}-eigenspace for $w \geq 0$.

Uglov defines canonical bases $\{G_v^+(\lambda, r)\}$ and $\{G_v^-(\lambda, r)\}$ in $\Lambda^{s+\frac{\infty}{2}}$. Interchanging the role of e and l, we can also define canonical bases $\{G_u^+(\lambda, r)\}$ and $\{G_u^-(\lambda, r)\}$. Then it is clear from the definition of these bases that

$$G_v^+(\lambda, r) = G_u^-(\lambda, r), \qquad G_v^-(\lambda, r) = G_u^+(\lambda, r). \tag{1}$$

3. Quasihereditary covers

The main references for this section are [GGOR03], [Rou08b] and [Ari08].

Let B be a finite-dimensional associative $\mathbf{k}$-algebra. A *cover* of B consists of a $\mathbf{k}$-linear abelian category $\mathfrak{A}$ with enough projectives, together with a projective object P of $\mathfrak{A}$ such that $B \cong \mathrm{End}_{\mathfrak{A}}(P)$ and the functor $\mathsf{F} = \mathrm{Hom}_{\mathfrak{A}}(P, -) : \mathfrak{A} \to B\text{-mod}$ is fully faithful on projectives. If in addition $\mathfrak{A}$ is a highest weight category in the sense of Cline, Parshall and Scott, we call $\mathfrak{A}$ a *highest weight cover* of B, and call A a *quasihereditary cover* of B.

As a typical example, the Schur algebra $\mathrm{End}_{\mathbf{k}\mathfrak{S}_m}(V^{\otimes m})$ of the symmetric group algebra $\mathbf{k}[\mathfrak{S}_m]$ is a quasihereditary cover of $\mathbf{k}[\mathfrak{S}_m]$, where V is a $\mathbf{k}$-vector space of dimension at least m and $\mathfrak{S}_m$ acts on $V^{\otimes m}$ by permutation of tensor factors.

3.1. Cyclotomic Schur algebras.

Let $q, Q_1, \dots, Q_l \in \mathbf{k}$. Let $\mathcal{S}_m(q; Q_1, \dots, Q_l)$ be the (full) cyclotomic q-Schur algebra over $\mathbf{k}$ associated with the set of all l-multicompositions of m, as defined in [DJM98]. (Note that the order of the Q_i's matters, in contrast to Ariki-Koike algebras.) Then $\mathcal{S}_m(q; Q_1, \dots, Q_l)$ is a quasihereditary cover of $\mathcal{H}_m(q; Q_1, \dots, Q_l)$ via the Schur functor

$$S \ : \ \mathcal{S}_m(q; Q_1, \dots, Q_l)\text{-mod} \to \mathcal{H}_m(q; Q_1, \dots, Q_l)\text{-mod}.$$

3.2. **Rational Cherednik algebras.** Let W be a complex reflection group of type $G(l,1,m)$ on an m-dimensional vector space $\mathfrak{h}$ over $\mathbf{C}$. Let $\mathcal{E}$ be the set of reflection hyperplanes of W. For $H \in \mathcal{E}$, we denote by W_H the pointwise stabilizer of H in W. We denote by $\epsilon_{H,j}$ the idempotent of $\mathbf{C}W_H$ corresponding to $(\mathrm{Res}_{W_H}(\det))^{-j}$:

$$\epsilon_{H,j} := \frac{1}{|W_H|} \sum_{w \in W_H} \det(w)^j w \in \mathbf{C}W_H.$$

We fix some parameters:

$$\boldsymbol{\kappa}^\circ = (\kappa_i)_{i \in \mathbf{Z}/l\mathbf{Z}}, \kappa_i \in \mathbf{C},\ \boldsymbol{h} = (h_i)_{i \in \mathbf{Z}/2\mathbf{Z}} \text{ with } h_0 = 0, h_i \in \mathbf{C}, \text{ and } \boldsymbol{\kappa} := (h_1; \boldsymbol{\kappa}^\circ).$$

For $H \in \mathcal{E}$ and $0 \le k < |W_H|$, put

$$C_{H,k} := \begin{cases} \kappa_k & \text{if } H = H_i, \\ h_k & \text{otherwise.} \end{cases}$$

Here, H_i is the reflection hyperplane for t_i which acts on the standard bases of $\mathfrak{h}$ trivially except the i-th coordinate base of $\mathfrak{h}$, in which case t_i acts as multiplication by $\exp(2\pi\sqrt{-1}/l)$. For $H \in \mathcal{E}$, put

$$\gamma_H := |W_H| \sum_{k=0}^{|W_H|-1} (C_{H,k+1} - C_{H,k})\epsilon_{H,k} \in \mathbf{C}W_H.$$

Then, those $\gamma_H, H \in \mathcal{E}$ and $\mathbf{C}W, S(\mathfrak{h}), S(\mathfrak{h}^*)$ define the rational Cherednik algebra (also known as rational double affine Hecke algebra) associated with W, $\mathfrak{h}$ and $\boldsymbol{\kappa}$. See [GGOR03, p.628,3.1] or [Ari08, § 4]. We denote this algebra by $\mathbf{H}_{\boldsymbol{\kappa}} = \mathbf{H}_{\boldsymbol{\kappa}}(W, \mathfrak{h})$.

Here, we are only concerned with the case that $\mathbf{H}_{\boldsymbol{\kappa}}$ has very small centre (namely, in the notation of [Rou05], the case $t = 1$).

By a theorem of Etingof-Ginzburg [Ari08, Lemma 4.8], we have a triangular decomposition $\mathbf{H}_{\boldsymbol{\kappa}} = S(\mathfrak{h}^*) \otimes_{\mathbf{C}} \mathbf{C}W \otimes_{\mathbf{C}} S(\mathfrak{h})$ as a vector space. Treating $S(\mathfrak{h}^*)$ and $S(\mathfrak{h})$ as "nilpotent subalgebras of universal enveloping algebra", we may define an analogue of BGG category $\mathcal{O}$ for $\mathbf{H}_{\boldsymbol{\kappa}}$. Let $\mathcal{O}_{\boldsymbol{\kappa}}(W, \mathfrak{h})$ be the full subcategory of $\mathbf{H}_{\boldsymbol{\kappa}}(W, \mathfrak{h})$-Mod consisting of the objects finitely generated as an $\mathbf{H}_{\boldsymbol{\kappa}}$-module and local nilpotent as $S(\mathfrak{h})$-modules.

In [GGOR03] it is shown that $\mathcal{O}_{\boldsymbol{\kappa}}(W, \mathfrak{h})$ is a highest weight category with simple objects indexed by the set $\mathrm{Irr}\, W$ of irreducible complex representations of W. The appropriate poset structure on $\mathrm{Irr}\, W$ for the highest weight structure is given in terms of $\boldsymbol{\kappa}^\circ$ (see [GGOR03]). This poset structure first appeared in [JMMO91] and [FLO+99]. Via Clifford theory the set $\mathrm{Irr}\, W$ may be identified with the set of l-multipartitions of size m.

3.3. **Rouquier's point of view.** In this subsection, we fix $e \ge 1$ and assume that $\ell = 0$. Following Rouquier we consider certain choices of the system of parameters $\boldsymbol{\kappa}$, indexed by *multicharges* $\mathbf{s} \in \mathbf{Z}^l$.

Given $\mathbf{s} = (s_1, s_2, \ldots, s_l) \in \mathbf{Z}^l$, we consider the category

$$\mathcal{O}_m[\mathbf{s}] := \mathcal{O}_{\boldsymbol{\kappa}}(W, \mathfrak{h}) \quad \text{with} \quad h_1 = \frac{1}{e}, \kappa_j = \frac{s_j}{e} - \frac{j}{l} \quad \text{for } j = 1, \ldots, l \text{ ——}(\diamond).$$

and the Ariki-Koike algebra

$$\mathcal{H}_m[\mathbf{s}] := \mathcal{H}_m(q; Q_1, \ldots, Q_l) \quad \text{with} \quad q = \zeta := \exp\left(\frac{2\pi\sqrt{-1}}{e}\right), Q_i = \zeta^{s_i} \quad \text{for } i = 1, \ldots, l.$$

Note that $\mathcal{H}_m[\mathbf{s}]$ depends only on the set $\{\overline{s}_1, \ldots, \overline{s}_l\}$ of residues modulo e. By [GGOR03], we have an Knizhnik-Zamolodchikov functor

$$\mathsf{KZ} : \mathcal{O}_m[\mathbf{s}] \to \mathcal{H}_m[\mathbf{s}]\text{-mod},$$

with $\mathsf{KZ} = \mathrm{Hom}_{\mathcal{O}_m[\mathbf{s}]}(P, -)$ for some projective object P of $\mathcal{O}_m[\mathbf{s}]$, and $\mathcal{O}_m[\mathbf{s}]$ is a highest weight cover of $\mathcal{H}_m[\mathbf{s}]$.

Rouquier explained that this cover coincides with the cyclotomic q-Schur algebra, as long as the multicharge $\mathbf{s}$ is 'dominant' and the parameters $Q_1, \ldots Q_l$ are distinct. He [Rou08b, Theorem 6.8] (cf. [Ari08, § 4]) proved the following result, using certain faithfulness properties of the functors KZ and S in the context of a general theory of highest weight covers.

Theorem 3 (Rouquier)**.** *Suppose that* $-1 \neq q \in \mathbf{C}$ *and that* $Q_1, \ldots, Q_l$ *are distinct. Suppose further that* $s_{i+1} >> s_i$ *for* $0 \leq i \leq l-2$. *Then,* $\mathcal{S}_m(q; Q_1, \ldots, Q_l)$-mod *and* $\mathcal{O}_m[\mathbf{s}]$ *are equivalent highest weight categories.*

Remark 4. *Using Bezrukavnikov-Etingof induction and restriction functors* [BE09]*, the commutativity between* KZ *functors and induction/restriction functors* [Sha09] *and Ringel duality* [Don98],[Mat03], [GGOR03, § 4], *we can handle some cases which are not treated in this theorem. For example, for type A,* i.e. *,* $l = 1$, *there is no restriction on q in this approach. This will be discussed elsewhere.*

Rouquier [Rou08b, Conjecture 5.6 & Remark 6.10] also expects the following:

Conjecture 5 (Rouquier)**.** *Suppose* $\mathbf{s}$ *and* $\mathbf{s}'$ *determine the same set of residues modulo e. Then the corresponding category* $\mathcal{O}$*'s are derived equivalent:*

$$\mathcal{D}^b(\mathcal{O}_\kappa[\mathbf{s}]) \cong \mathcal{D}^b(\mathcal{O}_\kappa[\mathbf{s}']).$$

So, according to Rouquier's Theorem 3 and Conjecture 5, the cyclotomic q-Schur algebras are special cases of categories $\mathcal{O}$'s of rational Cherednik algebras, and up to derived equivalence should account for all category $\mathcal{O}$'s.

3.4. **Categorification and conjecture on level-rank duality.** We now put

$$\mathcal{H}[\mathbf{s}] = \bigoplus_{m \geq 0} \mathcal{H}_m[\mathbf{s}] \qquad \text{and} \qquad \mathcal{O}[\mathbf{s}] = \bigoplus_{m \geq 0} \mathcal{O}_m[\mathbf{s}].$$

Rouquier has suggested that under the identification

$$\begin{array}{rcl} \mathcal{F}[\mathbf{s}]_{v=1} & \xrightarrow{\sim} & \mathbf{C} \otimes_{\mathbf{Z}} K(\mathcal{O}[\mathbf{s}]) \\ |\lambda, \mathbf{s}\rangle & \mapsto & [\Delta(\lambda)], \end{array}$$

Uglov's canonical basis vectors $G_v^+(\lambda, \mathbf{s})$ and $G_v^-(\lambda, \mathbf{s})$ in $\mathcal{F}[\mathbf{s}]$, specialised at $v = 1$, should be mapped to the classes $[T(\lambda)]$ and $[L(\lambda)]$ of characteristic tilting modules and simple modules [Rou08b][§6.5]. His conjecture is a natural extension of Ariki's theorem [Ari96], analogous to the Leclerc-Thibon conjecture [LT96],[VV99]. Let $\mathcal{L}[\emptyset, \mathbf{s}] := U_v(\widehat{\mathfrak{sl}}_e)|\emptyset, \mathbf{s}\rangle$ if $e > 1$ and $\mathcal{L}[\emptyset, \mathbf{s}] := H|\emptyset, \mathbf{s}\rangle$ if $e = 1$. Then the projection of $\mathcal{F}[\mathbf{s}]_{v=1}$ onto $\mathcal{L}[\emptyset, \mathbf{s}]_{v=1}$ is categorified by the KZ-functor:

$$\begin{array}{rccccl} & \mathcal{O}[\mathbf{s}] & \xrightarrow{\mathsf{KZ}} \!\!\!\!\rightarrow & \mathcal{H}[\mathbf{s}]\text{-mod} & \\ \text{‘decategorify’} & \Downarrow & & \Downarrow & \text{Ariki} \\ & \mathcal{F}[\mathbf{s}]_{v=1} & \twoheadrightarrow & \mathcal{L}[\emptyset, \mathbf{s}]_{v=1} & \end{array}$$

The decomposition of $\mathcal{L}[\emptyset, \mathbf{s}]$ into $U_v(\widehat{\mathfrak{sl}}_e)$-weight spaces corresponds to the decomposition

$$\mathcal{H}[\mathbf{s}]\text{-mod} = \bigoplus_{\mathbf{t} \in \mathbf{Z}^l(s), w \geq 0} \mathcal{H}[\mathbf{s}, \mathbf{t}]_w\text{-mod}.$$

into blocks [LM07]. Since the functor KZ induces an isomorphism of centres (see [GGOR03]), we have a corresponding block decomposition

$$\mathcal{O}[\mathbf{s}] = \bigoplus_{m \geq 0} \mathcal{O}_m[\mathbf{s}] = \bigoplus_{\mathbf{t} \in \mathbf{Z}^l(s), w \geq 0} \mathcal{O}[\mathbf{s}, \mathbf{t}]_w.$$

Following Uglov, given $\mathbf{s} = (s_1, \ldots, s_l) \in \mathbf{Z}^l(s)$, we define $\mathbf{s}' = (-s_l, \ldots, -s_1) \in \mathbf{Z}^l(-s)$. Now we can state our main conjecture.

Conjecture 6 (Level-Rank Duality)**.** $\mathcal{O}[\mathbf{s}, \mathbf{t}]_w$ *and* $\mathcal{O}[\mathbf{t}', \mathbf{s}']_w$ *are Koszul dual abelian categories :* $\mathcal{O}[\mathbf{s}, \mathbf{t}]_w^! \cong \mathcal{O}[\mathbf{t}', \mathbf{s}']_w$.

Remark 7. *The conjectured Koszul duality provides graded versions $\mathcal{O}^{gr}[\mathbf{s}]$ of the GGOR categories, and one would expect the identification*

$$\begin{array}{rcl} \mathcal{F}[\mathbf{s}] & \xrightarrow{\sim} & \mathbf{C}(v) \otimes_{\mathbf{Z}} K(\mathcal{O}^{gr}[\mathbf{s}])) \\ |\lambda, \mathbf{s}\rangle & \mapsto & [\Delta(\lambda)], \end{array}$$

to send $G_u^+(\lambda, \mathbf{s})$ and $G_u^-(\lambda, \mathbf{s})$ to the classes to graded characteristic tilting and simple modules respectively. Perhaps the most natural way to present the conjecture is the prediction of an equivalence

$$\mathsf{F} : \mathcal{D}^b(\mathcal{O}^{gr}[s,t]_w) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{O}^{gr}[t,s]_w)$$

such that $\mathsf{F}(X\langle 1\rangle) \cong \mathsf{F}(X)[-1]\langle -1\rangle$, preserving standard and costandard modules, and interchanging simple and tilting modules, in keeping with Rouquier's conjecture above and the equalities (1). The existence of F *would imply that $\mathcal{O}^{gr}[s,t]_w$ is equivalent to the module category over a* balanced algebra, *in the sense of Mazorchuk* [Maz09], [Maz10a], [Maz10b], *and then* F *would be the composition of Koszul and Ringel dualities.*

4. Hidden Hecke algebras

We now provide some evidence for Conjecture 6 in the case $l = 1$. We allow the possibility that $\mathbf{k}$ has positive characteristic. The level 1 cyclotomic q-Schur algebra $\mathcal{S}_m(q; Q_1)$ does not depend, up to isomorphism, on Q_1; it is simply the q-Schur algebra $\mathcal{S}_m(q)$.

Fix an integer $e \geq 2$ and recall the Fock space representation of the Kac-Moody algebra $\hat{\mathfrak{sl}}_e$. Let $\mathcal{F} = \oplus_\lambda \mathbf{C}\lambda$ be a complex vector space with basis indexed by the set of all partitions of all nonnegative integers. We define linear operators $E_0, \ldots, E_{e-1}, F_0, \ldots, F_{e-1}$ on $\mathcal{F}$ by

$$E_a\mu = \sum_{\lambda \to_a \mu} \lambda \qquad \text{and} \qquad F_a\lambda = \sum_{\lambda \to_a \mu} \mu,$$

where $\lambda \to_a \mu$ means the Young diagram of μ is obtained from that of λ by adding an node (i, j) such that $j - i \equiv a$ modulo e. These locally nilpotent endomorphisms extend to an action of $\hat{\mathfrak{sl}}_e$. Put $s_a := \exp(-F_a)\exp(E_a)\exp(-F_a)$, an automorphism of $\mathcal{F}$. Then for any partition λ we have $s_a\lambda = \pm\sigma_a\lambda$ for some partition $\sigma_a\lambda$. The permutations $\sigma_0, \ldots, \sigma_{e-1}$ define an action of the affine Weyl group $\hat{W}_e$ on the set of partitions.

Let $\mathbf{k}$ be a field (We always assume that $\mathbf{k}$ is a splitting field for the algebras we consider), let $q \in \mathbf{k}^\times$ and suppose that e is the quantum characteristic of $\mathbf{k}$ with respect to q:

$$e = \inf\{i \in \mathbf{N} \mid 1 + q + q^2 + \cdots + q^{i-1} = 0 \in \mathbf{k}\}.$$

The (full) q-Schur algebra $\mathcal{S}_m(q)$ is a quasihereditary $\mathbf{k}$-algebra with standard and simple modules $\Delta(\lambda)$ and $L(\lambda)$ indexed by partitions of m. Following [LLT96] and [LT96] we identify $\mathcal{F}$ with the sum of complexified Grothendieck groups of module categories of q-Schur algebras:

$$\begin{array}{rcl} \mathcal{F} & = & \bigoplus_{m \geq 0} \mathbf{C} \otimes_{\mathbf{Z}} K(\mathcal{S}_m(q)\text{-mod}) \\ \lambda & \leftrightarrow & [\Delta(\lambda)] \end{array}$$

The decomposition of $\mathcal{F}$ of into $\hat{\mathfrak{sl}}_e$-weight spaces coincides with the block decomposition of the q-Schur algebras. This is a restatement of 'Nakayama's conjecture': the blocks of $\mathcal{S}_m(q)$, $m \geq 0$ are classified by an e-core partition τ and a nonnegative integer w, where $\Delta(\lambda)$ and $L(\lambda)$ are in the block $B_{\tau,w}$ if λ has e-core τ and e-weight w. In Fock space terms, the space $[B]$ intersects with the simple $U(\hat{\mathfrak{sl}}_e)$-module $L(\Lambda_0 - w\delta)$ with highest weight $\Lambda_0 - w\delta$ in Misra-Miwa decomposition of $\mathcal{F}$ where δ is the null root.

Let $B = B_{\tau,w}$ and $a \in \{0, \ldots, e-1\}$. Then s_a restricts to an isomorphism $K(B\text{-mod}) \xrightarrow{\sim} K(\sigma_a(B)\text{-mod})$, where $\sigma_a B = B_{\sigma_a(\tau),w}$, and the induced action of $\hat{W}_e$ on blocks is transitive on blocks of e-weight w.

Theorem 8 ([CR08]). *The isomorphism*

$$s_a : K(B\text{-mod}) \xrightarrow{\sim} K(\sigma_a B\text{-mod})$$

lifts to an equivalence

$$\dot{s}_a : \mathcal{D}^b(B\text{-mod}) \cong \mathcal{D}^b(\sigma_a B\text{-mod})$$

of bounded derived categories.

For a precise concrete description of the derived categories of module categories over q-Schur algebras, we need some assumptions:

Definition 9. *Let B be a block algebra of $\mathcal{S}_m(q)$ of e-weight w. Write ℓ for the characteristic of* $\mathbf{k}$. *We say that B is weakly abelian if either $\ell = 0$ or $0 \leq w < \ell$. We say that B is abelian if in additon q lies in the prime subfield of* $\mathbf{k}$ *when $\ell \neq 0$.*

Remark 10. *If $q = 1$ then $\mathcal{S}_m(q)$ is the Schur algebra associated to the symmetric group $\mathfrak{S}_m$, and a block B of $\mathcal{S}_m(q)$ is abelian, in our sense, if and only if the corresponding block of $\mathfrak{S}_m$ has abelian defect groups. If $q \cdot 1_{\mathbf{k}} \neq 0$ for some power q of prime, then $\mathcal{S}_m(q)$ is the q-Schur algebra associated to the finite general linear group $GL_m(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ with q elements and a block B of $\mathcal{S}_m(q)$ is abelian if and only if the corresponding block of $GL_m(\mathbb{F}_q)$ has abelian defect groups. All of the theorems we state for abelian blocks should probably be true for weakly abelian blocks as well.*

Using the transitivity mentioned above and combining Theorem 8 with [CM10, Theorem 18] [1], we obtain the following result.

Theorem 11. *Suppose that B is abelian with e-weight w. Then we have an equivalence*

$$\mathcal{D}^b(B\text{-mod}) \cong \mathcal{D}^b(B_{\emptyset,1} \wr \mathfrak{S}_w\text{-mod}).$$

Theorem 11 means we have a reasonable handle on the derived categories of blocks of q-Schur algebras, at least in the abelian case, and suggests that even the module categories B-mod might be explained by $B_{\emptyset,1} \wr \mathfrak{S}_w$-mod.

On the other hand, many researchers expect that

Conjecture 12. *Any weakly abelian block B of $\mathcal{S}_m(q)$ is Koszul.*

If Conjecture 6 is true, then the Ariki-Koike algebras (of various levels) should appear in even in the representation theory of $\mathcal{S}_m(q)$. We now explain how that is indeed the case, and we realise the conjectural Ariki-Koike algebras concretely as extension algebras which we call "hidden Hecke algebras".

Given any block $B = B_{\tau,w}$ of a Schur algebra $\mathcal{S}_m(q)$ of any degree, consider the semisimple B-module

$$L_0^B := \bigoplus_{\lambda \vdash w} L(\tau + e\lambda)^{\bigoplus \dim S^\lambda},$$

where S^λ is the Specht module over $\mathbf{C}[\mathfrak{S}_w]$ corresponding to the partition λ. The following result shows that the L_0^B, viewed as complexes concentrated in degree 0, are permuted by the derived equivalences of Theorem 8.

Theorem 13. *Suppose that B is weakly abelian. Then for any a we have*

$$\dot{s}_a(L_0^B) \cong L_0^{\sigma_a(B)}.$$

More precisely, for any partition $\lambda \vdash w$,

$$\dot{s}_a L(\tau + e\lambda) \cong L(\sigma_a(\tau) + e\lambda).$$

In particular, the (partial) Yoneda algebra $\mathrm{Ext}_B^\bullet(L_0^B, L_0^B)$ depends only on the e-weight of B.

---

[1] in which we use the truth of Broué abelian defect conjecture on Rouquier blocks for $\mathbb{F}_\ell\mathfrak{S}_m$ [CK02] and $\overline{\mathbb{F}}_\ell GL_m(\mathbb{F}_q)$ [Tur02],[Miy01] where q is a power of prime and $\ell \nmid q$

Corollary 14. *Let B be an abelian block of weight w. Then we have an isomorphism*

$$A_w^B := \mathrm{Ext}_B^\bullet(L_0^B, L_0^B) \cong \mathbf{k}[x]/(x^e) \wr \mathfrak{S}_w$$

of graded algebras, with $\deg(x) = 2$ *and* $\deg(\mathfrak{S}_w) = 0$.

Remark 15. *We call A_w^B a level e hidden Hecke algebra arising from level $l = 1$. A sketch of the proof will be explained in the next subsection with an aid of "stubborn" property on simple modules $L(\tau + e\lambda)$ in Theorem 13 and a knowledge of "Rouquier blocks". Moreover, A_w^B is nothing but a very special Ariki-Koike algebra $A_w(e) = \mathcal{H}_w(1; 1, 1, \dots, 1)$. So, if we treat the case of characteristic 0, there should be a category $\mathcal{O}$ of RatDAHA which is a highest weight cover of $A_w(e)$.*

Based on Conjecture 6, in characteristic 0 we would expect a 'hidden Hecke algebra' in any block $\mathcal{B}$ of $\mathcal{O}[\mathbf{s}]$ is realized as an extension algebra $\mathrm{Ext}_{\mathcal{B}}^\bullet(L_{\mathcal{B}}, L_{\mathcal{B}})$. Here $L_{\mathcal{B}}$ would be a direct sum (with multiplicities) of all simple modules S in $\mathcal{B}$ with the property that $\mathrm{Ext}_{\mathcal{B}}^\bullet(B_0, S)$ is an injective (and obviously projective) module over the homological dual $\mathrm{Ext}_{\mathcal{B}}^\bullet(B_0, B_0)$, where B_0 is a semisimple module in $\mathcal{B}$ containing all non-isomorphic simple modules in $\mathcal{B}$.

We reformulate Conjecture 6 for ordinary q-Schur algebras so that it includes some positive characteristic cases as follows:

Conjecture 16. *If $w < \ell$ or $\ell = 0$, then B is Koszul, and the Koszul dual $B^!$ is a quasihereditary cover of $A_w(e)$ with respect to the projective $B^!$-module $P_B = \mathrm{Ext}^\bullet(B_0, L_0^B)$.*

Remark 17. *The image of $B^!$ under the functor $\mathsf{F} = \mathrm{Hom}(P_B, -)$ is the A_w^B-module*

$$M_B = \mathrm{Ext}^\bullet(L_0^B, B_0).$$

If the conjecture were true, then B may be recovered from M_B as follows:

$$B = \mathrm{End}_{A_w(e)}(M_B)^!.$$

Remark 18. *C is derived equivalent to B, then the hidden Hecke algebra A_w^C for C is isomorphic to $A_w(e) = A_w^B$ as we stated. So, our picture is very consistent with Rouquier's picture in § 3.3 if we assume that $B^!$ and $C^!$ are derived equivalent and Conjecture 16.*

$$\begin{tikzcd}
B^!\text{-mod} \arrow[dr, "\mathsf{KZ}_{B^!}"'] & & C^!\text{-mod} \arrow[dl, "\mathsf{KZ}_{C^!}"] \\
& A_w(e)\text{-mod} &
\end{tikzcd}$$

4.1. **Rouquier blocks.** As explained in Theorem 3, the category $\mathcal{O}[\mathbf{s}]$ is more tractible when $\mathbf{s}$ is a 'dominant multicharge', in that it can then be identified with the module category of a cyclotomic q-Schur algebra.

Via the Koszul duality of Conjecture 6, dominant multicharges correspond to certain e-cores, called Rouquier cores. The corresponding blocks are called Rouquier blocks (see, for example, [JLM06]).

4.1.1. *e-weight 1.* Suppose that B is a block of $\mathcal{S}_m(q)$ with e-weight $w = 1$. It is known that all e-weight 1 blocks are Rouquier and B is Morita equivalent to the principal block $B_{1,\emptyset}$ of $\mathcal{S}_e(q)$. The Loewy structures of the standard modules $\Delta(\lambda)$ and projective indecomposable modules $P(\lambda)$ over $B_{1,\emptyset}$ are :

$$\Delta(1^e) = L(1^e), \Delta(2, 1^{e-2}) = \begin{array}{c} L(2, 1^{e-2}) \\ L(1^e) \end{array}, \dots, \Delta(i, 1^{e-i}) = \begin{array}{c} L(i, 1^{e-i}) \\ L(i + 1^e - i - 1) \end{array}, \dots, \Delta(e) = \begin{array}{c} L(e) \\ L(e-1, 1) \end{array}.$$

$$P(1^e) = \begin{array}{c} L(1^e) \\ L(2, 1^{e-2}) \\ L(1^e) \end{array}, P(2, 1^{e-2}) = \begin{array}{c} L(2, 1^{e-2}) \\ L(1^e) L(3, 1^{e-3}) \\ L(2, 1^{e-2}) \end{array}, \dots,$$

$$P(i,1^{e-i}) = \begin{matrix} L(i,1^{e-i}) \\ L(i-1,1^{i-1})L(i+1,1^{e-i-1}) \\ L(i,1^{e-i}) \end{matrix}, \ldots, P(e) = \begin{matrix} L(e) \\ L(e-1,1) \end{matrix}.$$

Then, $L_0^B = L(e)$ and $A(1) \cong \mathrm{Ext}_B^\bullet(L(e)) \cong \mathbf{k}[x]/(x^e)$.

Put

$$M_1 := \bigoplus_{i=1}^{e-1} \mathrm{rad}^i\,(\mathbf{k}[x]/(x^e))\,\langle i\,\rangle.$$

Here, $\langle i\,\rangle$ indicates the i-shift of the grading, $(M\langle i\,\rangle)_j := M_{j-i}$. The Koszul dual $B^!$ is graded Morita equivalent to $Aus(e) := \mathrm{End}_{\mathbf{k}[x]/(x^e)}(M_1)$, which is the Auslander algebra of $\mathbf{k}[x]/(x^e)$. The truth of Conjecture 16 for this Auslander algebra is well known. Moreover, in [Rou08b], the other defect 1 blocks for any finite Weyl groups are listed, they are Brauer line algebras and the covers are treated. So, for defect 1 blocks, any Brauer line symmetric algebras, there is nothing left to prove.

4.1.2. *e-weight $w \geq 1$.* The next result implies that Conjecture 16 is true for at least one block for each e-weight $w > 0$.

Theorem 19. *Conjecture 16 is true for abelian Rouquier blocks. The Koszul dual of an abelian Rouquier block with e-weight w is Morita equivalent to* $\mathrm{Aus}(e) \wr \mathfrak{S}_w$ *(which is a special case of a cyclotomic Schur algebra).*

Remark 20. *Suppose that B is a Rouquier block with e-weight w. The ideas of the proof are:*

(1) *to use an A_w-module*

$$M := (M_1 \otimes V)^{\otimes w}$$

where V is an appropriate vector representation of ordinary or quantum general linear group whose semisimple rank is at least w.

(2) *to use the structural description of B, namely, B is Morita equivalent to $B_{1,\emptyset} \wr \mathfrak{S}_w$.* [CM10, p.72 Theorem 18].

Conjecture 16 says also that Conjecture 5(2) for level e > and rank $1 (=$ *the order of q) is Koszul dual to Broué's conjecture up to replacement of ℓ by the quantum characteristic e. Broué's conjecture here is proven by* [CR08]. *In Broué's conjecture algebras such as $B^{1,\emptyset} \wr \mathfrak{S}_w$ are called local blocks, which are Rouquier in our cases. So, $O_{\boldsymbol{\kappa}}(W,\mathfrak{h})$ with el-dominant charge $\boldsymbol{\kappa}$ and $\mathcal{S}_m(q;\overrightarrow{Q})_{\mathbf{C}}$ are regarded as local blocks in this sense and have nicer property than general blocks.*

By assuming characteristic zero, $\ell = 0$, we would like to see the wreath product structure $\mathrm{Aus}(e) \wr \mathfrak{S}_w$ *inside the rational Cherednik algebras of level e whose finite Hecke algebra part (*i.e. *, the image of* KZ*) is* $\mathbf{k}[x]/(x^e) \wr \mathfrak{S}_w$. *Note that we switched the roles of "level" and "rank" here. To avoid the confusion, we reput $l' := e$ and $e' := 1$. We consider the rational Cherednik algebra of $W = G(l',1,w)$ with parameters $h_1 = 1/e' = 1$ and $\mathbf{s}' = (s'_1,\ldots,s'_{l'})$. A rough sketch for seeing the wreath product structure goes as follows: First, on the multicharge $\mathbf{s}'$, we assume that*

$$s'_{i+1} >> s'_i \text{ for all } i \quad \text{———} \quad (\star).$$

We can take $\mathbf{H}_{\boldsymbol{\kappa}}$ such that $\boldsymbol{\kappa}$ satisfies $(\diamond)$ at § 3.4 by replacing e (resp. l) by $e' = 1$ (resp. $l' = e$). Next we consider another ratDAHA $\mathbf{H}_{\boldsymbol{\kappa}-1}$ whose $h_1 = 0$ and the other parts are given by $\kappa_i - 1$. [2]

Then, we can find an equivalence between $O_{\boldsymbol{\kappa}}(W,\mathfrak{h})$ and $O_{\boldsymbol{\kappa}-1}(W,\mathfrak{h})$ as long as we have a huge difference on s'_i's at $(\star)$. This can be checked by considering shift functors and spherical Hecke algebras. Indeed, note that by [DG10, Corollary 3.5][3] *and* [Val06, Lemma 4.18], *we know when we have the equivalence in question.*

Finally, by definition of $\mathbf{H}_{\boldsymbol{\kappa}-1}$ [4]*, we can see that*

$$\mathbf{H}_{\boldsymbol{\kappa}-1} = \mathbf{H}_{\boldsymbol{\kappa}-1}(G(l',1,1)) \wr \mathfrak{S}_w.$$

[2] Note that this doesn't satisfy $(\diamond)$ at § 3.4.

[3] The authors would like to thank S. Griffeth for explaining the results in [DG10, Corollary 3.5] to us.

[4] The authors would like to thank E. Vasserot for letting us notice a history that this Clifford theory, wreath product realization was already used by P. Etingof sometimes.

So, we can find κ such that $\mathcal{O}_\kappa(W,\mathfrak{h}) \cong \mathrm{Aus}(e) \wr \mathfrak{S}_w$-mod and the parameter choice of $(\diamond)$ at § 3.4 is satisfied for $\mathbf{H}_\kappa(W,\mathfrak{h})$. Therefore, the Koszul dual of Rouquier blocks are certainly in the category $\mathcal{O}$'s of higher level. Hence, in those Rouquier blocks cases, Conjecture 6 is true.

5. Numerical standard Koszul duality criterion

Let $A = \oplus_{i\geq 0} A_i$ be a graded algebra over an enough large splitting field $\mathbf{k}$, with A_0 semisimple. Assume that A is quasihereditary. Then there are unique (up to isomorphism) gradings on all 'special modules', so that we have homogeneous surjections $A \to P(x) \to \Delta(x) \to L(x)$ and injections $L(x) \to \nabla(x) \to I(x)$. Here, we took a standard notation: $\Delta(x)$ (resp. $\nabla(x)$, $L(x)$, $P(x)$ and $I(x)$) is the standard (resp. costandard, simple, projective indecomposable and injective indecomposable) module over A associated with $x \in \Lambda_A$, which is the index set for simple A-modules with a poset structure.

Recall from Agoston-Dlab-Lukacs[ÁDL03] that A is *standard Koszul* if all standard modules have linear projective resolutions, and all costandard modules have linear injective resolutions. Of course if there is a contravariant auto-equivalence of the module category fixing simples (or more generally inducing a automorphism of the poset of simples), then the condition on standards and costandards are equivalent.

There ought to be a statement of numerical standard Koszulity somewhere in the literature, mirroring Beilinson-Ginzburg-Soergel Numerical Koszul duality criterion [BGS96, Theorem 2.11.1]. The approach we take here is a sort of "square root" of their criterion.

Define matrices of power series in an indeterminate t as follows. We have graded decomposition matrices

$$D(t) = \left\{ \sum_{i\geq 0} \dim \mathrm{Homgr}(P(x)\langle i\rangle, \nabla(y)) t^i \right\}_{x,y}$$

and

$$D'(t) = \left\{ \sum_{i\geq 0} \dim \mathrm{Homgr}(\Delta(y), I(x)\langle i\rangle) t^i \right\}_{x,y}$$

as well as Vogan matrices

$$K(t) = \left\{ \sum_{i\geq 0} \dim \mathrm{Ext}^i(\Delta(x), L(y)) t^i \right\}_{x,y}$$

and

$$K'(t) = \left\{ \sum_{i\geq 0} \dim \mathrm{Ext}^i(L(y), \nabla(x)) t^i \right\}_{x,y}.$$

For the last two, extensions are taken in the ungraded category.

Then what we claim is that

Proposition 21 (Square Root Numerical Standard Koszul Duality Criterion)**.** *A is standard Koszul if and only if*

$$D(t)K(-t) = I \text{ and } D'(t)K'(-t) = I.$$

Proof. If there is a good contravariant duality as mentioned above, then $D(t) = D'(t)$ and $K(t) = K'(t)$, and the two numerical conditions are equivalent. (In fact for our main purpose, this is the case.)

To prove the claim consider a minimal graded projective resolution of $\Delta(x)$:

$$\ldots \to P^1 \to P^0 \to \Delta(x).$$

We write

$$P^i = \sum_y P(y) \otimes_k M(y)^i,$$

where $M(y)^i = \oplus_j M(y)^i_j$ is a graded $\mathbf{k}$-vector space.

By construction $M(y)^i \cong \mathrm{Ext}^i(\Delta(x), L(y))^*$, and so the coefficient of t^n in the (x,z)-entry of the matrix identity $K(-t)D(t) = I$ translates into the equation

$$\sum_y \sum_{i=0}^{n} (-1)^i \mathrm{Homgr}(P(y), \nabla(z)\langle i\rangle) \otimes M(y)^{n-i} = 0$$

in the Grothendieck group of the category of graded $\mathbf{k}$-vector spaces, for all $n > 0$.

By minimality of the resolution, $M(y)^i_j \neq 0$ only if $j \geq i$. We prove that $D(t)K(-t) = I$ implies that the resolution is linear, *i.e.* that $M(y)^i = M(y)^i_i$ for all y and i, by induction on i. It is obvious for $i = 0$, so let $n > 0$. The resolution is an exact sequence with terms in the subcategory of graded modules with good filtrations, so we obtain via the functor $\mathrm{Homgr}(-, \nabla(z)\langle n\rangle)$ an exact sequence of graded vector spaces

$$\bigoplus_y \mathrm{Homgr}(P(y), \nabla(z)) \otimes M(y)^n_n \to$$

$$\bigoplus_y \mathrm{Homgr}(P(y), \nabla(z)\langle 1\rangle) \otimes M(y)^{n-1} \to \ldots \to \bigoplus_y \mathrm{Homgr}(P(y), \nabla(z)\langle n\rangle) \otimes M(y)^0,$$

where we have used the induction hypothesis that $M(y)^i = M(y)^i_i$ for $i < n$. Passing to the Gronthendieck group and comparing with the equation above, we obtain $M(y)^n = M(y)^n_n$ for all y, as desired.

The dual statement that $D'(t)K'(-t) = I$ implies the existence of linear injective resolutions of costandard modules is proved similarly. □

We denote by $\langle k\rangle$ the shift functor of A-grmod, that is $(M\langle k\rangle)_j := M_{j-k}$. Our convention here is identical with Beilinson-Ginzburg-Soergel [BGS96] and is opposite to Ariki [Ari09] in which notation $[-k]$ is ours. By the work of Brundan-Kleshchev [BK09] from the advantages of the Khovanov-Lauda-Rouquier quiver Hecke algebra [KL09],[Rou08a] the Iwahori-Hecke algebra $\mathcal{H}_r := \mathcal{H}_r(q; Q)$ of type A_{r-1} with parameter q whose quantum characteristic is e is known to be graded. (Here, $\mathcal{H}_r$ is independent of Q.) The grading structure on $\mathcal{H}_r$ is well-explained in [Ari09].

Since our note heavily depends on Ariki's result in [Ari09] and we selfishly assume that the readers are familiar with Ariki's paper, here, before any explanation we shall make a list of major differences between his notation and ours:

- Our $\mathcal{H}_r$ is the opposite ring of Ariki's $\mathcal{H}_r$ since we work mainly in the left module category.
- Ariki's coefficient field F is our $\mathbf{k}$.
- Ariki's graded right Young module $Y(\lambda)$ is our graded left Young module Y^λ, up to the above opposite ring issue.
- Ariki's shift functor $(-)[k]$ for graded modules is our $(-)\langle -k\rangle$.
- Ariki's $\mathcal{S}_{d,m}$ is our $\mathcal{S}(d,m)$.
- Ariki's $e^+_{\lambda\mu}(v)$ is our $d_{\lambda^t\mu^t}(v)$ where t indicates the transpose.
- Ariki's $H^0(\lambda)$ (resp. $W(\lambda)$) is our $\nabla_n(\lambda)$ (resp. $\Delta_n(\lambda)$) for $\mathcal{S}(n,r)$ for $r = |\lambda|$.

5.1. **Anti-automorphism and contravariant duality.** Let $(-)^\circ : \mathcal{H}_n\text{-grmod} \to \text{grmod-}\mathcal{H}_n$ be the contravariant duality functor as in [Ari09, Definition 2.4]. Then,

$$M\langle i\rangle^\circ = M^\circ\langle i\rangle \text{ for any } i. \tag{2}$$

Let $\sharp$ be the involutory anti-automorphism of $\mathcal{H}_n$ at [Ari09, Definition 2.6]. As in [Ari09, Definition 2.7], for $M \in \mathcal{H}_n$-grmod, define $M^{-\sharp} \in \text{grmod-}\mathcal{H}_n$ by

$$M^{-\sharp} = \bigoplus_{k\in\mathbf{Z}} M^{-\sharp}_k, \; M^{-\sharp}_k := (M^\sharp)_{-k}.$$

Then,

$$M\langle i\rangle^{-\sharp} = M^{-\sharp}\langle -i\rangle \text{ for any } i. \tag{3}$$

Let $(-)^*$ be the composition $((-)^\circ)^{-\sharp} : \mathcal{H}_n\text{-grmod} \to \mathcal{H}_n\text{-grmod}$, which is contravariant. By (2) and (3), we have

$$M\langle i\rangle^* = M^*\langle -i\rangle \text{ for any } i. \tag{4}$$

We call a graded module U selfdual if $U \cong U^*$ as graded modules.

Let Y^λ be the graded Young module in [Ari09], which is the opposite object of $Y(\lambda)$ in Ariki's notation.

The argument in [Ari09, p.18,l.11-18] says that

Lemma 22. *Ariki's graded Young module Y^λ is selfdual as a graded module.*

If $V \cong V^*$ as graded modules, by (4) we have

$$\mathrm{Hom}(V, V\langle k\rangle) \cong \mathrm{Hom}(V^*, V^*\langle k\rangle) = \mathrm{Hom}(V^*\langle -k\rangle, V^*) = \mathrm{Hom}((V\langle k\rangle)^*, V^*). \tag{5}$$

Let M be Ariki's graded q-tensor space module, which is thought of a graded version of the r-fold q-tensor space $V_n^{\otimes r} = \bigoplus_{\lambda=(\lambda_1,\cdots,\lambda_n)\models r} M^\lambda$ of n-dimensional vector representation V_n where M^λ is the q-permutation module on the Hecke subalgebra corresponding to a parabolic subgroup $\mathfrak{S}_\lambda$. Ariki gives a grading structure on M by using his graded Young modules, and he define his graded q-Schur algebra $\mathcal{S}(n,r)$ by

$$\mathcal{S}(n,r) := \bigoplus_{k\in\mathbf{Z}} \mathrm{Hom}_{\mathcal{H}_r}(M, M\langle k\rangle).$$

Note that we shall make a special section 7 and there Lemma 26 and Corollary 28 ensure that Ariki's assumptions $n \geq r$ and $e \geq 4$ for his grading can be removable in a suitable sense. Moreover, when we forget the grading, we have that $\mathcal{S}(r,r) = \mathcal{S}_r(q;Q_1) = \mathcal{S}_r(q)$ under the notation § 3.1 and § 4.

Since we have a canonical anti-isomorphism $J = \sum_k J_k$ of $\bigoplus_k \mathrm{Hom}(M, M\langle k\rangle)$ onto $\bigoplus_k \mathrm{Hom}((M\langle k\rangle)^*, M^*)$ induced by $*$, by (5) and taking $V := M$, we have an anti-automorphism of $\mathcal{S}(n,r)$. (we still denote it by J).

For $X \in \mathcal{S}(n,r)$-grmod, we give a left graded module structure on $X^\circ := \bigoplus_{k\in\mathbf{Z}}(X^\circ)_k$ where $(X^\circ)_k = \mathrm{Hom}_{\mathbf{k}}(X_k, \mathbf{k})$ as follows: $f \in X^\circ$, $s \in \mathcal{S}(n,r)$ and $x \in X$, $(s\cdot f)(x) := f(J(s)\cdot x)$. In this way, by twisting X° by J we obtain the contravariant functor $(-)^* : \mathcal{S}(n,r)\text{-grmod} \to \mathcal{S}(n,r)\text{-grmod}$.

By the property (4), we have that

$$X\langle i\rangle^* = X^*\langle -i\rangle \text{ for any } i \text{ and any } X \in \mathcal{S}(n,r)\text{-grmod}. \tag{6}$$

One can observe that the duality $*$ fixes any graded $\mathcal{S}(n,r)$-module $L_n(\lambda)$, up to isomorphism.

6. Kazhdan-Lusztig polynomials, Quantum GL_n and Koszulity

Let $G = G(n)$ be the quantum general linear group with semisimple rank n over a field $\mathbf{k}$ [Don98]. We write $X^+(n) = \mathbf{N}_0\varpi_1 \oplus \cdots \oplus \mathbf{N}_0\varpi_{n-1} \oplus \mathbf{Z}\varpi_n$ for the dominant weights for GL_n where ϖ_i is the i-th fundamental weight $\epsilon_1 + \cdots + \epsilon_i$. Let $\Lambda(n,r) := \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{N}_0^n \mid \sum_i \lambda_i = r, \lambda_j \geq \lambda_{j+1} \text{ for } j = 1, \ldots n-1\} \subset X^+(n)$ with $\lambda = \sum_i \lambda_i \epsilon_i$. This is the set of dominant weights for $\mathrm{End}_{\mathcal{H}_r}(V_n^{\otimes r})$, called, polynomial weights, where $V_n^{\otimes}$ is the vector representation of the quantum general linear group $G(n)$ with parameter q and rank n. [Don98]. Let $\Lambda(r) := \Lambda(r,r)$.

The main aim of this section is to obtain the assumption of our "square root" (Proposition 21) of the Beilinson-Ginzburg-Soergel numerical Koszul duality criterion [BGS96], which is beautifully equipped in the orthogonality relation of parabolic Kazhdan-Lusztig polynomials and their inverse.

In this section, we certainly need to take the Lusztig conjecture on quantum groups at roots of unity, solved by Kazhdan-Lusztig [KL93, KL94a, KL94b] and Kashiwara-Tanisaki[KT95], and the equations (7) into account for our aim. So, we need to assume that $\ell = 0$ or Lusztig conjecture for algebraic groups in positive characteristic (or ℓ is huge [AJS94]).

For simplicity, we only work in characteristic zero $\ell = 0$ in this note.

By [Don96, § 4], we have that

$$\text{for any } M, N \in \mathcal{S}(n,r)\text{-mod and any } k \geq 0, \mathrm{Ext}^i_{G(n)}(M,N) \cong \mathrm{Ext}_{\mathcal{S}(n,r)}(M,N). \tag{7}$$

For $\lambda, \mu \in \Lambda(n,r)$, we consider Vogan polynomials $p_{\mu\lambda}(v) \in \mathbf{N}_0[v]$:

$$p_{\mu\lambda}(v) := \sum_{i \geq 0} \dim \operatorname{Ext}^i_{\mathcal{S}(n,r)}(L_n(\lambda), \nabla_n(\mu))v^i. \tag{8}$$

And, as in [Don98],

$$\operatorname{Ext}^i_{S(n,r)}(L_n(\lambda), \nabla_n(\mu)) \cong \operatorname{Ext}^i_{S(n,r)}(\Delta_n(\mu), L_n(\lambda)).$$

Here, the extension groups for above three equations are taken in the ungraded module category.

In this note we use Soergel's convention [Soe97] for parabolic Kazhdan-Lusztig polynomials (for example, the quadratic relation for the affine Hecke algebra of type GL_r is taken as $(T_i + v)(T_i - v^{-1}) = 0$). Thanks to the works of Vogan, Gabbar, Joseph, Andersen[And86, And83], Cline-Parshall-Scott[CPS92, CPS93] and Kaneda[Kan87], [Jan03, C.2 Proposition] says that the Vogan polynomial above is a parabolic Kazhdan-Lusztig polynomial. The statement on [Jan03, C.2 Proposition] is not about quantum groups, but the statement on the quantum groups in characteristic zero is easier than in positive characteristic and the proof is completely parallel to the algebraic group case.

Let $(d_{\lambda\mu}(v))_{\lambda\mu\in\Lambda(r,r)}$ be the Leclerc-Thibon v-deformed decomposition matrix of $\mathcal{S}(r,r)$ in the level 1 v-deformed Fock space over the affine quantum group $U_v(\hat{\mathfrak{sl}}_e)$ in [LT96].Here, our $d_{\lambda\mu}(v)$ is Ariki's $e^+_{\lambda^t\mu^t}(v)$ (which is not Ariki's $d_{\lambda\mu}(v)$). Let $(c_{\mu\lambda}(v))_{\lambda\mu\in\Lambda(r,r)}$ be the inverse matrix of $(d_{\lambda\mu}(v))_{\lambda\mu\in\Lambda(r,r)}$. By the work of Varagnolo and Vasserot [VV99] the Leclerc-Thibon conjecture [LT96] is affirmatively settled and $d_{\lambda\mu}(v)$ is known to be a parabolic Kazhdan-Lusztig polynomial in Soergel's convention.

We have

$$c_{\mu^t\lambda^t}(-t) = p_{\mu\lambda}(t). \tag{9}$$

Here, for $\nu \in \Lambda(n,r)$ ν^t means the conjugate or the transposed partition of ν.

In [Ari09], Ariki considers the graded multiplicity $[\Delta_n(\lambda) : L_n(\mu)\langle k\rangle]$ for $\lambda, \mu \in \Lambda(n,r)$ that is equal to $[W(\lambda) : L(\mu)[-k]]$ in Ariki's notation . And, Ariki obtained under the conditions $e \geq 4$ and $n \geq r$ that

$$d_{\lambda^t\mu^t}(v) = \sum_{k\in\mathbf{Z}} (\Delta_n(\lambda) : L_n(\mu)\langle k\rangle)v^k. \tag{10}$$

Before going further, we need to mention again that we may remove Ariki's assumptions $e \geq 4$ and $n \geq r$ in a suitable sense. This will be done at § 7. So, we assume that we have a grading structure on all q-Schur algebras which are consistent with parabolic Kazhdan-Lusztig polynomials. Namely, we will have the equation (10) even for the cases $e = 2, 3$.

Now, we also know that $\dim \operatorname{Homgr}(\Delta_n(\lambda), I_n(\mu)\langle k\rangle) = [\Delta_n(\lambda) : L_n(\mu)\langle k\rangle]$. Since $\Lambda(n,r)$ is a cosaturated subset of $\Lambda(r,r)$ (see [Don98, Appendix] for the definition of cosaturation) on the highest weight structure for $\mathcal{S}(n,r)$, by putting $A := \mathcal{S}(n,r)$ under the notation at Proposition 21, we have that

$$D(t)K(-t) = I \text{ and } D'(t)K'(-t) = I.$$

So, by Proposition 21, we have

Theorem 23. *The graded q-Schur algebra $\mathcal{S}(n,r)$ is standard Koszul. And, its Koszul dual is a quasihereditary.* [5]

Remark 24. *In Corollary 28, small $n < r$ will be treated. So, the theorem above is also valid for small $n \leq r$ and $e = 2, 3$.*

[5] At the end of [CPS93, § 3], Cline, Parshall and Scott wrote that the Koszulity above can be shown if we can assume the existence of a graded Kazhdan-Lusztig theory. And, they also left some comments like that if we start from a graded quasihereditary algebra A with a graded Kazhdan-Lusztig theory, then they wrote that $A^!$ is a quasihereditary with a graded Kazhdan-Lusztig theory and its details will appear elsewhere. In an Oberwolfach meeting 2006, when the authors asked L. Scott on the Koszulity, he told the authors that he believes that q-Schur algebras are Koszul in characteristic zero or in the case that Lusztig's conjecture holds (perhaps, before the authors started studying any representation theory).

7. TRUNCATION AND THE CASES $e = 2, 3$

We supply some missing proofs. In [Ari09], Ariki succeed in giving a grading structure on q-Schur algebra except the cases $e = 2, 3$, like q is -1 or a primitive 3-rd root of unity. The reason that Ariki needs to assume $e \neq 2, 3$ is that he uses an adjoint functor of the Schur functor and it behaves bad in the case $e = 2, 3$. In this section, we extend Ariki's result to those excluded cases by using Ariki's result for big $e > 3$ and the runner removal Morita equivalence in [CM10].

To do this, we need the following well known basic lemma.

Lemma 25. *Let A and B are Morita equivalent finite dimensional algebras. Suppose that A is a graded algebra. Then, we can give a grading on B such that B is graded Morita equivalent to A.*

Note that for any $n \geq r$, $\mathcal{S}(n, r)$ is Morita equivalent to $\mathcal{S}(r, r)$. So, we may always assume that $n \leq r$ even for the treatment of grading thanks to Lemma 25.

Suppose that $\mathcal{S}(r, r)$ is graded for a grading fixed M.

We take a cosaturated subset Λ of $\Lambda(r, r)$ and take an idempotent 1_Λ of $\mathcal{S}(r, r)$ corresponding to the sum of projective modules $P_r(\lambda), \lambda \in \Lambda$. Here, $P_r(\lambda)$, namely, the idempotent corresponding to Y^λ must be carefully chosen since we are giving the grading on M via homogeneous Y^λ's in Ariki's construction of graded q-tensor space module. (This is an invisible choice: we don't know any simultaneous canonical definition of grading structure on M at the moment since finding a closed formula for the multiplicity of Y^λ in M itself is a very difficult problem, which surely depends on the characteristic of $\mathbf{k}$. See [Don98, Appendix] for the definition of cosaturation.) We can deduce that $1_\Lambda \mathcal{S}(r, r) 1_\Lambda$ is a graded quasihereditary algebra. In particular, since $\Lambda(n, r), n \leq r$ is a cosaturated subset of $\Lambda(r, r)$ for any $n \leq r$, we know that

Lemma 26. *for any n, $\mathcal{S}(n, r)$ is graded quasihereditary*

and we have an equality on graded decomposition numbers $[\Delta_N(\lambda) : L_N(\mu)\langle k \rangle] = [\Delta_n(\lambda) : L_n(\mu)\langle k \rangle]$ for any $\lambda, \mu \in \Lambda(n, r)$. (The extension group comparison for $\mathcal{S}(N, r)$ and $\mathcal{S}(n, r)$ is well known [Don98, Appendix] by an aid of the Grothendieck spectral sequence.)

For an algebra A, we denote by Λ_A for the set of indices of a complete set of simple A-modules.

We take another quantum parameter $q' \in \mathbf{k}$. We denote the quantum characteristic of q' by e'. We write $\Delta'_n(\lambda)$ (resp. $\Delta'_n(\lambda)$, $P'_n(\lambda)$) for the simple (resp. Weyl, projective indecomposable) module over a q'-Schur algebra of semisimple rank n corresponding to a highest weight λ.

In [CM10, Theorem 2] the following was shown (slightly weaker statement):

Theorem 27. *Suppose that $e' > e$.*

For any abelian block algebra A of q-Schur algebra of semisimple rank n with e-weight $w > 0$, we have that there exists a block algebra B of q'-Schur algebra of semisimple rank n with e'-weight w such that

- *there exists an equivalence* $\mathsf{F} : A\text{-mod} \cong B\text{-mod}$,
- *there exists an bijection* $(-)^+ : \Lambda_A \cong \Lambda_B$,
- $\mathsf{F}L_n(\lambda) \cong L'_n(\lambda^+)$, $\mathsf{F}\Delta_n(\lambda) \cong \Delta'_n(\lambda^+)$ *and* $\mathsf{F}P_n(\lambda) \cong P'_n(\lambda^+)$.

In particular,

$$\mathrm{End}_{\mathcal{H}(A)}\left(\bigoplus_{\lambda \in \Lambda_A} Y^\lambda\right) \cong \mathrm{End}_{\mathcal{H}(B)}\left(\bigoplus_{\lambda^+ \in \Lambda_B} Y^{\lambda^+}\right) \tag{11}$$

where $\mathcal{H}(C)$ is the Hecke algebra of $\mathfrak{S}_{|\nu|}$ for some $\nu \in \Lambda_C$ with the identical parameter with C's and Y^ν is the Young module over $\mathcal{H}(C)$ corresponding to ν.

By applying Ariki's results in [Ari09] to RHS of (11) and by Lemma 25, we have

Corollary 28. *All abelian q-Schur algebra blocks are graded. In particular, if in addition $\ell = 0$, then all q-Schur algebras are graded, quasihereditary, standard Koszul and their graded decomposition numbers are coincident with Lascoux-Leclerc-Thibon v-decomposition numbers in the level* 1 *v-deformed Fock space over* $U_v(\hat{\mathfrak{sl}}_e)$.

8. Some more evidences

In this section, we pick up more evidences related to Conjecture 6. The first numerical evidence is a consistency between orthogonality relations of "square root" Proposition 21 of numerical Koszul duality criterion [BGS96] and of canonical basis ((doubly) parabolic Kazhdan-Lusztig polynomials) [Ugl00]. In this subsection, we shall find something different from this.

8.1. Theorem 27 [CM10, Theorem 2] says that one can change the rank or the quantum characteristic e for q-Schur algebras, more precisely, any q-Schur algebra module category with e is a quotient category as well as a subcategory of a q-Schur algebra module category with bigger $e' > e$. Taking this point of view and Conjecture 6 into account, we can guess that we may change the level l as follows:

Conjecture 29. *Fix $m > 0$. Let W_k be the complex reflection group $(\mathbf{Z}/k\mathbf{Z}) \wr \mathfrak{S}_m$ of type $G(k, 1, m)$. We have two embeddings*

$$r : \mathrm{Irr}\, W_l \ni \chi^{\lambda} \mapsto \chi^{(\lambda,0)} \in \mathrm{Irr}\, W_{l+1} \text{ and } s : \mathrm{Irr}\, W_l \ni \chi^{\lambda} \mapsto \chi^{(0,\lambda)} \in \mathrm{Irr}\, W_{l+1}.$$

In the same way, we have associated embeddings for $\boldsymbol{\kappa}^{\circ}$ in $(l+1)$-tuples, and the same for $\boldsymbol{\kappa}$ by preserving the extra entry h_1. Take $f \in \{r, s\}$.

(1) *If $(f(\mathrm{Irr}\, W_l), \boldsymbol{\kappa})$ is a saturated subset of $(\mathrm{Irr}\, W_{l+1}, f(\boldsymbol{\kappa}))$, then $\mathcal{O}_{\boldsymbol{\kappa}}(W_l)$ is a subcategory of $\mathcal{O}_{f(\boldsymbol{\kappa})}(W_{l+1})$.*
(2) *If $(f(\mathrm{Irr}\, W_l), \boldsymbol{\kappa})$ is a cosaturated subset of $(\mathrm{Irr}\, W_{l+1}, f(\boldsymbol{\kappa}))$, then $\mathcal{O}_{\boldsymbol{\kappa}}(W_l)$ is a quotient category of $\mathcal{O}_{f(\boldsymbol{\kappa})}(W_{l+1})$.*

Remark 30. (1) *See* [Don98, Appendix] *for the definition of saturation and cosaturation.*
(2) *For cyclotomic q-Schur algebras in characteristic* 0 *(which should be special cases of categories $\mathcal{O}$'s of RatDAHA in Rouquier's view point), we can prove this conjecture.*
(3) *From the points of views § 3.3,3.4, there should be two embeddings of canonical bases/global bases of level l into the ones of level $l+1$. K. Iijima* [Iij11, Theorem A & B] *proved this evidence by purely combinatorics.*

Acknowledgments. *The project of this paper started in 2008 at Tambara Workshop on Algebras in Lie Theory organized by S. Ariki. The authors are grateful for his invitation. In 2010, an opportunity for giving a talk at The 10th International Conference by Graduate School of Mathematics, Nagoya University was given to the second author by S. Ariki as a nightwatchman. The authors would like to thank him for granting these opportunities and for his interest in the present note and his patience and encouragement in its development.*

The authors would like to thank I. Gordon, S. Griffeth, T. Kuwabara, V. Mazorchuk, H. Nakajima, R. Rouquier, P. Shan, T. Suzuki and E.Vasserot for fruitful discussions.

References

[ÁDL03] I. Ágoston, V. Dlab, and E. Lukács. Quasi-hereditary extension algebras. *Algebr. Represent. Theory*, 6(1):97–117, 2003.
[AJS94] H. H. Andersen, J. C. Jantzen, and W. Soergel. Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p. *Astérisque*, (220):321, 1994.
[And83] H. H. Andersen. Filtrations of cohomology modules for Chevalley groups. *Ann. Sci. École Norm. Sup. (4)*, 16(4):495–528 (1984), 1983.
[And86] H. H. Andersen. An inversion formula for the Kazhdan-Lusztig polynomials for affine Weyl groups. *Adv. in Math.*, 60(2):125–153, 1986.
[Ari96] S. Ariki. On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$. *J. Math. Kyoto Univ.*, 36:789–808, 1996.
[Ari08] S. Ariki. Finite dimensional Hecke algebras. In *Trends in representation theory of algebras and related topics*, EMS Ser. Congr. Rep., pages 1–48. Eur. Math. Soc., Zürich, 2008.
[Ari09] S. Ariki. Graded q-Schur algebras, 2009.
[BE09] R. Bezrukavnikov and P. Etingof. Parabolic induction and restriction functors for rational Cherednik algebras. *Selecta Math. (N.S.)*, 14(3-4):397–425, 2009.
[BGS96] A. Beilinson, V. Ginzburg, and W. Soergel. Koszul duality patterns in representation theory. *J. Amer. Math. Soc.*, 9(2):473–527, 1996.
[BK09] J. Brundan and A. Kleshchev. Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras. *Invent. Math.*, 178(3):451–484, 2009.
[CK02] J. Chuang and R. Kessar. Symmetric groups, wreath products, Morita equivalences, and Broué's abelian defect group conjecture. *Bull. London Math. Soc.*, 34:174–184, 2002.

[CM10] J. Chuang and H. Miyachi. Runner removal Morita equivalences. In T. Shoji et al, editor, *Representation Theory of Algebraic Groups and Quantum Groups*, volume 284 of *Progress in Mathematics*, pages 55–79. Birkhäuser, 2010. ISBN: 978-0-8176-4696-7.

[CPS92] E. Cline, B. Parshall, and L. Scott. Infinitesimal Kazhdan-Lusztig theories. In *Kazhdan-Lusztig theory and related topics (Chicago, IL, 1989)*, volume 139 of *Contemp. Math.*, pages 43–73. Amer. Math. Soc., Providence, RI, 1992.

[CPS93] E. Cline, B. Parshall, and L. Scott. Abstract Kazhdan-Lusztig theories. *Tohoku Math. J. (2)*, 45(4):511–534, 1993.

[CR08] J. Chuang and R. Rouquier. Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification. *Ann. of Math. (2)*, 167(1):245–298, 2008.

[DG10] C. Dunkl and S. Griffeth. Generalized Jack polynomials and the representation theory of rational Cherednik algebras. *Selecta Math. (N.S.)*, 16(4):791–818, 2010.

[DJM98] R. Dipper, G. James, and A. Mathas. Cyclotomic q-Schur algebras. *Math. Z.*, 229(3):385–416, 1998.

[DM02] R. Dipper and A. Mathas. Morita equivalences of Ariki-Koike algebras. *Math. Z.*, 240(3):579–610, 2002.

[Don96] S. Donkin. Standard homological properties for quantum GL_n. *J. Algebra*, 181(1):235–266, 1996.

[Don98] S. Donkin. *The q-Schur Algebra*. LMS Lecture Note Series 253. Cambridge Univ. Press, 1998.

[FLO+99] O. Foda, B. Leclerc, M. Okado, J.-Y. Thibon, and T. A. Welsh. Branching functions of $A^{(1)}_{n-1}$ and Jantzen-Seitz problem for Ariki-Koike algebras. *Adv. Math.*, 141(2):322–365, 1999.

[Fre82] I. B. Frenkel. Representations of affine Lie algebras, Hecke modular forms and Korteweg-de Vries type equations. In *Lie algebras and related topics (New Brunswick, N.J., 1981)*, volume 933 of *Lecture Notes in Math.*, pages 71–110. Springer, Berlin, 1982.

[GGOR03] V. Ginzburg, N. Guay, E. Opdam, and R. Rouquier. On the category $\mathcal{O}$ for rational Cherednik algebras. *Invent. Math.*, 154(3):617–651, 2003.

[Iij11] K. Iijima. A comparison of q-decomposition numbers in the q-deformed Fock spaces of higher levels, 2011.

[Jan03] J.C. Jantzen. *Representations of Algebraic Groups*, volume 107 of *Mathematical Surveys and Monographs*. Amer. Math. Soc., second edition, 2003.

[JLM06] G. James, S. Lyle, and A. Mathas. Rouquier blocks. *Math. Z.*, 252(3):511–531, 2006.

[JMMO91] M. Jimbo, K. C. Misra, T. Miwa, and M. Okado. Combinatorics of representations of $U_q(\widehat{\mathfrak{sl}}(n))$ at $q = 0$. *Comm. Math. Phys.*, 136(3):543–566, 1991.

[Kac90] V. G. Kac. *Infinite dimensional Lie algebras*. Cambridge University Press, 3rd edition, 1990.

[Kan87] M. Kaneda. On the inverse Kazhdan-Lusztig polynomials for affine Weyl groups. *J. Reine Angew. Math.*, 381:116–135, 1987.

[KL93] D. Kazhdan and G. Lusztig. Tensor structures arising from affine Lie algebras. I, II. *J. Amer. Math. Soc.*, 6(4):905–947, 949–1011, 1993.

[KL94a] D. Kazhdan and G. Lusztig. Tensor structures arising from affine Lie algebras. III. *J. Amer. Math. Soc.*, 7(2):335–381, 1994.

[KL94b] D. Kazhdan and G. Lusztig. Tensor structures arising from affine Lie algebras. IV. *J. Amer. Math. Soc.*, 7(2):383–453, 1994.

[KL09] M. Khovanov and A. D. Lauda. A diagrammatic approach to categorification of quantum groups. I. *Represent. Theory*, 13:309–347, 2009.

[KT95] M. Kashiwara and T. Tanisaki. Kazhdan-Lusztig conjecture for affine Lie algebras with negative level. *Duke Math. J.*, 77:21–61, 1995.

[LLT96] A. Lascoux, B Leclerc, and J.-Y. Thibon. Hecke algebras at roots of unity and crystal bases of quantum affine algebras. *Commun. Math. Physics*, 181:205–263, 1996.

[LM07] Sinéad Lyle and Andrew Mathas. Blocks of cyclotomic Hecke algebras. *Adv. Math.*, 216(2):854–878, 2007.

[LT96] B. Leclerc and J.-Y. Thibon. Canonical bases of q-deformed Fock spaces. *Internat. Math. Res. Notices*, (9):447–456, 1996.

[Mat03] A. Mathas. Tilting modules for cyclotomic Schur algebras. *J. Reine Angew. Math.*, 562:137–169, 2003.

[Maz09] V. Mazorchuk. Applications of the category of linear complexes of tilting modules associated with the category $\mathcal{O}$. *Algebr. Represent. Theory*, 12(6):489–512, 2009.

[Maz10a] V. Mazorchuk. Koszul duality for stratified algebras. I. Balanced quasi-hereditary algebras. *Manuscripta Math.*, 131(1-2):1–10, 2010.

[Maz10b] V. Mazorchuk. Koszul duality for stratified algebras II. Standardly stratified algebras. *J. Aust. Math. Soc.*, 89(1):23–49, 2010.

[Miy01] H. Miyachi. *Unipotent Blocks of Finite General Linear Groups in Non-defining Characteristic*. PhD thesis, Chiba univ., March. 2001.

[Rou05] R. Rouquier. Representations of rational Cherednik algebras. In *Infinite-dimensional aspects of representation theory and applications*, volume 392 of *Contemp. Math.*, pages 103–131. Amer. Math. Soc., Providence, RI, 2005.

[Rou06] R. Rouquier. Derived equivalences and finite dimensional algebras. In *International Congress of Mathematicians. Vol. II*, pages 191–221. Eur. Math. Soc., Zürich, 2006.

[Rou08a] R. Rouquier. 2-Kac-Moody algebras, 2008.
[Rou08b] R. Rouquier. q-Schur algebras and complex reflection groups. *Mosc. Math. J.*, 8(1):119–158, 184, 2008.
[Sha09] P. Shan. Crystals of Fock spaces and cyclotomic rational double affine Hecke algebras, 2009.
[Soe97] W. Soergel. Kazhdan-Lusztig polynomials and a combinatoric for tilting modules. *Representation Theory*, 1:83–114, 1997.
[Tur02] W. Turner. Equivalent blocks of finite general linear groups in non-describing characteristic. *J. Algebra*, 247(1):244–267, 2002.
[Ugl00] D. Uglov. Canonical bases of higher-level q-deformed Fock spaces and Kazhdan-Lusztig polynomials. In *Physical combinatorics (Kyoto, 1999)*, volume 191 of *Progr. Math.*, pages 249–299. Birkhäuser Boston, Boston, MA, 2000.
[Val06] R. Vale. *On category $\mathcal{O}$ for the rational Cherednik algebra of the complex reflection group* $(\mathbb{Z}/\ell\mathbb{Z}) \wr S_n$. PhD thesis, University of Glasgow, 2006.
[VV99] M. Varagnolo and E. Vasserot. On the decomposition matrices of the quantized Schur algebra. *Duke Math. J.*, 100(2):269–297, 1999.
[Yvo06] X. Yvonne. A conjecture for q-decomposition matrices of cyclotomic v-Schur algebras. *J. Algebra*, 304(1):419–456, 2006.

(J. Chuang) Centre for Mathematical Science, City University, Northampton Square, London EC1V 0HB United Kingdom.
E-mail address: `j.chuang@city.ac.uk`

(H. Miyachi) Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya Aichi, 464-8602 Japan
E-mail address: `miyachi@math.nagoya-u.ac.jp`