RUNNER REMOVAL MORITA EQUIVALENCES

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Dedicated to Ken-ichi Shinoda and Toshiaki Shoji on the occasions of their 60-th birthdays

CONTENTS

1. INTRODUCTION

Gordon James and Andrew Mathas [JM02] showed that certain decomposition numbers of Iwahori-Hecke algebras of symmetric groups and of the associated q-Schur algebras at different complex roots of unity are equal. The purpose of this paper is to interpret these equalities as consequences of Morita equivalences.

Our results extend to q-Schur algebras over fields of positive characteristic, as long as the parameter q is in the prime subfield and we restrict to blocks 'of abelian defect'. The idea of the proof is to deal first with certain distinguished blocks, the *Rouquier blocks*, and then use special derived equivalences, called *perverse equivalences*, that arise from $\mathfrak{sl}_2$-categorifications, to make a link to other blocks. This second part is an inductive step, which is based on a result proven in a forthcomming paper [CRa].

For the Rouquier blocks, the case of ground fields of characteristic 0 is more difficult, contrary to the usual expectation. For the only known method to prove structure theorems for these blocks is via local representation theory of finite groups. Over fields of positive characteristic, we require finite general linear groups – hence the insistence that the parameter q lies in the ground field. To obtain the result in characteristic 0 we use a lifting argument; a more conceptual method would be desirable.

We formulate our main result in such a way as to also allow comparison between Schur algebras at different roots of unity of the same order. Therefore as a bonus we deduce that the q-Schur algebra $\mathcal{S}_{\Bbbk,q}(r)$ over a field $\Bbbk$ is isomorphic to the q'-Schur algebra $\mathcal{S}_{\Bbbk,q'}(r)$ *as* $\Bbbk$*-algebras* if $\Bbbk$ has characteristic 0 and q and q' have the same multiplicative order.

The relevant combinatorial operations used by James and Mathas, the addition of runners on abaci and the reverse procedure of 'runner removal', have very natural interpretations in terms of alcove geometry. This is very consistent with F. Goodman's remark on James-Mathas's results. We aren't able to exploit this point of view in our proof, because for the Rouquier blocks the combinatorics of partitions and abaci seem better suited. Nevertheless, in the final section we take a stab at a possible analogue of the main theorem for quantized enveloping algebras at complex roots of unity.

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2. Notations and Background.

2.1. Partitions and Fock space. We associate to any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)$ its Young diagram $\{(i,j) \mid 1 \leq i \leq \lambda_j\} \in \mathbb{N} \times \mathbb{N}$. We denote by λ^{tr} the *conjugate* partition; its Young diagram is obtained from that of λ by interchanging the coordinates i and j.

Fix an integer $e \geq 2$. Given $a \in \{0, \ldots, e-1\}$, and partitions λ and μ we write $\lambda \to_a \mu$ if the Young diagram of μ is obtained from that of λ by adding an extra (i,j) such that $j - i \equiv a$ modulo e. Let $\mathcal{F} = \oplus_\lambda \mathbb{C}\lambda$ be a complex vector space with basis indexed by the set of all partitions of all nonnegative integers. We define linear operators $\mathfrak{e}_0, \ldots, \mathfrak{e}_{e-1}, \mathfrak{f}_0, \ldots, \mathfrak{f}_{e-1}$ on

$\mathcal{F}$ by

$$\mathfrak{e}_a\lambda = \sum_{\mu \to_a \lambda} \mu \qquad \text{and} \qquad \mathfrak{f}_a\lambda = \sum_{\lambda \to_a \mu} \mu.$$

These locally nilpotent endomorphisms extend to an action of the Kac-Moody algebra $\widehat{\mathfrak{sl}}_e$. Put $s_a := \exp(-\mathfrak{f}_a)\exp(\mathfrak{e}_a)\exp(-\mathfrak{f}_a)$, an automorphism of $\mathcal{F}$. Then for any partition λ we have $s_a\lambda = \pm\sigma_a(\lambda)$ for some partition $\sigma_a(\lambda)$. The permutations $\sigma_0, \ldots, \sigma_{e-1}$ define an action of the affine Weyl group on the set of partitions.

Following James, consider an abacus with e half-infinite vertical runners, labelled $\rho_0, \ldots, \rho_{e-1}$ from left to right. On runner i we may put beads in positions labelled $i, i+e, i+2e, \ldots$ from top to bottom. Let $d \geq 0$. Any partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_d)$ with at most d nonzero parts may be represented by placing d beads at positions $\lambda_1 + d - 1, \lambda_2 + d - 2, \ldots, \lambda_d$. Note that λ may be recovered easily from the resulting configuration.

Sliding a bead one place up a runner into an unoccupied position corresponds to removing a rim e-hook from the Young diagram of λ. Repeating this process until no longer possible, say w times, we obtain a partition called the e-core of λ, and we say that λ has weight w. In going from the partition to its core, we remember how many times each bead on runner i is moved as a partition $\lambda^{(i)}$. The resulting e-tuple $\overline{\lambda} = \{\lambda^{(0)}, \ldots, \lambda^{(e-1)}\}$ is called the e-quotient of λ.

The actions of $\mathfrak{e}_a$, $\mathfrak{f}_a$, and σ_a described above have an easy interpretation on an abacus with d beads. Let $i \in \{0, \ldots, e-1\}$ such that $i \equiv a + d$ modulo e. Then $\lambda \to_a \mu$ if and only if (the abacus representation) of μ can be obtained from that of λ by moving a bead in ρ_{i-1} one position to the right into an unoccupied position in ρ_i. And σ_a acts by the interchanging the configuration of beads on runners ρ_{i-1} and ρ_i.

2.2. **Representations of Schur algebras.** Let $\Bbbk$ be a field and let $q \in \Bbbk^\times$. Let $e = e(q)$ be the least integer $i \geq 2$ such that $1 + q + \cdots + q^{i-1} = 0$ in $\Bbbk$. Hence e is the characteristic of $\Bbbk$ if $q = 1$, and is the multiplicative order of q otherwise. According to Kleshchev, e is sometimes called a 'quantum characteristic' for Hecke algebras.

The Schur algebra $\mathcal{S}_{\Bbbk,q}(d,r)$ is a quasihereditary algebra with simple modules $L(\lambda)$ indexed by the partitions of r with at most d parts, with respect to the dominance order. The Weyl module $\Delta(\lambda)$ has simple head isomorphic to $L(\lambda)$ and any composition factor of its radical is isomorphic to $L(\mu)$ for some $\mu \lhd \lambda$; it is characterized up to isomorphism the largest module by these properties.

If $d \geq \tilde{d}$ there exists a *Green's idempotent* f in $\mathcal{S}_{\Bbbk,q}(d,r)$ and a canonical isomorphism $f\mathcal{S}_{\Bbbk,q}(d,r)f \cong S_{\Bbbk,q}(\tilde{d},r)$ (see [Gre80]). The exact functor $\mathcal{S}_{\Bbbk,q}(d,r)\text{-mod} \to \mathcal{S}_{\Bbbk,q}(\tilde{d},r)\text{-mod} : M \mapsto fM$ sends $L(\lambda)$ to the corresponding simple module $\tilde{L}(\lambda)$ if λ has at most $\tilde{d}$ parts, and to 0 otherwise. Similarly

$f\Delta(\lambda) \cong \tilde{\Delta}(\lambda)$ if λ has at most $\tilde{d}$ parts. In particular if $\tilde{d} \geq r$ then the functor is an equivalence preserving labels.

Two simple modules $L(\lambda)$ and $L(\mu)$ are in the same block of $\mathcal{S}_{\Bbbk,q}(r) := \mathcal{S}_{\Bbbk,q}(r,r)$ if and only if λ and μ have the same e-core. Note that they are then necessarily of the same e-weight. So the blocks of $\mathcal{S}_{\Bbbk,q}(r)$, $r \geq 0$ are classified by pairs (τ, w) where τ is an e-core partition, i.e a partition which is its own e-core, and $w \geq 0$.

We may identify the Fock space $\mathcal{F}$ with the sum of complexified Grothendieck groups of module categories of q-Schur algebras:

$$\begin{aligned} \mathcal{F} &= \bigoplus_{r\geq 0} \mathbb{C} \otimes_{\mathbb{Z}} K(\mathcal{S}_{\Bbbk,q}(r)\text{-mod}) \\ \lambda &\leftrightarrow \quad [\Delta(\lambda)] \end{aligned}$$

Let B be a block and let $a \in \{0, \ldots, e-1\}$. Then s_a restricts to an isomorphism $K(B\text{-mod}) \xrightarrow{\sim} K(\dot{B}\text{-mod})$ for some block $\dot{B}$ of the same e-weight; we write $s_a B = \dot{B}$ and say that B and $\dot{B}$ form a *Scopes pair*. The induced action of the affine Weyl group on the set of all blocks of $\mathcal{S}_{\Bbbk,q}(r), r \geq 0$ is transitive on blocks of a fixed weight.

We remark that by 'Scopes pair' we mean an arbitrary $[w : k]$ pair of blocks (in the original terminology of Scopes [Sco91]); we do not place the restriction $w \leq k$.

Definition 1. *A block B with e-core τ and weight w of $\mathcal{S}_{\Bbbk,q}(r)$ is called a* Rouquier block *if there is some d such that in the d-bead abacus representation of τ, in any pair of adjacent runners there are at least $w-1$ more beads on the righthand runner. The e-core τ is called a* Rouquier core *relative to w.*

Clearly for all $w \geq 0$ there exist Rouquier blocks of weight w. So it is convenient to first prove that a statement about blocks is true for the Rouquier blocks, and then use the affine Weyl group action to show that it holds for all blocks.

3. The main result

3.1. **James-Mathas construction.** To state the main theorem we need to describe a map on partitions, due to James and Mathas. Let $2 \leq e \leq e'$, $d \geq 0$, and $\alpha \in \{0, \ldots, e\}$. Given a partition λ with at most d nonzero parts, add $e' - e$ empty runners between $\rho_{\alpha-1}$ and ρ_α in the abacus representation of λ on an e-runner abacus with d beads. The new configuration on an e'-runner abacus represents a partition which we call λ^+.

To take an example, let $e = 3$, $e' = 5$, $d = 3$ and $\alpha = 3$. Then the operation $\lambda \mapsto \lambda^+$ adds two empty runners on the right of the abacus. So, e.g., if $\lambda = (5, 4, 3)$ then $\lambda^+ = (9, 6, 5)$; see Figure 1 below.

James and Mathas showed that the map $\lambda \mapsto \lambda^+$ links the representation theory of Schur algebras at complex primitive e-th and e'-th roots of unity in

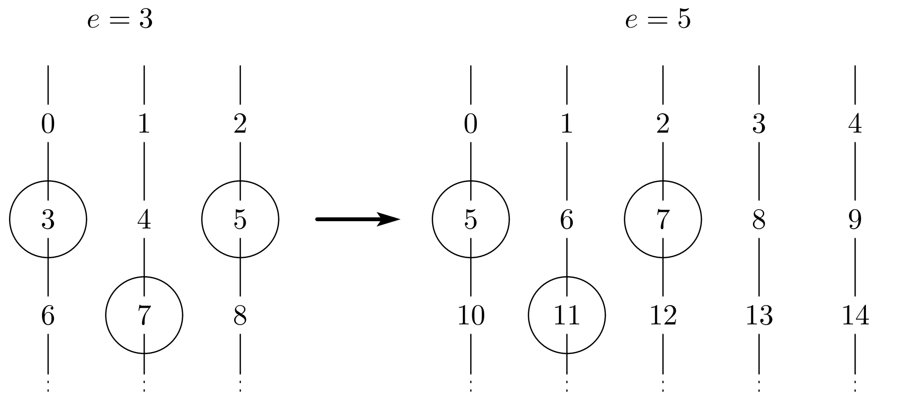

FIGURE 1. Adding two runners.

a precise way, obtaining equalities of decomposition numbers. Our theorem below interprets their result in terms of Morita equivalences of blocks.

3.2. **Statement of the theorem.** Fix $q, q' \in \Bbbk^\times$, let $e = e(q)$ and $e' = e(q')$, and assume that $e \leq e'$. Let B be a block ideal of $\mathcal{S} := \mathcal{S}_{\Bbbk,q}(r)$. Let Λ be the set of partitions λ of r such that $B \cdot L(\lambda) \neq 0$.

In what follows we shall be considering a number of blocks which are defined in terms of B; we indicate this relationship notationally by using decorations on 'B'. To avoid confusion we will usually use the same decorations for modules and for the poset of labels of simples.

Fix $d \leq r$, and $\alpha \in \{0, \dots, e\}$. Let $\underline{\Lambda}$ be the subset of Λ consisting of partitions with at most d parts. We sometimes denote by $l(\lambda)$ the number of nonzero parts of λ. Assume that $\underline{\Lambda}$ is nonempty. Then there exists r' and a block B' of $\mathcal{S}' := \mathcal{S}_{\Bbbk,q'}(r', r')$ such that for all $\lambda \in \underline{\Lambda}$ we have $|\lambda^+| = r'$ and $\lambda^+ \in \Lambda' := \{\mu \mid B' \cdot L'(\mu)\} \neq 0$.

Defining $\underline{\Lambda}'$ to be the set of partitions in Λ' with at most d parts, we have a bijection

$$\underline{\Lambda} \xrightarrow{\sim} \underline{\Lambda}' : \lambda \mapsto \lambda^+$$

preserving the dominance relation.

Now there exists an idempotent $f \in \mathcal{S}$ and an isomorphism $f\mathcal{S}f \cong \underline{\mathcal{S}}$, where $\underline{\mathcal{S}} := \mathcal{S}_{\Bbbk,q}(d, r)$. We define $\underline{B} := fBf$, a sum of blocks of $\underline{\mathcal{S}}$. Then we have a quotient functor B-mod $\to \underline{B}$-mod : $M \mapsto fM$. The simple modules $\underline{L}(\lambda)$ of $\underline{B}$ are indexed by $\underline{\Lambda}$. The Weyl module corresponding to $\lambda \in \underline{B}$ is denoted $\underline{\Delta}(\lambda)$.

Let $\underline{\mathcal{S}}' := \mathcal{S}_{\Bbbk,q'}(d, r')$, and then define $\underline{B}'$ analogously to $\underline{B}$. We use $\underline{L}'$ and $\underline{\Delta}'$ to denote the simple modules and Weyl modules of $\underline{B}'$.

Theorem 2. *Suppose that one of the following holds:*

- $\Bbbk$ *has characteristic* 0.
- $\Bbbk$ *has characteristic* $\ell > e$ *and* $q, q' \in \mathbb{F}_\ell$ *and the weight of* B *is strictly less than* ℓ.

Then there exists an equivalence

$$\mathsf{F} : \underline{B}\text{-mod} \xrightarrow{\sim} \underline{B}'\text{-mod}$$

such that

$$\mathsf{F}\underline{L}(\lambda) \cong \underline{L}'(\lambda^+)$$

and

$$\mathsf{F}\underline{\Delta}(\lambda) \cong \underline{\Delta}'(\lambda^+)$$

for all $\lambda \in \underline{\Lambda}$.

Remark 3.
1. *Since the map* $\lambda \mapsto \lambda^+$ *preserves the dominance order, the statement concerning* Δ*-sections is an immediate consequence of the statement about simples.*
2. *For a given block* B, *one can always choose* $d = r$, *so that* $\underline{\Lambda} = \Lambda$, $\underline{\mathcal{S}} = \mathcal{S}$ *and* $\underline{B} = B$. *On the other hand if* $e' > e$ *then* $\underline{\Lambda}'$ *is always strictly smaller than* Λ', *so that* $\underline{B}'$-mod *is a proper quotient of* B'-mod.
3. *In case* $1 \neq q \in \mathbb{F}_\ell$ *and the weight of* B *is strictly less than* ℓ, *we may always take* $q' = 1$, *thus obtaining Morita equivalences between blocks of q-Schur algebras and blocks of ordinary Schur algebras. In particular, we have reduced the verification of James's conjecture on decomposition numbers* ([Jam90, §4], [Mat99, p.117-118, 6.37&2nd paragraph of p.118], [Lus80]) *to the case* $q = 1$, i.e. *to the case of ordinary Schur algebras.*
4. *If* $q = 1$ *and* $\Bbbk$ *has positive characteristic* ℓ, *then* B *is a block of an ordinary Schur algebra, and its weight is strictly less than* ℓ *if and only if the corresponding block of a symmetric group has abelian defect groups. So the restriction on the weight of* B *in Theorem 2 should be regarded as an 'abelian defect' condition* [Bro90, 6.2. Question], [Bro92, 4.9. Conjecture], [BMM93], *which is a blockwise refinement of the assumption of* [Mat99, 6.37] *and is milder than any known assumptions in any literatures on James's conjecture.*
5. *We expect that the hypotheses on* q, q' *and the weight of* B *in positive characteristic are not necessary. They just reflect our current state of knowledge on the structure of Rouquier blocks of q-Schur algebras, which constitute the base case of our inductive proof; the inductive step is valid in general. For example, a positive resolution of Turner's remarkable conjectures on Rouquier blocks with nonabelian defect groups* [Tur05] *would remove the restriction on the weight of* B.

If $e = e'$ the map $\lambda \mapsto \lambda^+$ is the identity. We have rigged the statement of the theorem and its proof to include this special case, and obtain the following corollary.

Corollary 4.
- *Suppose that* $\Bbbk$ *has characteristic* 0 *and that* q *and* q' *are primitive* e*-th roots of unity in* $\Bbbk$*. Then for all* r *and* d *we have an isomorphism* $\mathcal{S}_{\Bbbk,q}(d,r) \xrightarrow{\sim} \mathcal{S}_{\Bbbk,q'}(d,r)$ **of $\Bbbk$-algebras**.
- *Suppose* $\Bbbk$ *has characteristic* $\ell > e$ *and that* $q, q' \in \mathbb{F}_\ell$*. Then for all* r *and* d*, corresponding blocks of* $\mathcal{S}_{\Bbbk,q}(d,r)$ *and* $\mathcal{S}_{\Bbbk,q'}(d,r)$ *of weight strictly less than* ℓ *are isomorphic* **as $\Bbbk$-algebras**.

Proof. Take the notation and hypotheses of Theorem 2. For all $\lambda \in \underline{\Lambda}$, the formal characters of $\Delta(\lambda)$ and $\Delta'(\lambda)$ are equal; hence $\dim \underline{\Delta}(\lambda) = \dim \underline{\Delta}'(\lambda)$. Since the decomposition numbers of $\underline{B}$ and $\underline{B}'$ are 'the same', we deduce that $\dim \underline{L}(\lambda) = \dim \underline{L}'(\lambda)$. But we already know that $\underline{B}$ and $\underline{B}'$ are Morita equivalent so conclude that $\underline{B}$ and $\underline{B}'$ are isomorphic as $\Bbbk$-algebras.

If the characteristic of $\Bbbk$ is 0, we may sum over all blocks to obtain isomorphisms of Schur algebras. □

3.3. **Decomposition numbers.** We now formulate the numerical consequences of Theorem 2. In characteristic 0 we recover weak versions of the theorem of James and Mathas [JM02], and the related result of Fayers [Fay07]. They prove equalities of the v-decomposition numbers defined by Lascoux, Leclerc and Thibon (see [Lec02]), which specialize at $v = 1$, via the theorem of Varagnolo-Vasserot [VV99] (and Ariki [Ari96]), to our formulas. Our approach has the advantage of also being valid in positive characteristic, as long as the parameter q is in the ground field and we are in an 'abelian defect' situation.

In comparing out statement with those of [JM02] and [Fay07], it is important to keep in mind that our labelling of modules is conjugate to theirs. In particular, our result is more directly related to Fayers's.

Theorem 5. *Keep the notation and assumptions of Theorem 2. We have, for all* $\lambda, \mu \in \underline{\Lambda}$*,*

$$[\Delta(\lambda) : L(\mu)] = [\Delta'(\lambda^+) : L'(\mu^+)]$$

and

$$[\Delta(\lambda^{\mathrm{tr}}) : L(\mu^{\mathrm{tr}})] = [\Delta'((\lambda^{\mathrm{tr}})^+) : L'((\mu^{\mathrm{tr}})^+)].$$

Proof. The first equality is equivalent to

$$[\underline{\Delta}(\lambda) : \underline{L}(\mu)] = [\underline{\Delta}'(\lambda^+) : \underline{L}'(\mu^+)],$$

which is an immediate consequence of Theorem 2.

For $\lambda \in \Lambda$, let $T(\lambda)$ be the corresponding indecomposable tilting module of B. If $\lambda \in \underline{\Lambda}$, the image $fT(\lambda)$ under the Schur functor is an indecomposable tilting module of $\underline{B}$. By the conclusion of the Main theorem 2, we have equalities of filtration multiplicities, for any $\lambda \in \underline{\Lambda}$,

$$\begin{aligned}(T(\mu) : \Delta(\lambda))_{\mathcal{S}(r,r)} &= (fT(\mu) : f\Delta(\lambda))_{\mathcal{S}(d,r)} \\ &= (f'T(\mu^+) : f'\Delta'(\lambda^+))_{\mathcal{S}'(d,r')} \\ &= (T(\mu^+) : \Delta'(\lambda^+))_{\mathcal{S}'(r',r')}.\end{aligned}$$

On the other hand, Donkin's formula [Don98, Proposition 4.1.5] tells us that

$$[\Delta(\lambda^{\mathrm{tr}}) : L(\mu^{\mathrm{tr}})] = (T(\mu) : \nabla(\lambda)),$$

where $\nabla(\lambda)$ is the dual Weyl module associated to λ. Here we may safely replace $\nabla(\lambda)$ by $\Delta(\lambda)$, since contragradient duality for the q-Schur algebra sends $T(\mu)$ to itself and $\nabla(\lambda)$ to $\Delta(\lambda)$. Thus the second equality in the statement of the theorem is proved. □

3.4. **Truncation functors.** In this subsection we give an application of truncation functors which will be used as a reduction step in the proof of the main theorem. The main source is [Don98, 4.2].

In this subsection, we denote by $G = G(n)$ the quantum general linear group with simple roots $\Pi = \{\alpha_1, \dots, \alpha_{n-1}\}$. For any subset Σ of Π, we have an associated standard Levi subgroup G_Σ. We denote by X_Σ the set of dominant weights for G_Σ. To specify the group for standard, costandard, simple, and tilting modules, we again attach subscripts in the notation Δ, ∇, L, T, e.g., $\Delta_\Sigma(\lambda), \nabla_\Sigma(\lambda), L_\Sigma(\lambda), T_\Sigma(\lambda)$, and simply write $\Delta_m(\lambda), \nabla_m(\lambda), L_m(\lambda), T_m(\lambda)$ if for modules over $G(m)$. In the case they are modules over the full parent group G we sometimes will not attach any subscripts, simply writing $\Delta(\lambda), \nabla(\lambda), L(\lambda), T(\lambda)$, etc.

For a standard Levi subgroup G_Σ of G and a dominant weight λ of G_Σ, we denote by $\mathrm{Tr}^\lambda_\Sigma$ the Harish-Chandra (λ, Σ)-truncation functor (see [Don98, p.86], or [Jan03, p.181,2.11] for its slightly different description); given a G-module V we define $\mathrm{Tr}^\lambda_\Sigma V = \bigoplus_{\mu \in X, \lambda-\mu \in \mathbb{Z}\Sigma} V_\mu$, a G_Σ-module. The truncation functor satisfies the following properties.

(1) $\mathrm{Tr}^\lambda_\Sigma$ is exact,
(2) $\mathrm{Tr}^\lambda_\Sigma \nabla(\mu) = \begin{cases} \nabla_\Sigma(\mu), & \text{if } \lambda - \mu \in \mathbb{Z}\Sigma, \\ 0, & \text{otherwise} \end{cases}$,
(3) $\mathrm{Tr}^\lambda_\Sigma L(\mu) = \begin{cases} L_\Sigma(\mu), & \text{if } \lambda - \mu \in \mathbb{Z}\Sigma, \\ 0, & \text{otherwise} \end{cases}$

The following can be found in [Don98, p.89]: Fix $\lambda \in X_\Pi$. If a module U is filtered by Δ's, a module V is filtered by ∇'s and every weight of U and V is less than or equal to λ, then the map

$$\mathrm{Hom}_G(U, V) \to \mathrm{Hom}_{G_\Sigma}(\mathrm{Tr}^\lambda_\Sigma U, \mathrm{Tr}^\lambda_\Sigma V) \text{ is surjective.} \tag{1}$$

For $\lambda \in X_\Pi$ and $\mu \in X_\Sigma$ we have

$$(T_\Sigma(\lambda) : \nabla_\Sigma(\mu)) = \begin{cases} (T(\lambda) : \nabla(\mu)), & \text{if } \mu \in X_\Pi \text{ and } \lambda - \mu \in \mathbb{Z}\Sigma; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We have the following application of the truncation functor.

Proposition 6. *Suppose that $\mathcal{T}$ is a cosaturated subset of the set of partitions of r sharing a common e-core, where e is the quantum characteristic of G. Further suppose that there exists a weight σ for $G(k)$ such that $T(\sigma) = L(\sigma)$ and such that for any $\lambda \in \mathcal{T}$ there exists a partition $(\lambda^{\mathrm{tr}})^b$*

such that $\lambda^{\mathrm{tr}} = \sigma \cup (\lambda^{\mathrm{tr}})^b$. *Here, the notation* $\cup$ *is taken from page 6 of Macdonald's text book.* [1]

Put $\mathrm{tr}\mathcal{T} := \{\lambda^{\mathrm{tr}} \mid \lambda \in \mathcal{T}\}$ *and fix an integer* m *such that* $m \geq \max\{l(\lambda^b) | \lambda \in \mathrm{tr}\mathcal{T}\}$.

Then,

$$\mathrm{End}_{G(k+m)}\left(\bigoplus_{\lambda\in\mathrm{tr}\mathcal{T}} T_{k+m}(\lambda)\right) \cong \mathrm{End}_{G(m)}\left(\bigoplus_{\lambda\in\mathrm{tr}\mathcal{T}} T_m(\lambda^b)\right). \tag{3}$$

Here, via the isomorphism (3), the idempotent of $\mathrm{End}_{G(k+m)}(T_{k+m}(\lambda))$ *corresponds to the idempotent of* $\mathrm{End}_{G(k+m)}(T_m(\lambda^b))$.

Proof. Since $T(\sigma) = L(\sigma)$, we know that

$$\mathrm{End}_{G(m)}\left(\bigoplus_{\lambda\in\mathrm{tr}\mathcal{T}} T_m(\lambda^b)\right) \cong \mathrm{End}_{G(k)\times G(m)}\left(\bigoplus_{\lambda\in\mathrm{tr}\mathcal{T}} T_k(\sigma)\boxtimes T_m(\lambda^b)\right) \tag{4}$$

Then, by (1) and by taking $\mathrm{Tr}^{\tau}_{\Sigma(k,m)}$ into account where τ is the maximum of λ^{tr} for all $\lambda \in \mathcal{T}$ and $\Sigma(k,m) = \{\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_{k+m-1}\}$, we know that there is a surjection of $\mathrm{End}_{G(k+m)}(\bigoplus_{\lambda\in\mathcal{T}} T_{k+m}(\lambda))$ onto the RHS of (4). So, it suffices to count the dimensions of the endomorphism rings in question. The dimension of LHS of (3) is equal to $\sum_{\lambda,\nu\in\mathrm{tr}\mathcal{T}}\sum_{\mu\lhd\lambda}(T_{k+m}(\lambda) : \nabla_{k+m}(\mu))(T_{k+m}(\nu) : \nabla_{k+m}(\mu))$. And, the dimension of LHS of (4) is equal to $\sum_{\lambda,\nu\in\mathrm{tr}\mathcal{T}}\sum_{\mu^b\lhd\lambda^b}(T_m(\lambda^b) : \nabla_m(\mu^b))(T_m(\nu^b) : \nabla_m(\mu^b))$. Then, the cosaturation condition on $\mathcal{T}$ and the assumption on the unique decomposition $\lambda^{\mathrm{tr}} = \sigma \cup (\lambda^{\mathrm{tr}})^b$, and (2) ensure that each non-zero term in the second summations in the dimension formulas for (3) and (4) match up each other. $\square$

3.5. **Proof of the main theorem.** In this subsection we reduce the main theorem to the case of large d (relative to the block B) using the results of the following sections, on Scope pairs and Rouquier blocks.

Because the orbit of the block B under the affine Weyl group action contains a Rouquier block, there exists a sequence of blocks $B_0, \ldots, B_s = B$, such that B_0 is a Rouquier block, and any two successive blocks form a Scopes pair.

Now choose c such that all partitions in all blocks B_i have at most $\tilde{d} = d + ec$ parts. Now define blocks $\widetilde{B}'$ and $\underline{\widetilde{B}'}$ analogously to B' and $\underline{B}'$, replacing d by $\tilde{d}$. Then by induction, with base case §5, which proves the main result for Rouquier blocks, and inductive step §4.2, we deduce that the main theorem holds for B, as long as we replace d by $\tilde{d}$. In other words we have an equivalence

$$\tilde{\mathsf{F}} : B\text{-mod} \xrightarrow{\sim} \underline{\widetilde{B}'}\text{-mod}$$

[1] In our set up the first entry $(\lambda^{\mathrm{tr}})^b_1$ is at most $\sigma_{l(\sigma)}$. So, it's just a concatenation of two partition. For example, $(9,8,7) \cup (3,2,1) = (9,8,7,3,2,1)$.

such that

$$\tilde{\mathsf{F}}L(\lambda) \cong \underline{L}'(\widetilde{\lambda}^+)$$

for all $\lambda \in \Lambda$. Here $\widetilde{\lambda}^+$ is defined similarly to λ^+, except we use abaci with $\tilde{d}$ beads rather than d beads. Note that $\widetilde{\lambda}^+$ has $ce - \alpha$ more parts than λ^+. (Recall that the extra empty runner in λ^+ is inserted between $\rho_{\alpha-1}$ and ρ_α.)

Let $\sigma = ((d+ce-\alpha)^{e'-e}, \ldots, (d+2e-\alpha)^{e'-e}, (d+e-\alpha)^{e'-e})$. Then for all $\lambda \in \underline{\Lambda}$ we have

$$(\widetilde{\lambda}^+)^{\mathrm{tr}} = \sigma \cup (\lambda^+)^{\mathrm{tr}}. \tag{5}$$

Furthermore σ is minimal amongst partitions in its block having at most $c(e'-e)$-parts. Hence we have $T_{c(e'-e)}(\sigma) = L_{c(e'-e)}(\sigma)$.

Let $m := \max\{l((\lambda^+)^{\mathrm{tr}}) \mid \lambda \in \underline{\Lambda}\}$.

For all $\lambda, \mu \in \underline{\Lambda}$, we have

$$\begin{aligned} \mathrm{Hom}_B(P(\lambda), P(\mu)) &\cong \mathrm{Hom}_{B''}(P(\widetilde{\lambda}^+), P(\widetilde{\mu}^+)) \\ &\cong \mathrm{Hom}_{G(c(e'-e)+m)}(T((\widetilde{\lambda}^+)^{\mathrm{tr}}), T((\widetilde{\mu}^+)^{\mathrm{tr}})) \\ &\cong \mathrm{Hom}_{G(c(e'-e)+m)}(T(\sigma \cup (\lambda^+)^{\mathrm{tr}}), T(\sigma \cup (\mu^+)^{\mathrm{tr}})) \\ &\cong \mathrm{Hom}_{G(m)}(T((\lambda^+)^{\mathrm{tr}}), T((\mu^+)^{\mathrm{tr}})) \\ &\cong \mathrm{Hom}_{B'}(P(\lambda^+), P(\mu^+)), \end{aligned}$$

where the first isomorphism is deduced from the equivalence $\tilde{\mathsf{F}}$, the second and last isomorphisms by Ringel selfduality [Don98], the third by Proposition 6 and the fourth by (5). Here, the quantum characteristic of the G''s is e'. Now the main theorem follows immediately.

4. Scopes pairs

In this section we carry out the inductive step of the proof of the main theorem. Here we make an additional assumption that d is large, in the sense that all the partitions in the blocks we are considering have at most d parts.

After reviewing the notion of a perverse equivalence [Rou06, §2.6], we spell out the inductive step. A key point here is that the combinatorics of the perverse equivalences are independent of the 'quantum characteristic' $e(q)$; a proof is relegated to the end of the section.

4.1. **Perverse equivalences.** We work in the context of finite-dimensional algebras, sufficient for our application. Roughly speaking, a perverse equivalence is a derived equivalence filtered by shifted Morita equivalences; see [Rou06, §2.6], [CRa] for more details. Let A and $\dot{A}$ be two finite-dimensional algebras and $\mathcal{S}$ (resp. $\dot{\mathcal{S}}$) the set of isomorphism classes of finite-dimensional simple A-modules (resp. simple $\dot{A}$-modules).

Definition 7. *An equivalence* $\mathsf{G} : D^b(A\text{-mod}) \xrightarrow{\sim} D^b(\dot{A}\text{-mod})$ *is* perverse *if there is*

- *a filtration* $\emptyset = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_r = \mathcal{S}$
- *a filtration* $\emptyset = \dot{\mathcal{S}}_0 \subset \dot{\mathcal{S}}_1 \subset \cdots \subset \dot{\mathcal{S}}_r = \dot{\mathcal{S}}$
- *and a function* $p : \{1, \ldots, r\} \to \mathbb{Z}$

such that

- G *restricts to equivalences* $D^b_{\mathcal{A}_i}(A\text{-mod}) \xrightarrow{\sim} D^b_{\dot{\mathcal{A}}_i}(\dot{A}\text{-mod})$
- $\mathsf{G}[-p(i)]$ *induces equivalences* $\mathcal{A}_i/\mathcal{A}_{i-1} \xrightarrow{\sim} \dot{\mathcal{A}}_i/\dot{\mathcal{A}}_{i-1}$.

Here $\mathcal{A}_i$ *(resp.* $\dot{\mathcal{A}}_i$*) is the Serre subcategory of* A-mod *(resp.* $\dot{A}$-mod*) generated by* $\mathcal{S}_i$ *(resp.* $\dot{\mathcal{S}}_i$*), and by* $D^b_{\mathcal{A}_i}(A\text{-mod})$ *we mean the full subcategory of* $D^b(A\text{-mod})$ *whose objects are complexes with homology modules belonging to* $\mathcal{A}_i$.

The following basic result implies that the filtration $\mathcal{S}_\bullet$ and the perversity p determine $\dot{A}$ (and $\dot{\mathcal{S}}_\bullet$), up to Morita equivalence.

Proposition 8. *Let* $\mathsf{G} : D^b(A\text{-mod}) \xrightarrow{\sim} D^b(\dot{A}\text{-mod})$ *and* $\mathsf{G}' : D^b(A\text{-mod}) \xrightarrow{\sim} D^b(\dot{A}'\text{-mod})$ *be perverse. If* $\mathcal{S}'_\bullet = \mathcal{S}_\bullet$ *and* $p' = p$*, then the composition* $\mathsf{G}'\mathsf{G}^{-1}$ *restricts to an equivalence* $\dot{A}$-mod $\xrightarrow{\sim} \dot{A}'$-mod.

Indeed, the composition is $G'G^{-1}$ is also a perverse equivalence with respect to p'' where p'' is identically zero. Then an easy inductive argument shows that $G'G^{-1}$ restricts to an equivalence $\dot{\mathcal{A}}_i \xrightarrow{\sim} \dot{\mathcal{A}}'_i$.

Now let $\mathcal{A} = \oplus_{r\geq 0}\mathcal{S}_{\Bbbk,q}(r)$-mod. Let $a \in \{0, \ldots, e-1\}$. Given M, a $\mathcal{S}_{\Bbbk,q}(r-1)$-module lying in a block B, let $\tilde{M}$ be the corresponding $\mathcal{S}_{\Bbbk,q}(r, r-1)$-module, under the canonical equivalence $\mathcal{S}_{\Bbbk,q}(r, r-1)$-mod $\xrightarrow{\sim} \mathcal{S}_{\Bbbk,q}(r-1)$-mod. Tensoring with the natural representation $V = \Bbbk^r$ of $\mathcal{S}_{\Bbbk,q}(r, 1) = \mathrm{End}(V)$ we obtain a $\mathcal{S}_{\Bbbk,q}(r)$-module $V \otimes \tilde{M}$. We define F_aM to be the projection of $V \otimes \tilde{M}$ onto the sum of blocks C of $\mathcal{S}_{\Bbbk,q}(r)$ such that $K(C\text{-mod}) \cap \mathfrak{f}_a(K(B\text{-mod})) \neq 0$.

This recipe defines an exact endofunctor F_a on $\bigoplus_{r\geq 0}\mathcal{S}_{\Bbbk,q}(r)$-mod lifting the action of $\mathfrak{f}_a$ on $\mathcal{F}$. Let E_a be a left adjoint to F_a. Then E_a is also a right adjoint to F_a, and E_a lifts the action of $\mathfrak{e}_a$ on $\mathcal{F}$.

The adjoint pair (E_a, F_a) may be used to define a $\mathfrak{sl}_2$-categorification on $\mathcal{A}$ [CR08]. We obtain as a result derived equivalences between blocks in Scopes pairs, which are known to be perverse, by [CRa]. Here are the details: Let B be a block of $\mathcal{S}_{\Bbbk,q}(r)$, and put $\dot{B} = s_aB$, a block of $\mathcal{S}_{\Bbbk,q}(\dot{r})$, for some $\dot{r}$. Let Λ and $\dot{\Lambda}$ be the sets of partitions labelling simple modules in B and $\dot{B}$.

Proposition 9. *There exists an equivalence* $\mathsf{G} : D^b(B\text{-mod}) \xrightarrow{\sim} D^b(\dot{B}\text{-mod})$ *such that*

- G *is perverse with respect to the filtrations*

$$\Lambda_j = \{\lambda \in \Lambda \mid F_a^j L(\lambda) = 0\} \qquad \textit{and} \qquad \dot{\Lambda}_j = \{\lambda \in \dot{\Lambda} \mid E_a^j L(\lambda) = 0\}$$

  and the perversity $p(j) = j - 1$.
- G *induces the isomorphism* $s_a : K(B\text{-mod}) \xrightarrow{\sim} K(\dot{B}\text{-mod})$.

- G *(and G^{-1}) are induced by complexes of functors, each of which is a direct summand of a composition of E_a's and F_a's.*

Remark 10. *The filtrations above may also be defined by the formulas*

$$\Lambda_j = \{\lambda \in \Lambda \mid \widetilde{F}_a^j L(\lambda) = 0\} \qquad \text{and} \qquad \dot{\Lambda}_j = \{\lambda \in \dot{\Lambda} \mid \widetilde{E}_a^j L(\lambda) = 0\},$$

where $\widetilde{F}_a$ and $\widetilde{E}_a$ are the Kleshchev/Kashiwara operators*:*

$$\widetilde{F}_a(M) := \mathrm{Soc}(F_a(M)), \qquad \widetilde{E}_a(M) := \mathrm{Soc}(E_a(M))$$

[Kle95a, Kle95b, Kle96]*.*

In very special cases we can deduce Morita equivalences of a type first discovered by Scopes for symmetric groups. The following corollary is a slightly new and a slight extension of [EMS94] and [Don94, §5].

In [EMS94] and [Don94, §5] they dealt with 1-Schur algebras not for the q-Schur algebras where $q \neq 1$. But, the basic ideas given below are (at least at the level of combinatorics) identical with theirs.

Corollary 11 (Scopes's Morita Equivalences)**.** *Fix $d \geq 0$. Suppose that for all $\lambda \in \dot{\Lambda}$, the d-bead abacus representation of λ contains no beads in ρ_i, where $i \equiv a + d$ modulo e. Then there is an equivalence $\underline{B}$-mod $\xrightarrow{\sim}$ $\underline{\dot{B}}$-mod between the blocks $\underline{B}$ and $\underline{\dot{B}}$ of $\mathcal{S}_{\Bbbk,q}(d,r)$ and $\mathcal{S}_{\Bbbk,q}(d,\dot{r})$ corresponding to B and $\dot{B}$, that sends $\underline{L}(\lambda)$ to $\underline{L}(\sigma_a\lambda)$ (and $\underline{\Delta}(\lambda)$ to $\underline{\Delta}(\sigma_a\lambda)$).*

Proof. Let $\mathsf{G} : D^b(B\text{-mod}) \xrightarrow{\sim} D^b(\dot{B}\text{-mod})$ be the perverse equivalence provided by Proposition 9. Then G restricts to an equivalence $\mathsf{G}_1 : \mathcal{A}_1 \xrightarrow{\sim} \dot{\mathcal{A}}_1$ between the Serre subcategories

$$\mathcal{A}_1 := \{M \in B\text{-mod} \mid F_a M = 0\} \quad \text{and} \quad \dot{\mathcal{A}}_1 := \{M \in \dot{B}\text{-mod} \mid E_a M = 0.\}$$

Let $\lambda \in \underline{\Lambda}$. Then by the assumption that there are no beads in ρ_i we have $\mathfrak{f}_a\lambda = 0$. Thus $\Delta(\lambda) \in \mathcal{A}_1$, and $[\mathsf{G}_1\Delta(\lambda)] = [\Delta(\sigma_a\lambda)]$, since $s_a\lambda = \pm\sigma_a\lambda$ and the sign is determined to be $+$ as G_1 is an equivalence. It follows by induction that for all $\lambda \in \underline{\Lambda}$ we have $\mathsf{G}_1 P(\lambda) \cong P(\sigma_a\lambda)$. This implies the desired statements, since $\underline{B}$ and $\underline{\dot{B}}$ are Morita equivalent to $\mathrm{End}\left(\bigoplus_{\lambda\in\underline{\Lambda}} P(\lambda)\right)$ and $\mathrm{End}\left(\bigoplus_{\lambda\in\underline{\dot{\Lambda}}} P(\lambda)\right)$ respectively. □

4.2. **Inductive step.** Consider a Scopes pair of blocks B and $\dot{B} = \sigma_a B$. We assume that Theorem 2 is true for B and aim to verify it for $\dot{B}$.

We *shall also assume* that all partitions in Λ and $\dot{\Lambda}$ have at most d parts. Hence $\Lambda = \underline{\Lambda}$, $\dot{\Lambda} = \underline{\dot{\Lambda}}$, $B = \underline{B}$ and $\dot{B} = \underline{\dot{B}}$. Let $i \in \{0, \dots, e-1\}$ so that $i \equiv a + d \mod e$. We may assume that $\alpha \neq i \mod e$ and therefore that $\dot{B}' = \sigma_{a'}B'$, where $a' \in \{0, \dots, e'-1\}$ such that $i \equiv a' + d \mod e'$. If $\alpha = i < e$ we reduce to the case $\alpha = i + 1$ by the following observation: $B'_\alpha = \sigma_{\alpha+e'-e} \cdots \sigma_{\alpha+1} B'_{\alpha+1}$, and there exists an equivalence $\underline{B'_{\alpha+1}}$-mod $\xrightarrow{\sim}$ $\underline{B'_\alpha}$-mod sending $\underline{L}'(\lambda^{+,\alpha+1})$ to $\underline{L}'(\lambda^{+,\alpha})$ and $\underline{\Delta}'(\lambda^{+,\alpha+1})$ to $\underline{\Delta}'(\lambda^{+,\alpha})$ for all

$\lambda \in \underline{\Lambda}$, obtained by $e' - e$ applications of Corollary 11. If $\alpha = e$ and $i = 0$ we can reduce to the case $\alpha = e - 1$ by a similar argument.

By Proposition 9 we have a perverse equivalence $\mathsf{G} : D^b(B\text{-mod}) \xrightarrow{\sim} D^b(\dot{B}\text{-mod})$ specified by the filtration $\Lambda_j = \{\lambda \in \Lambda \mid F_a^j L(\lambda) = 0\}$ and the perversity $p(j) = j - 1$. We have a parallel situation for B' and $\dot{B}'$, a perverse equivalence $\mathsf{G}' : D^b(B'\text{-mod}) \xrightarrow{\sim} D^b(\dot{B}'\text{-mod})$ with respect to the filtration $\Lambda'_j = \{\lambda \in \Lambda' \mid F_{a'}^j L'(\lambda) = 0\}$ and perversity $p(j) = j - 1$. Moreover $[\mathsf{G}\Delta(\lambda)] = \pm[\Delta(\sigma_a \lambda)]$ for all $\lambda \in \Lambda$. An analogous statement holds for G'.

By a result of Cline, Parshall and Scott [CPS82] (see also [PS88]) the exact functor $B'\text{-mod} \to \underline{B}'\text{-mod} : M \mapsto fM$ induces an equivalence of triangulated categories

$$\frac{D^b(B'\text{-mod})}{D^b_{\mathcal{E}}(B'\text{-mod})} \xrightarrow{\sim} D^b(\underline{B}'\text{-mod}),$$

where $\mathcal{E}$ is the Serre subcategory of B'-mod generated by $L(\lambda)$, $\lambda \in \Lambda' \setminus \underline{\Lambda}'$. An analogous statement holds for the dot versions.

We claim that G' restricts to an equivalence $D^b_{\mathcal{E}}(B'\text{-mod}) \xrightarrow{\sim} D^b_{\dot{\mathcal{E}}}(\dot{B}'\text{-mod})$. An equivalent statement is that it restricts to an equivalence between the left perpendicular categories, *i.e.* the full triangulated subcategories of $D^b(B'\text{-mod})$ and $D^b(\dot{B}'\text{-mod})$ generated by $\{P'(\lambda) \mid \lambda \in \underline{\Lambda}'\}$ and $\{P'(\lambda) \mid \lambda \in \underline{\dot{\Lambda}}'\}$. To see that this latter statement is true recall (Proposition 9) that G' is induced by a complex of functors, each of which is a direct summand of a composition of powers of $E_{a'}$ and $F_{a'}$. Because of our assumption that all partitions in Λ and $\dot{\Lambda}$ have at most d parts, and that $\alpha \neq i$ mod e, the map between Grothendieck groups induced by any such direct summand functor sends $\sum_{\lambda \in \underline{\Lambda}'} \mathbb{Z}[\Delta'(\lambda)]$ into $\sum_{\lambda \in \underline{\dot{\Lambda}}'} \mathbb{Z}[\Delta'(\lambda)]$, and therefore $\sum_{\lambda \in \underline{\Lambda}'} \mathbb{Z}[P'(\lambda)]$ into $\sum_{\lambda \in \underline{\dot{\Lambda}}'} \mathbb{Z}[P'(\lambda)]$. A similar reasoning applies to G'^{-1}, and the claim follows.

Hence G' induces a perverse equivalence

$$\underline{\mathsf{G}}' : D^b(\underline{B}'\text{-mod}) \xrightarrow{\sim} D^b(\underline{\dot{B}}'\text{-mod})$$

with respect to the filtration $\underline{\Lambda}'_j = \{\lambda \in \underline{\Lambda}' \mid F_a^j L'(\lambda) = 0\}$ and the perversity $p(j) = j - 1$.

By assumption we have a Morita equivalence

$$\mathsf{F} : \underline{B}\text{-mod} \xrightarrow{\sim} \underline{B}'\text{-mod}$$

such that $\mathsf{F}\underline{L}(\lambda) \cong \underline{L}'(\lambda^+)$ and $\mathsf{F}\underline{\Delta}(\lambda) \cong \underline{\Delta}'(\lambda^+)$ for all $\lambda \in \Lambda$. Moreover by Lemma 12, proved in the following subsection, the bijection $\underline{\Lambda} \xrightarrow{\sim} \underline{\Lambda}' : \lambda \mapsto \lambda^+$ restricts to bijections $\underline{\Lambda}_j \xrightarrow{\sim} \underline{\Lambda}'_j$ for all j. By Proposition 8 we deduce that the composition

$$\underline{\mathsf{G}}'\mathsf{F}\mathsf{G}^{-1} : D^b(\dot{B}\text{-mod}) \xrightarrow{\sim} D^b(\underline{\dot{B}}'\text{-mod})$$

restricts to an equivalence

$$\dot{\mathsf{F}} : \dot{B}\text{-mod} \xrightarrow{\sim} \dot{\underline{B}}'\text{-mod}\,.$$

Moreover $[\dot{\mathsf{F}}\Delta(\lambda)] = [\underline{\Delta}'(\lambda^+)]$ for all $\lambda \in \dot{\Lambda}$. By unitriangularity of the decomposition matrices of B and $\underline{B}'$, it follows that $\dot{\mathsf{F}}L(\lambda) \cong \underline{L}'(\lambda^+)$ for all $\lambda \in \dot{\Lambda}$, and therefore that $\dot{\mathsf{F}}\Delta(\lambda) \cong \underline{\Delta}'(\lambda^+)$ for all $\lambda \in \dot{\Lambda}$.

4.3. **Comparison of crystals.** The aim of this subsection is to complete the inductive step by proving Lemma 12, which was used above to get a good compatibility between filtrations.

Let $a \in \{0, \dots, e-1\}$. For any partition λ, we define

$$\begin{aligned} \varphi_a(\lambda) \;&:=\; \max\{k \geq 0 \mid (\widetilde{F}_a)^k(L(\lambda)) \neq 0\} \\ &=\; \max\{k \geq 0 \mid {F_a}^k(L(\lambda)) \neq 0\} \end{aligned}$$

and similarly

$$\begin{aligned} \varphi'_a(\lambda) \;&:=\; \max\{k \geq 0 \mid (\widetilde{F}_a)^k(L'(\lambda)) \neq 0\} \\ &=\; \max\{k \geq 0 \mid {F_a}^k(L'(\lambda)) \neq 0\} \end{aligned}$$

(cf[HK02, p.85]), where $\widetilde{F}_a$ is the Kleshchev/Kashiwara operator, defined in Remark 10. Remember that $L(\lambda)$ is a simple module over a Schur algebra with parameter q while for $L'(\lambda)$ the parameter is q'.

Lemma 12. *Let $i \in \{0, \dots, e-1\}$ and define a and a' as in §4.2. Then*

$$\varphi_a(\lambda) = \varphi'_{a'}(\lambda^+)$$

for any $\lambda \in \underline{\Lambda}$.

Proof. We explain below why the statement holds if λ is e-restricted. But first we shall assume the truth of this special case and deduce the statement for arbitrary λ.

We can write any partition λ uniquely as $\lambda = \lambda^{e\text{-res}} + e\tilde{\lambda}$, where $\lambda^{e\text{-res}}$ is e-restricted. In terms of the abacus $\lambda^{e\text{-res}}$ is obtained from λ by repeatedly moving a bead up a runner, say from position h to $h-e$, where positions $h-e, \dots, h-1$ are unoccupied. This description makes it clear that $(\lambda^{e\text{-res}})^+ = (\lambda^+)^{e'\text{-res}}$.

By Steinberg's tensor product theorem [Don98, p.65] (cf. [Lus89, 7.4],[DD91], [PW91, 11.7]) we have $L(\lambda) \cong L(\lambda^{e\text{-res}}) \otimes L(e\tilde{\lambda})$. More generally tensoring with $L(e\tilde{\lambda})$ sends modules over a block of $\mathcal{S}_{q,\Bbbk}(r - e|\tilde{\lambda}|)$ to modules over the block of $\mathcal{S}_{q,\Bbbk}(r)$ corresponding to the same e-core. Hence for any k the isomorphism $V^{\otimes k} \otimes L(\lambda) \cong V^{\otimes k} \otimes L(\lambda^{e\text{-res}}) \otimes L(e\tilde{\lambda})$, after a projection onto appropriate blocks, gives an isomorphism ${F_a}^k(L(\lambda)) \cong {F_a}^k(L(\lambda^{e\text{-res}})) \otimes L(e\tilde{\lambda})$. Thus we deduce that $\varphi_a(\lambda) = \varphi_a(\lambda^{e\text{-res}})$.

Putting together the pieces, we have, for any λ,

$$\varphi_a(\lambda) = \varphi_a(\lambda^{e\text{-res}}) = \varphi'_{a'}((\lambda^{e\text{-res}})^+) = \varphi'_{a'}((\lambda^+)^{e'\text{-res}}) = \varphi'_{a'}(\lambda^+).$$

Now we return to the special case of e-restricted partitions. One can define a summand $F_{\mathcal{H},a} : \mathcal{H}_{\Bbbk,q}(\mathfrak{S}_{r-1})\text{-mod} \to \mathcal{H}_{\Bbbk,q}(\mathfrak{S}_r)\text{-mod}$ of the induction functor between Hecke algebras of type A, analogous to F_a. One then has obvious analogues $\widetilde{F}_{\mathcal{H},a}$ and $\varphi_{\mathcal{H},a}$ of $\widetilde{F}_a$ and φ_a.

After Kleshchev's branching rule appeared, Brundan [Bru98] extended Kleshchev's result to Hecke algebras of type A and showed that for any simple $\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_{r-1})$-module L the $\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_r)$-module $\widetilde{F}_{\mathcal{H},a}(L)$ is simple. Since there is a commutative diagram of functors

$$\begin{array}{rccc} F_a : & \mathcal{S}_{\Bbbk,q}(r-1)\text{-mod} & \to & \mathcal{S}_{\Bbbk,q}(r)\text{-mod} \\ & \downarrow & & \downarrow \\ F_{\mathcal{H},a} : & \mathcal{H}_{\Bbbk,q}(\mathfrak{S}_{r-1})\text{-mod} & \to & \mathcal{H}_{\Bbbk,q}(\mathfrak{S}_r)\text{-mod} \end{array},$$

where the horizontal arrows are Schur functors, this implies that $\varphi_a(\lambda) = \varphi_{\mathcal{H},a}(\lambda)$ for any e-restricted λ.

Brundan also showed that for e-restricted partitions $\varphi_{\mathcal{H},a}(\lambda)$ is the number of conormal indent a-nodes for λ. We won't define this combinatorial notion here; we just observe that his result implies that $\varphi_{\mathcal{H},a}(\lambda)$ depends only on the configuration of beads on runners ρ_{i-1} and ρ_i in the abacus representation of λ, where $i \in \{0, \ldots, e-1\}$ is such that $i \equiv a + d \mod e$. We deduce that for any e-restricted partition λ,

$$\varphi_a(\lambda) = \varphi_{\mathcal{H},a}(\lambda) = \varphi'_{\mathcal{H},a'}(\lambda) = \varphi'_{a'}(\lambda),$$

as desired.

$\square$

5. Rouquier blocks

We now complete the proof of Theorem 2 by handling the base case: Rouquier blocks.

5.1. **Wreath product interpretation of Rouquier blocks.** In this subsection we assume that $ch(\Bbbk) = \ell > w > 0$, and that q is a prime power not congruent to 0 or 1 modulo ℓ. So $e = e(q)$ is the multiplicative order of $q \cdot 1_{\mathbb{F}_\ell}$. We denote by $\mathbf{L}_\lambda$ the standard Levi subgroup of $\mathbf{GL}_{|\lambda|}(\mathbb{F}_q)$ corresponding to the Young subgroup $\mathfrak{S}_\lambda$. In [Tur02, p.250 Lemma 1 & p.249,Theorem 1] and [Miy01, p.30 Lemma 5.0.6 & p.31 Theorem 5.0.7] (cf [Tur05, Theorem 71]) the $\mathbf{GL}_n(\mathbb{F}_q)$ analogue of the main theorem in [CK02, Theorem 2] is proved independently:

Theorem 13. *Suppose γ is a Rouquier e-core with respect to w. Put $r := ew + |\gamma|$ and $\mathbf{G} := \mathbf{GL}_r(\mathbb{F}_q)$. Put $\mathbf{L} := \mathbf{L}_{(e^w,|\gamma|)}$, a Levi subgroup of $\mathbf{G}$. Then $\mathbf{L}$ has a parabolic complement $\mathbf{P} = \mathbf{L}\mathbf{U}_\mathbf{L}$ in $\mathbf{G}$. Put $\mathbf{I} := N_\mathbf{G}(\mathbf{L}) \cong \mathbf{L} \rtimes \mathfrak{S}_w$. So, $\Bbbk[\mathbf{I}] \cong \Bbbk[\mathbf{L}] \rtimes \mathfrak{S}_w$.*

There exists a $(\Bbbk[\mathbf{G}], \Bbbk[\mathbf{I}])$-bimodule M such that

(1) *M is a direct summand of $\Bbbk[\mathbf{G}/\mathbf{U}_\mathbf{L}] \otimes_{\Bbbk[\mathbf{L}]} \Bbbk[\mathbf{I}]$ as a $(\Bbbk[\mathbf{G}], \Bbbk[\mathbf{I}])$-bimodule.*

(2) *$M \otimes_{A^\sharp} M^\vee \cong A$ and $M^\vee \otimes_A M \cong A^\sharp$, where A is the Rouquier unipotent block of $\Bbbk[\mathbf{G}]$ with e-core γ and $A^\sharp$ is the unipotent block of $\Bbbk[\mathbf{I}]$ with $(w+1)$-tuple of e-cores $(\emptyset^w, \gamma)$.*
(3) *M is left projective as well as right projective.*

5.2. **Images of some modules via the equivalence.** In order make a connection to q-Schur algebras we need to identify the images of modules under the equivalence in Theorem 13. We retain the assumptions on q and ℓ of the previous section. For $\lambda \vdash n$, we denote by $S(\lambda)$ the Specht $\Bbbk\,\mathbf{GL}_n(\mathbb{F}_q)$-module corresponding to λ (see [Jam84],[Jam86]), by $D(\lambda)$ its unique simple quotient, by $P(\lambda)$ the projective cover of $D(\lambda)$ and by $X(\lambda)$ the Young $\Bbbk\,\mathbf{GL}_n(\mathbb{F}_q)$-module corresponding to λ (see [DJ89] for the definition.)

The labels of simple modules over the principal block $B_0(\Bbbk[\mathbf{GL}_e(\mathbb{F}_q)\wr\mathfrak{S}_w])$ are given by e-multipartitions of w, which we denote by $\mathcal{MP}(e,w)$. This goes as follows: The principal block $B_0(\mathbf{GL}_e(\mathbb{F}_q))$ has e non-isomorphic simple modules $\{D(\mu) \mid \mu \text{ is a hook partition of } e\}$. For an e-tuple of non-negative integers $\underline{m} = (m_1, \ldots, m_e)$ such that $\sum_i m_i = w$, define $S(\underline{m})$ to be the $\mathbf{GL}_e(\mathbb{F}_q)^w$-module $\boxtimes_{i=1}^e (S(e-i+1, 1^{i-1})^{\boxtimes m_i})$. So, the Young subgroup $\mathfrak{S}_{\underline{m}}$ acts on $S(\underline{m})$. For an e-multipartition (= e-tuple of partitions) $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(e)})$, put $m_i := |\lambda^{(i)}|$ and then define the Specht $B_0(\Bbbk[\mathbf{GL}_e(\mathbb{F}_q) \wr \mathfrak{S}_w])$-module $S(\underline{\lambda})$ by

$$\mathrm{Ind}_{\mathbf{GL}_e(\mathbb{F}_q)^w \rtimes \mathfrak{S}_{\underline{m}}}^{\mathbf{GL}_e(\mathbb{F}_q)\wr\mathfrak{S}_w} \left(\boxtimes_{i=1}^e (S(e-i+1, 1^{i-1})^{\boxtimes m_i}) \otimes_{\Bbbk} S^{\lambda^{(i)}} \right).$$

If we replace $S(\star)$ by $D(\star)$ (resp. $X(\star)$), then we obtain simple modules (resp. Young modules) over $\mathbf{GL}_e(\mathbb{F}_q) \wr \mathfrak{S}_w$.

So, the labels of simple modules over $A^\sharp \cong B_0(\Bbbk[\mathbf{GL}_e(\mathbb{F}_q) \wr \mathfrak{S}_w]) \boxtimes B_\gamma$, where B_γ is the simple block algebra corresponding to the unipotent character indexed by e-core γ, are also given by $\mathcal{MP}(e,w)$.

Define a map $(-)^\natural : \mathcal{MP}(e,w) \to \mathcal{MP}(e,w)$ by

$$((\overline{\lambda})^\natural)_i := \begin{cases} \lambda^{(i)} & \text{if } i+e \text{ is even,} \\ (\lambda^{(i)})^{\mathrm{tr}} & \text{otherwise.} \end{cases}$$

Here, the 0-th runner of an e-quotient is treated as the 1st entry of an e-multipartition.

By Hida-Miyachi [Miy01] the images of simple, Specht, Young and projective indecomposable modules over $A^\sharp$ via M are determined explicitly.

Theorem 14 (Hida-Miyachi). *For $K \in \{D, S, X, P\}$ and any $\lambda \in \Lambda$,*

$$K(\lambda) \cong M \otimes_{A^\sharp} K(\overline{\lambda}^\natural).$$

Remark 15. *The proof of this result will given in the appendix, thanks to Akihiko Hida's permission. At the level of combinatorics of indices λ for modules $S(\lambda), D(\lambda), X(\lambda), P(\lambda)$ the result above is identical with* [CT03] *up to replacing p in Chuang-Tan by e in Hida-Miyachi.*

For a finite group $\mathbf{H}$ with $|\mathbf{H}|^{-1} \in \Bbbk$, put

$$\mathbf{e_H} := \frac{1}{|\mathbf{H}|}\sum_{h\in\mathbf{H}} h.$$

Let $\mathbf{B}$ (resp. $\mathbf{B_L}$) be a standard Borel subgroup of $\mathbf{G}$ (resp. $\mathbf{L}$).

Put $B := \mathbf{e_B}A\mathbf{e_B}$ and $B^\sharp := \mathbf{e_{B_L}}A^\sharp\mathbf{e_{B_L}}$. Then, by the origin of Iwahori-Hecke algebras we can regard B (resp. $B^\sharp$) as a block of $\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_r)$ (resp. $(\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_e)^{\otimes w} \rtimes \Bbbk[\mathfrak{S}_w]) \otimes_\Bbbk \mathcal{H}_{\Bbbk,q}(\mathfrak{S}_{|\gamma|})$).

By applying Schur functors, we obtain the Hecke algebra version of Theorem 14 (cf [Tur05, Theorem 78]):

Corollary 16. *$Y^\lambda \cong N \otimes_{B^\sharp} Y^{(\overline{\lambda})^\natural}$ for any $\lambda \in \Lambda$ where $Y^\lambda = \mathbf{e_B}X_e(\lambda)$, $Y^{\overline{\nu}} = \mathbf{e_{B_L}}X(\overline{\nu})$ and $N = \mathbf{e_B}M\mathbf{e_{B_L}}$.*

Remark 17. *Here, we label Young modules according to Dipper-James's convention. Namely, Y^λ is a unique indecomposable direct summand of the q-permutation module $\mathrm{Ind}_{\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_\lambda)}^{\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_r)}(\mathrm{ind})$ such that Y^λ contains S^λ as a unique submodule where S^λ is Dipper-James's q-Specht module and* ind *is the index representation.*

5.3. **Proof of main theorem : initial step.** In this subsection, our assumptions on $\Bbbk$ and q are as in the statement of Theorem 2: $q \in \mathbb{F}_\ell^\times$ and $\ell > w$, or $q \in \mathbb{C}$, *i.e.* we include the cases $ch(\Bbbk) = 0$ or $e = ch(\Bbbk) > w$. Let γ be the Rouquier e-core with respect to $w > 0$. Put $r = ew + |\gamma|$. We denote by A^w (resp. A_w) the Rouquier block of $\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_r)$ (resp. $\mathcal{S}_{\Bbbk,q}(r)$). We denote $B_0(\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_e))^{\boxtimes w} \boxtimes_\Bbbk B^\gamma$ (resp. $B_0(\mathcal{S}_{\Bbbk,q}(e))^{\boxtimes w} \boxtimes_\Bbbk B_\gamma$) by B^w (resp. B_w) where B^γ (resp. B_γ) is the defect zero simple block algebra of $\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_{|\gamma|})$ (resp. $\mathcal{S}_{\Bbbk,q}(|\gamma|)$) corresponding to the e-core γ. $\mathfrak{S}_w$ acts on both B^w and B_w by permuting the $\mathcal{H}_{\Bbbk,q}(\mathfrak{S}_e)$-components and the $\mathcal{S}_{\Bbbk,q}(e)$-components respectively. We denote $B^w \rtimes \Bbbk[\mathfrak{S}_w]$ (resp. $B_w \rtimes \Bbbk[\mathfrak{S}_w]$) by C^w (resp. C_w). Similar to the case $GL_e(\mathbb{F}_q) \wr \mathfrak{S}_w$, by Clifford theory we can construct a standard C_w-module $\Delta(\underline{\lambda})$ for an e-multipartition $\underline{\lambda}$: $\Delta(\underline{\lambda}) := \mathrm{Ind}_{B_w\rtimes\mathfrak{S}_{\underline{m}}}^{B_w\rtimes\mathfrak{S}_w}\left(\bigboxtimes_{i=1}^{e}(\Delta(i,1^{e-i})^{\boxtimes m_i}) \otimes_\Bbbk S^{\lambda^{(i)}}\right)$ where $m_i = |\lambda^{(i)}|$ for $i = 1,\ldots,e$.

By replacing Δ by any $K \in \{L,\nabla,P,I,T\}$, we can construct $K(\underline{\lambda})$.

Now, we unify the results in [CT03] on Schur algebras, Hida-Miyachi [Miy01] on finite general linear groups and a new result in characteristic zero into q-Schur algebras as follows:

Theorem 18. *There exists an (A^w, C^w)-bimodule M^w such that*

1. *M^w is a direct summand of $A^w \otimes_{B^w} C^w$,*
2. *$M^w \otimes_{C^w} -$ is an equivalence between A^w-mod and C^w-mod.*
3. *$Y^\lambda \cong M^w \otimes_{C^w} Y^{(\overline{\lambda})^\natural}$ for any λ such that $A^w S^\lambda \neq 0$.*

So, we have an equivalence $\mathsf{G} : A_w\text{-mod} \to C_w\text{-mod}$ such that $\mathsf{G}(K(\lambda)) \cong K((\overline{\lambda})^\natural)$ for any $K \in \{L,\Delta,\nabla,P,I,T\}$ and any λ so that $A_w \cdot \Delta(\lambda) \neq 0$.

Proof. The case $ch(\Bbbk) = e > 0$ is already treated in [CT03], and the statement is true. Next we look at the case $ch(\Bbbk) > e > 0$. However, this is nothing but Theorem 13 and Corollary 16.

The only remaining claim is to prove the statement for the case $ch(\Bbbk) = 0$. We leave the proof of the existence of M^w in characteristic zero to Section 6. There, by considering a modular system $(\Bbbk, \mathcal{O}, \mathbb{F})$ where $ch(\Bbbk) = 0$, $\mathcal{O}$ is a complete discrete valuation ring with maximal ideal π, $\mathcal{O}/(\pi) \cong \mathbb{F}, ch(\mathbb{F}) > 0$, and $q \in \Bbbk$ and $\overline{q} = q + (\pi) \in \mathbb{F}$ both have order $e > 0$, we shall realize M^w as a lifting of a $(\mathcal{H}_{\mathbb{F},\overline{q}}(\mathfrak{S}_r), \mathcal{H}_{\mathbb{F},\overline{q}}(\mathfrak{S}_e) \wr \mathfrak{S}_w \boxtimes \mathcal{H}_{\mathbb{F},\overline{q}}(\mathfrak{S}_{|\gamma|}))$-bimodule $M^w_{\mathbb{F}}$ which satisfies statements (1), (2) and (3). Since A^w, B^w and C^w are liftings of the corresponding algebras over $\mathbb{F}$, statement (2) is clear. By statement (1), we can ensure that M^w sends Young modules to Young modules. So, statement (3) follows from a simple chasing of characters, *i.e.* the weights (partitions) λ. $\square$

Define Λ_w to be the set of partitions whose e-weight is w and e-core is Rouquier with respect to w. Similarly, define Λ'_w for e'.

Define ι to be the embedding $\mathcal{MP}(e, w)$ into $\mathcal{MP}(e', w)$ by $\iota(\overline{\nu})_{i+e'-e} := (\overline{\nu})_i$ for $i = 1, \ldots, e$ and $(\overline{\nu})_j = \emptyset$ for $j \le e' - e$.

Next, we define an embedding ι of Λ_w into Λ'_w by $\iota(\lambda) := \nu$ where the e'-quotient of ν is $\iota(\overline{\lambda})$.

From now on, we suppose that e' is the quantum characteristic of $q' \cdot 1_{\mathbb{F}_\ell}$ in $\Bbbk$ so that $e' \ge e$, γ' is a Rouquier e'-core with respect to w. Put $r' = e'w + |\gamma'|$.

Lemma 19. (1) *Let f be an idempotent of C'_w such that if $\underline{\lambda}^{(i)} = \emptyset$ for all $i = 1, \ldots, e' - e$, then $fL'(\underline{\lambda}) \ne 0$, otherwise $fL'(\underline{\lambda}) = 0$. Then, fC'_wf is Morita equivalent to C_w.*

(2) *There exists an idempotent ξ of A'_w such that $\xi L'(\iota(\lambda)) \ne 0$ for any $\lambda \in \Lambda_w$ and $\xi A'_w \xi$ is Morita equivalent to A_w.*

Proof. Recall the definition of $C'_w = B'_w \rtimes \mathfrak{S}_w \cong (B_0(\mathcal{S}_{\Bbbk,q'}(e')) \wr \mathfrak{S}_w) \boxtimes B_\gamma$. Put B' (resp. B) to be $B_0(\mathcal{S}_{\Bbbk,q'}(e'))$ (resp. $B_0(\mathcal{S}_{\Bbbk,q}(e))$.) Take an idempotent ξ of B' such that $\xi L'(i, 1^{e'-i}) = 0$ for any $i \le e' - e$ and $\xi L'(j, 1^{e'-j}) \ne 0$ for any $j > e' - e$. Then, $\xi B' \xi$ is Morita equivalent to B. Indeed, we can show this by the fact that the both B and B' are Brauer line tree algebras with no exceptional vertex. Let $f := \xi \boxtimes \cdots \boxtimes \xi \boxtimes 1_\gamma$ be the idempotent of $B^{\boxtimes w} \boxtimes B_\gamma$. So, fB'_wf is Morita equivalent to B_w. Now, by taking wreath products, we know that (1) is clear.

By Theorem 18, we have an idempotent j of C'_w corresponding to ξ so that $j \cdot \mathsf{G}(L(\iota(\lambda))) \ne 0$ for any λ. By definition of ι we know that for $\mu \in \Lambda_w$, $\mu^{(i)} = 0$ for all $0 \le i \le e' - e - 1$ iff $\mu = \iota(\lambda)$ for some $\lambda \in \Lambda$. So, this means that j satisfies the condition of (1). $\square$

Define $B, B', \underline{B}, \underline{B}', \Lambda, \Lambda', \underline{\Lambda}, \underline{\Lambda}'$ as in Section 2, taking α to be 0.

Let f'' to be an idempotent of A'_w corresponding to $\bigoplus_{\underline{\Lambda} \in \Lambda} P(\iota(\lambda))$.

By an argument similar to Corollary 11, one can show the following:

Lemma 20. *Suppose $B = A_w$. Then, there exists an equivalence*

$$\mathsf{H} : f''A'_w f''\text{-mod} \to \underline{B}'\text{-mod}$$

such that

$$\mathsf{H}(f''\Delta'(\iota(\mu))) \cong \underline{\Delta}'(\mu^+) \text{ and } \mathsf{H}(f''L'(\iota(\mu))) \cong \underline{L}'(\mu^+) \text{ for any } \mu \in \underline{\Lambda}.$$

Proposition 21. *Suppose that $A_w = B$. There exists an equivalence F : $\underline{B}$-mod $\to \underline{B}'$-mod such that $\mathsf{F}(\underline{\Delta}(\lambda)) \cong \underline{\Delta}'(\lambda^+)$ and $\mathsf{F}(\underline{L}(\lambda)) \cong \underline{L}'(\lambda^+)$ for any $\lambda \in \underline{\Lambda}$. Namely, the main theorem is true for Rouquier blocks.*

Proof. By Lemma 19, we know that there exists an idempotent ξ of A'_w such that $\xi L'(\lambda^+) \neq 0$ for any $\lambda \in \Lambda$ and $\xi A'_w \xi$ is Morita equivalent to A_w. Let f be the idempotent of A_w such that $fA_w f$ is a subalgebra of $\mathcal{S}_{\Bbbk,q}(d,r)$, *i.e.* f kills $L(\mu)$ for $l(\mu) > d$ and $fL(\lambda) \neq 0$ for $l(\lambda) \leq d$. Similarly, we can find that the property of an idempotent f'' of A'_w defined above is that $f''A'_w f''$ is a subalgebra of $\mathcal{S}_{\Bbbk,q}(d,r')$, *i.e.* f'' kills $L'(\mu)$ for $l(\mu) > d$ and $f''L'(\lambda) \neq 0$ for $l(\lambda) \leq d$ and $\xi = f'' + \xi'$ for some idempotent ξ'. Let T be a functor from $fA_w f$-mod to $f''A'_w f''$-mod such that $\mathsf{T}(f\Delta(\lambda)) \cong f''\Delta'(\iota(\lambda))$, *i.e.* T is a restriction of the equivalence of Lemma 19.

Since $\max\{l(\lambda) \mid \lambda \in \underline{\Lambda}\} = \max\{l(\lambda^+) \mid \lambda \in \underline{\Lambda}\} = d$ by Theorem 18 and by definition of ι, we know that $fL(\lambda) \neq 0$ iff $f'L'(\iota(\lambda)) \neq 0$, and we know that the dominance order of $\underline{\Lambda}$ is preserved by the ι map and the dominance order of $\underline{\Lambda}'$ is preserved by the $+$ map. Therefore, the composition of equivalences T and H in Lemma 20 is an equivalence between $\underline{B}$-mod and $\underline{B}'$-mod, which satisfies the conditions on the images of Δ-sections. □

6. Lifting Morita equivalences

In this section we supply a missing argument for the proof of Theorem 18. We will choose an appropriate modular system to work in with the help of the following lemmas.

Let $\zeta_n = \exp(2\pi i/n) \in \mathbb{C}$, and denote by $\Phi_n(x) \in \mathbb{Z}[x]$ the n-th cyclotomic polynomial.

Lemma 22. *Suppose that $a \in \mathbb{Z}[\zeta_e]$, $a \neq 0$ and $e > 1$. Then, there exists a prime number ℓ and a homomorphism $\bar{\cdot} : \mathbb{Z}[\zeta_e] \to \mathbb{F}_\ell$ such that $q := \overline{\zeta_e}$ is a primitive e-th root of unity in $\mathbb{F}_\ell$ and so that $\overline{a} \neq 0$.*

Proof. Choose $f(x) \in \mathbb{Z}[x]$ such that $f(\zeta_e) = a$. Then, $a \neq 0$ implies $\Phi_e(x) \nmid f(x)$. Since $\Phi_e(x)$ is monic, there exist $Q(x)$ and $r(x) \neq 0$ in $\mathbb{Z}[x]$ so that

(1) $\deg(r) < \deg(\Phi_e)$
(2) $f(x) = Q(x)\Phi_e(x) + r(x)$.

By Dirichlet's Theorem there exist infinitely many prime numbers $\ell > 0$ such that $e \mid \ell - 1$. Choose one such ℓ so that $\overline{r}(x) \neq 0$. We complete the proof of the lemma by showing that there exists a primitive e-th root of unity q in $\mathbb{F}_\ell$ satisfying $\overline{f}(q) \neq 0$. If not, $(x - q)$ divides $\overline{f}(x)$ in $\mathbb{F}_\ell[x]$ for all primitive e-th roots of unity q. Hence, $\overline{\Phi_e}(x)$ divides $\overline{f}(x)$ in $\mathbb{F}_\ell[x]$. So, by (2) above, we deduce that $\overline{\Phi_e}(x)$ divides $\overline{r}(x) \neq 0$ in $\mathbb{F}_\ell[x]$, contradicting (1). □

Lemma 23. *Let Γ be a $\mathbb{Z}[\zeta_e]$-algebra, free and of finite rank over $\mathbb{Z}[\zeta_e]$, and let X be a Γ-lattice of finite rank. There exists a prime ℓ and a homomorphism $\bar{\cdot} : \mathbb{Z}[\zeta_e] \to \mathbb{F}_\ell$ such that $q := \overline{\zeta_e}$ is a primitive e-root of unity in $\mathbb{F}_\ell$ and*

$$\dim \mathrm{End}_{\mathbb{Q}(\zeta_e)\otimes_{\mathbb{Z}[\zeta_e]}\Gamma}(\mathbb{Q}(\zeta_e) \otimes_{\mathbb{Z}[\zeta_e]} X) = \dim \mathrm{End}_{\mathbb{F}_\ell\otimes_{\mathbb{Z}[\zeta_e]}\Gamma}(\mathbb{F}_\ell \otimes_{\mathbb{Z}[\zeta_e]} X).$$

Proof. Let $\bar{\cdot} : \mathbb{Z}[\zeta_e] \to \mathbb{F}_\ell$ be a homomorphism, and for $R \in \{\mathbb{Q}(\zeta_e), \mathbb{Z}[\zeta_e], \mathbb{F}_\ell\}$ put $RX = R\otimes_{\mathbb{Z}[\zeta_e]} X$ and $R\Gamma = R\otimes_{\mathbb{Z}[\zeta_e]} \Gamma$. Let I be a finite set of generators for Γ as a $\mathbb{Z}[\zeta_n]$-algebra (we could for example take I to be a basis). Then $\mathrm{End}_R\,\Gamma(RX)$ is the kernel of the R-homomorphism f_R of $\mathrm{End}_R(RX)$ into $\bigoplus_{g\in I} \mathrm{End}_R(RX)$ defined by

$$f_R(x)_g := x \circ g - g \circ x \in \mathrm{End}_R(RX).$$

Let M_f be the matrix representing $f_{\mathbb{Z}[\zeta_e]}$ with respect to some chosen $\mathbb{Z}[\zeta_e]$-bases. Let $a \in \mathbb{Z}[\zeta_e]$ be the product of all nonzero minors of M_f. By Lemma 22 we may choose $\bar{\cdot} : \mathbb{Z}[\zeta_e] \to \mathbb{F}_\ell$ such that $\overline{\zeta_e}$ is a primitive e-root of unity in $\mathbb{F}_\ell$ and $\overline{a} \neq 0$. Then the ranks of M_f as a matrix over $\mathbb{Q}(\zeta_e)$ and over $\mathbb{F}_\ell$ are the same, and it follows that $\mathrm{End}_{\mathbb{F}_\ell\Gamma}(\mathbb{F}_\ell X)$ and $\mathrm{End}_{\mathbb{Q}(\zeta_e)\Gamma}(\mathbb{Q}(\zeta_e)X)$ have the same dimension. □

We're ready to return to Theorem 18. Let γ be a Rouquier e-core with respect to w. Put $r = ew + |\gamma|$. For any domain R and any $\zeta \in R^\times$ let $A_{R,\zeta} = \mathcal{H}_{R,\zeta}(\mathfrak{S}_r)$, $B_{R,\zeta} = \mathcal{H}_{R,\zeta}(\mathfrak{S}_{(e^w,\gamma)})$ and $C_{R,\zeta} = B_{R,\zeta} \rtimes R[\mathfrak{S}_w]$. Further, let $\Gamma_{R,\zeta} = A_{R,\zeta} \otimes_R C_{R,\zeta}$, and let $X_{R,\zeta}$ be the $\Gamma_{R,\zeta}$-module $A_{R,\zeta} \otimes_{B_{R,\zeta}} C_{R,\zeta}$.

Now consider in particular $\Gamma = \Gamma_{\mathbb{Z}[\zeta_e],\zeta_e}$ and $X = X_{\mathbb{Z}[\zeta_e],\zeta_e}$, and choose a prime ℓ and a homomorphism $\bar{\cdot} : \mathbb{Z}[\zeta_e] \to \mathbb{F}_\ell$ according to Lemma 23. Note that X is free over $\mathbb{Z}[\zeta_e]$ since $A = A_{\mathbb{Z}[\zeta_e],\zeta_e}$ and $C = C_{\mathbb{Z}[\zeta_e],\zeta_e}$ are free over $B = B_{\mathbb{Z}[\zeta_e],\zeta_e}$.

Let $\mathcal{O}$ be the completion of $\mathbb{Z}[\zeta_e]$ at the kernel of $\bar{\cdot}$, so that $\mathcal{O}$ is a complete discrete valuation ring. By Lemma 23, the natural embedding $\mathbb{F}_\ell\,\mathrm{End}_\Gamma(X) \hookrightarrow \mathrm{End}_{\Gamma_{\mathbb{F}_\ell,q}}(X_{\mathbb{F}_\ell,q})$ must be an isomorphism. We saw in the proof of Theorem 18 that there exists a summand $M_{\mathbb{F}_\ell,q}$ of $X_{\mathbb{F}_\ell,q}$ with certain good properties. A projection onto this summand determines an idempotent of $\mathrm{End}_{\Gamma_{\mathbb{F}_\ell,q}}(X_{\mathbb{F}_\ell,q})$ which we may lift to an idempotent of $\mathrm{End}_\Gamma(X)$. We obtain in this way a summand M of X, with the property that $\mathbb{F}_\ell M \cong M_{\mathbb{F}_\ell,q}$. Passing now to the quotient field $\Bbbk$ of $\mathcal{O}$, we obtain a $\Gamma_{\Bbbk,\zeta_e}$-module $\Bbbk M$ that settles Theorem 18 in the characteristic 0 case.

We end this section by remarking that the lifting technique we have employed is applicable in some other situations involving 'Rouquier-like' blocks in Hecke algebras of other types. The set up is as follows:

Let W be a finite Weyl group. Let W_L be a parabolic subgroup of W. Take a subgroup $\overline{N} \subset N_W(W_L)/W_L$.

Let $A_{R,\zeta} = \mathcal{H}_{R,\zeta}(W)$, $B_{R,\zeta} = \mathcal{H}_{R,\zeta}(W_L)$ and $C_{R,\zeta} = B_{R,\zeta} \rtimes R[\overline{N}]$. Exactly as above we let $\Gamma_{R,z} = A_{R,\zeta} \otimes_R C_{R,\zeta}$, and let $X_{R,\zeta}$ be the $\Gamma_{R,\zeta}$-module $A_{R,\zeta} \otimes_{B_{R,\zeta}} C_{R,\zeta}$. Recall that the classification of blocks of $A_{R,\zeta}$, where R is a field, only depends on the multiplicative order of ζ provided that the characteristic of R is either 0 or sufficiently large ([FS82], [FS89], [DJ87], [DJ92], [GR97]).

Proposition 24. *We fix e. Let $\mathfrak{A}$ be a block of $A_{\mathbb{Q}(\zeta_{2e}),\zeta_e}$ and $\mathfrak{C}$ be a block of $C_{\mathbb{Q}(\zeta_{2e}),\zeta_e}$. Suppose that for all sufficiently large primes ℓ there exist a primitive e-th root of unity q in $\mathbb{F}_\ell$ and an $(A_{\mathbb{F}_\ell,q}, C_{\mathbb{F}_\ell,q})$ -bimodule M_ℓ such that*

1. *M_ℓ is a direct summand of $X_{\mathbb{F}_\ell,q}$*
2. *M_ℓ induces a Morita equivalence between the blocks of $A_{\mathbb{F}_\ell,q}$ and $C_{\mathbb{F}_\ell,q}$ that correspond to $\mathfrak{A}$ and $\mathfrak{C}$.*

Then, there exists a direct summand M_0 of $X_{\mathbb{Q}(\zeta_{2e}),\zeta_e}$ inducing a Morita equivalence between $\mathfrak{A}$ and $\mathfrak{C}$.

Example 25. *Let $W(X_r)$ be the finite Weyl group of type X_r. Put $\zeta := \sqrt{-1}$, a primitive 4-th root of unity. Put $A_{R,\zeta} = \mathcal{H}_{R,\zeta}(W(E_6))$, $B_{R,\zeta} = \mathcal{H}_{R,\zeta}(W(D_4))$, and $C_{R,\zeta} = \mathcal{H}_{R,\zeta,1}(W(F_4)) = B_{R,\zeta} \rtimes R[\mathfrak{S}_3]$ where the parameters of $\mathcal{H}_{R,\zeta,1}(W(F_4))$ are ζ and 1. Define $\Gamma_{R,\zeta}$ and $X_{R,\zeta}$ as above. Suppose that $q \cdot 1_{\mathbb{F}_\ell} \in \mathbb{F}_\ell$, $\ell \mid q^2+1$, and ℓ is sufficiently large.* [2] *Then, by Geck's result on Schur index* [Gec03] *and the equivalence on blocks for finite Chevalley groups $E_6(q)$ and $D_4(q) \rtimes \mathfrak{S}_3$ ($D_4(q)$ with a triality automorphism group) in* [Miy08] *there exists an $(A_{\mathbb{F}_\ell,q}, C_{\mathbb{F}_\ell,q})$-bimodule M_ℓ such that*

1. *M_ℓ is a direct summand of $X_{\mathbb{F}_\ell,q}$.*
2. *M_ℓ induces a Morita equivalence between the prinicipal blocks of $A_{\mathbb{F}_\ell,q}$ and $C_{\mathbb{F}_\ell,q}$.*

So, by Proposition 24, we have the corresponding result in characteristic zero.

7. Quantized enveloping algebras

7.1. **Guessing an analogue of the main theorem.** The main theorem suggests an analogous statement for quantized enveloping algebras. Before stating it we introduce the necessary notation, following Jantzen [Jan03].

Let $\mathfrak{g}$ be a reductive complex Lie algebra. Let k be a commutative ring and q an invertible element of k. Let $U_{q,k} = U_{\mathcal{A}} \otimes_{\mathcal{A}} k$, where $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ and $U_{\mathcal{A}}$ is the divided powers integral form of the quantized enveloping algebra of $\mathfrak{g}$.

[2] Here, the assumption 'ℓ : sufficiently large' is used to make sure that $\mathbb{F}_\ell$ is a splitting field for the principal ℓ-blocks of corresponding finite Chevalley groups $E_6(q)$ and $D_4(q)$. So, we require [Gec03]. The claimed equivalence in finite Chevalley groups does always exist in characteristic $\ell > 3$ such that $\ell \mid q^2 + 1$ by some extension of $\mathbb{F}_\ell$.

Assume that k is a field of characteristic 0, and q is a primitive e-th root of unity, where e is odd and 3 doesn't divide e if $\mathfrak{g}$ has a component of type G_2. Let $U_{q,k}$-mod be the category of finite-dimensional $U_{q,k}$-modules of type I (see [Jan03, p.523] for definition).

For each dominant weight $\lambda \in X(T)_+$ there is a simple $U_{q,k}$ module $L_q(\lambda)$ of type I with highest weight λ, which is unique up to isomorphism. Every object of $U_{q,k}$-mod has a composition series with factors of the form $L_q(\lambda)$.

Let W_e be the affine Weyl group, acting on $X(T) \otimes_{\mathbf{Z}} \mathbf{R}$. Let

$$\bar{C}_e = \{\lambda \in X(T) \otimes_{\mathbf{Z}} \mathbf{R} \mid 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq e \text{ for all } \alpha \in R^+\}$$

be the closure of the standard e-alcove (see [Jan03, p.233]); it is a fundamental domain for this action.

Let q' be a primitive e'-th root of unity in k, and let $U_{q',k}$ be the corresponding quantum group in obvious fashion. We place the same restrictions on e' as we do on e above, and further we assume $e' \geq e$.

There is an isomorphism $f : W_e \to W_{e'}$ sending $s_{\beta,ne}$ to $s_{\beta,ne'}$ for all $\beta \in R$ and $n \in \mathbf{Z}$. Here, we use the notation $s_{\beta,r}$ of [Jan03, p.232, 6.1]. Note that the actions of $w \in W_e$ and $f(w) \in W_{e'}$ on $X(T)$ are different!

The inclusion of $\bar{C}_e$ into $\bar{C}_{e'}$ sends some walls of $\bar{C}_e$ into the interior of $\bar{C}_{e'}$, so that affine Weyl group stabilizers are not preserved. To 'correct' this consider a injective map $\iota : \bar{C}_e \hookrightarrow \bar{C}_{e'}$ with the property that for all $\lambda \in \bar{C}_e$ and $\alpha \in R^+$, we have $\langle \lambda + \rho, \alpha^\vee \rangle = \langle \iota(\lambda) + \rho, \alpha^\vee \rangle$ unless $\langle \lambda + \rho, \alpha^\vee \rangle = e$, in which case $\langle \iota(\lambda) + \rho, \alpha^\vee \rangle = e'$. Such a map always exists, and is unique if G is semisimple. Moreover we can always choose it to be the identity map on the interior of $\bar{C}_e$.

Lemma 26. *Let $\lambda \in X(T) \cap \bar{C}_e$ and $w \in W_e$. Then $w \cdot \lambda \in X(T)_+$ if and only if $f(w) \cdot \iota(\lambda) \in X(T)_+$.*

Guess 27. *There is a full k-linear embedding*

$$F : U_{q,k}\text{-mod} \to U_{q',k}\text{-mod}$$

such that for all $\lambda \in X(T) \cap \bar{C}_e$ and $w \in W_e$ with $w \cdot \lambda \in X(T)_+$, we have

$$F(L_q(w \cdot \lambda)) \cong L_{q'}(f(w) \cdot \iota(\lambda)).$$

The image of F is a sum of blocks of $U_{q',k}$-mod.

7.2. **The case $\mathfrak{gl}_d$.** The aim of this section is to confirm that Guess 27 is correct for $\mathfrak{g} = \mathfrak{gl}_d$.

7.2.1. *Weights, abaci and affine Weyl group actions.* We begin by making a link, in the $\mathfrak{gl}_d$ case, between the map $w \mapsto f(w)$ appearing above and the James-Mathas operation $\lambda \mapsto \lambda^+$. To this end we describe the dot actions of the affine Weyl group on weights in terms of abaci. We keep the notation in 7.1, taking the usual presentation

$$X(T)_+ = \{\lambda = \lambda_1\varepsilon_1 + \ldots + \lambda_d\varepsilon_d \in X(T) \mid \lambda_1 \geq \ldots \geq \lambda_d\}$$

for the set of dominant weights for $\mathfrak{gl}_d$.

The weights lying in the closure of the standard e-alcove are

$$X(T) \cap \bar{C}_e = \{\lambda \in X(T) \mid \ 0 \le (\lambda_i + d - i) - (\lambda_j + d - j) \le e \text{ for all } i < j\}.$$

Here ρ is usually taken to be half the sum of the positive weights. But it does no harm to work with a normalized version that has the same inner products with roots; so we have taken $\rho = (d-1)\varepsilon_1 + (d-2)\varepsilon_2 + \cdots + 0\varepsilon_d$.

We fix a ρ-shifted identification of $X(T)$ with $\mathbf{Z}^d$, sending $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_d\varepsilon_d \in X(T)$ to $(\lambda_1 + d - 1, \lambda_2 + d - 2, \ldots, \lambda_d) \in \mathbf{Z}^d$. This leads to an identification $\bar{C}_e \xrightarrow{\sim} \{\beta \in \mathbf{Z}^d \mid \beta_1 \ge \ldots \ge \beta_d \text{ and } \beta_1 - \beta_d \le e\}$. In this picture the action of the affine Weyl group $W_e \cong \mathbf{Z}^{d-1} \rtimes \mathfrak{S}_d$ on $X(T) = \mathbf{Z}^d$ is as follows:

$$\begin{aligned} \sigma.(\beta_1, \ldots, \beta_d) &= (\beta_{\sigma(1)}, \ldots, \beta_{\sigma(d)}) \\ (m_1, \ldots, m_{d-1}).(\beta_1, \ldots, \beta_d) &= (\beta_1 + em_1, \beta_2 + em_2 - em_1, \ldots, \beta_d - em_{d-1}) \end{aligned}$$

where $\sigma \in \mathfrak{S}_n$, $(m_1, \ldots, m_{d-1}) \in \mathbf{Z}^{d-1}$ and $(\beta_1, \ldots, \beta_d) \in \mathbf{Z}^d$. This is conveniently represented using James's abacus, with e runners and with d beads *labelled* by $1, \ldots, n$. The action of $\mathfrak{S}_d$ is given by permutating the labels, and that of $\mathbf{Z}^{d-1}$ by moving beads up and down runners. From this description the following lemma, making the combinatorial connection between Theorem 2 and Guess 27, is immediate. We say that a weight $\lambda \in X(T)$ is *polynomial* if λ is dominant and $\lambda_d \ge 0$, i.e. if $(\lambda_1, \ldots, \lambda_d)$ is a partition.

Lemma 28. *Let $\lambda \in X(T) \cap \bar{C}_e$ and $w \in W_e$, and suppose that $w.\lambda = \mu - (me - \lambda_d)(1^d)$ for some polynomial weight μ and some integer m. Then $f(w).\iota(\lambda) = \mu^{+,e} - (me' - \lambda_d)(1^d)$.*

7.2.2. *Proof for $\mathfrak{gl}_d$.* Here, we shall confirm Guess 27 for $\mathfrak{g} = \mathfrak{gl}_d$ by appealing to Theorem 2. We denote by $\mathcal{P}_0^e$ the category of polynomial representations over $U_{q,k}(\mathfrak{gl}_d)$, i.e. the full subcategory of $U_{q,k}(\mathfrak{gl}_d)$-mod consisting of modules with composition factors of the form $L_q(\lambda)$ where λ is a polynomial weight.

Tensoring with the representation $\det^{-m} = L(-m, \ldots, -m)$ induces a self-equivalence of $U_{q,k}(\mathfrak{gl}_d)$-mod sending $L(\lambda)$ to $L(\lambda - m(1^d))$; denote by $\mathcal{P}_m^e$ the essential image of $\mathcal{P}_0^e$ under this equivalence. Then we have an exhaustive filtration $\mathcal{P}_0^e \subset \mathcal{P}_1^e \subset \ldots$ of $U_{q,k}(\mathfrak{gl}_d)$-mod.

It is known that $\mathcal{P}_0^e$ is equivalent to $\bigoplus_{r=0}^{\infty} \mathcal{S}_{k,q^2}(d,r)$-mod, in a way preserving labels on simple modules. Hence by the main theorem 2 there exists a full embedding $G_0 : \mathcal{P}_0^e \hookrightarrow \mathcal{P}_0^{e'}$ sending $L_q(\mu)$ to $L_{q'}(\mu^+)$ for all polynomial weights μ.

We get for each $m \ge 0$ a corresponding embedding $G_m : \mathcal{P}_{me}^e \to \mathcal{P}_{me'}^{e'}$, such that $G_m(L(\mu - me(1^d))) \cong L(\mu^+ - me'(1^d))$ for any polynomial weight

μ. Since $\mu^+ - me'(1^d) = (\mu - me(1^d))^+$, we have a commutative diagram:

$$\begin{array}{cccc} G_m: & \mathcal{P}^e_{me} & \hookrightarrow & \mathcal{P}^{e'}_{me'} \\ & \cup & & \cup \\ G_{m-1}: & \mathcal{P}^e_{(m-1)e} & \hookrightarrow & \mathcal{P}^{e'}_{(m-1)e'}. \end{array}$$

By taking the limit $m \to \infty$, we have a full embedding

$$G := \lim_{m\to\infty} G_m : U_{q,k}(\mathfrak{gl}_d)\text{-mod} = \bigcup_{m=0}^{\infty} \mathcal{P}^e_{me} \to U_{q',k}(\mathfrak{gl}_d)\text{-mod} = \bigcup_{m=0}^{\infty} \mathcal{P}^{e'}_{me'}$$

This isn't quite the functor we want; a slight adjustment is required. By the linkage principle, $U_{q,k}(\mathfrak{gl}_d)$-mod $= \oplus_{\lambda\in\bar{C}\cap X(T)}\mathcal{M}_{W_e.\lambda}$, where $\mathcal{M}_{W_e.\lambda}$ is the full subcategory consisting of modules with composition factors of the form $L_q(w.\lambda)$. Let Z be the self equivalence of $U_{q,k}(\mathfrak{gl}_d)$-mod whose restriction to $\mathcal{M}_{W_e.\lambda}$ is tensoring with $\det^{\lambda_d}$. One can define a self equivalence Z' of $U_{q',k}(\mathfrak{gl}_d)$-mod analogously.

Let $F := Z'GZ^{-1} : U_{q,k}(\mathfrak{gl}_d)$-mod $\to U_{q',k}(\mathfrak{gl}_d)$-mod. Then for each $\lambda \in X(T) \cap \bar{C}_e$, the functor F restricts to an equivalence $\mathcal{M}_{W_e.\lambda} \xrightarrow{\sim} \mathcal{M}_{W_{e'}.\lambda}$ sending $L_q(\mu-(me-\lambda_d)(1^d))$ to $L_{q'}(\mu^+-(me'-\lambda_d)(1^d))$ for any polynomial weight μ. By Lemma 28, this implies that $F(L_q(w.\lambda)) \cong L_{q'}(f(w).\iota(\lambda))$, as desired.

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