Error exponents for entanglement concentration

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Abstract

Asymptotic entanglement concentration with exponentially decreasing error probability is discussed. Distillable entanglement is derived as a function of an error exponent. The formula links the upper bound of distillable entanglement, which is the well-known entropy of entanglement, with the lower bound attained in deterministic concentration. A strong converse of asymptotic entanglement concentration is also presented.

Key words: entanglement, entanglement concentration, error exponent, type

1 Introduction

Quantification of entanglement is the key to understanding of its full potential as an indispensable resource for quantum information processing, in superdense coding [1], quantum teleportation [2], quantum cryptography [3], and quantum computing [4]. The essentials of bipartite pure-state entanglement have already been revealed both in finite regimes and in the asymptotic limit: the intimate connection between the mathematical theory of majorization and entanglement manipulation [5, 6, 7, 8], and the existence of a unique measure of entanglement in the asymptotic limit [9, 10].

One way of quantifying entanglement is to estimate the number of Bell pairs,

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{1}$$

distilled from a given entangled state by local operations and classical communication (LOCC). Though the above quantity of distillable entanglement can be defined for mixed states, we deal with only pure states here. In order to make use of partially entangled states for quantum teleportation, we need to convert them into maximally entangled states by LOCC. The process is called entanglement concentration, whose efficiency in the asymptotic limit is the focus of this paper.

The unique measure of bipartite pure-state entanglement gives the limitation on the efficiency of entanglement concentration. Suppose we share n identical copies of a partially entangled state

$$|\phi\rangle = \sum_{i=1}^{d} \sqrt{p_i}|i\rangle|i\rangle, \tag{2}$$

where the Schmidt coefficients squared are arranged in decreasing order, i.e., $p_1 \geq p_2 \geq \cdots \geq p_d \geq 0$, and sum to one. Bennett *et al.* [9] proved that the maximum number of Bell pairs distilled per copy from $|\phi\rangle^{\otimes n}$ is given by

$$E_{entropy}(\phi) = -\sum_{i=1}^{d} p_i \log p_i, \tag{3}$$

in the asymptotic limit, $n \to \infty$. (Logarithms are taken to base two throughout this paper.) They imposed the condition that the success probability of entanglement concentration tends to one in the asymptotic limit, i.e.,

$$P_{success} = 1 - \epsilon, \tag{4}$$

where

$$\epsilon \to 0 \quad \text{as} \quad n \to \infty. \tag{5}$$

With this restriction, the maximum attainable entanglement yield is proved to be Eq. (3).

On the other hand, many researches on entanglement concentration have been undertaken from various viewpoints [7, 8, 11, 12, 13, 14]. Among other things, the bound on entanglement yield in deterministic concentration [13]

$$E_{det}(\phi) = -\log p_1, \tag{6}$$

gives another quantification of entanglement. The restriction of *deterministic* means that the process succeeds with probability one both in finite regimes and in the asymptotic limit.

Though the quantities $E_{entropy}$ and E_{det} give entanglement yield in the asymptotic limit, where both processes succeed with probability one, the two quantities do not coincide. The main purpose of this paper is to find out the reason for the discrepancy. We will see that it is caused by the difference of the rate error probabilities decrease when n tends to infinity in both concentration processes. Roughly speaking, while we obtain $E_{entropy}$ when error probability decreases slowly, we obtain E_{det} when it decreases rapidly. We will assume that the error probability exponentially decreases, and represent the rate by the *exponent* of the error probability. This is a common approach in the information sciences, and will allow us to 'tune' between the two extremes just mentioned.

In the derivation of $E_{entropy}$, we use the asymptotic equipartition property [15]. However, a detailed analysis of the asymptotic behavior requires more powerful mathematical tools; namely, the method of types [15, 16], which makes it possible to calculate the probabilities of rare events and derive stronger results than when we focus only on typical sequences.

The argument via the method of types gives entanglement yield as a function of an error exponent and reveals the missing link between $E_{entropy}$ and E_{det}. In addition, we will also see that the success probability exponentially decreases when we try to distill more entanglement than $E_{entropy}$ (strong converse).

2 Asymptotic entanglement concentration

This section presents asymptotic entanglement concentration with exponentially decreasing failure probability. Suppose we wish to distill a maximally entangled state with the greatest possible Schmidt number L_n from n identical copies of $|\phi\rangle$, i.e., $|\phi\rangle^{\otimes n} = \sum_{\mathbf{i}} \sqrt{p^n(\mathbf{i})}|\mathbf{i}\rangle|\mathbf{i}\rangle$, where $p^n(\mathbf{i})$ is the n -i.i.d extension of p_i.

Let P_{L_n} be the optimal success probability with which we distill a maximally entangled state of size L_n from $|\psi\rangle^{\otimes n}$. We assume that the failure probability, $1 - P_{L_n}$ decreases exponentially as the number of the entangled pairs n increases. Then, the first order coefficient in the exponent of the failure probability in the asymptotic limit is called an *error exponent*, r, which is defined as

$$r = \lim_{n\to\infty} -\frac{1}{n}\log(1 - P_{L_n}). \tag{7}$$

Intuitively, this means that the error probability behaves as 2^{-nr}.

We formulate the maximum number of Bell pairs distilled per copy in the asymptotic limit

$$E = \lim_{n\to\infty} \frac{1}{n}\log L_n, \tag{8}$$

as a function of the error exponent r by using the Shannon entropy $H(p)$ and the relative entropy $D(p \parallel q)$, i.e.,

$$H(p) = -\sum_{i=1}^{d} p_i \log p_i, \tag{9}$$

and

$$D(p \parallel q) = \sum_{i=1}^{d} p_i \log \frac{p_i}{q_i}, \tag{10}$$

where p and q are probability distributions.

Theorem 1 *In entanglement concentration of n identical copies of $|\phi\rangle = \sum_{i=1}^{d}\sqrt{p_i}|i\rangle|i\rangle$, i.e., $|\phi\rangle^{\otimes n}$, if we assume the failure probability decreases exponentially as the number of copies n increases, then the number of Bell pairs distilled per copy is given by*

$$E(r) = \min_{q:D(q\|p)\le r} \left\{D(q \parallel p) + H(q)\right\}, \tag{11}$$

where $r = \lim_{n\to\infty} -\frac{1}{n}\log(1 - P_{L_n})$ is an error exponent.

This theorem connects the following two facts on the distillable entanglement of bipartite pure states E:

1. If we allow error probability that vanishes in the asymptotic limit, E cannot exceed $H(p)$ [9].
2. If we stick to deterministic strategies even in finite regimes (i.e., no error is allowed), E is equal to $-\log p_1$ [13].

Equation (11) provides the missing link between them, i.e., $E_{entropy} = H(p) = \lim_{r\to 0} E(r)$ and $E_{det} = -\log p_1 = \lim_{r\to\infty} E(r)$.

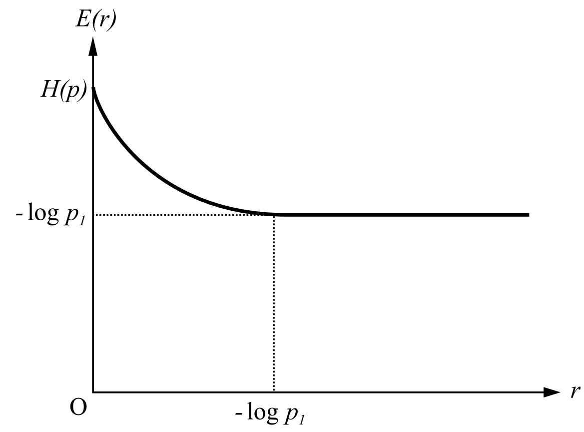

Figure 1: Entanglement yield in asymptotic entanglement concentration with an error exponent r. The horizontal axis represents an error exponent. The vertical axis represents the number of Bell pairs distilled per copy in the asymptotic limit: $E(r) = \min_{q:D(q\|p)\le r}\{D(q \parallel p) + H(q)\}$.

3 Strong converse

In this section, conversely, we discuss asymptotic entanglement concentration with exponentially decreasing *success* probability, which will finally lead to the strong converse of asymptotic entanglement concentration.

Suppose we distill a maximally entangled state of size L_n^* from $|\phi\rangle^{\otimes n}$. We assume the success probability $P_{L_n^*}$ decreases exponentially as the number of the entangled pairs n increases. Then, the first order coefficient in the exponent of the success probability in the asymptotic limit is defined as

$$r = \lim_{n\to\infty} -\frac{1}{n}\log P_{L_n^*}. \tag{12}$$

We derive the maximum number of Bell pairs distilled per copy in the asymptotic limit, E^*, as a function of the exponent r.

Theorem 2 *In entanglement concentration of n identical copies of $|\phi\rangle = \sum_{i=1}^{d}\sqrt{p_i}|i\rangle|i\rangle$, i.e., $|\phi\rangle^{\otimes n}$, if the success probability decreases exponentially as the number of copies n increases, then the number of Bell pairs distilled per copy is given by*

$$E^*(r) = \max_{q:D(q\|p)\le r} H(q). \tag{13}$$

where $r = \lim_{n\to\infty} -\frac{1}{n}\log P_{L_n^*}$.

We conclude from Theorem 2 that if we try to distill a maximally entangled state of size greater than $H(p)$, then the success probability exponentially decreases (strong converse). This was observed in Ref. [11], but here we are able to derive the *exact* error rate.

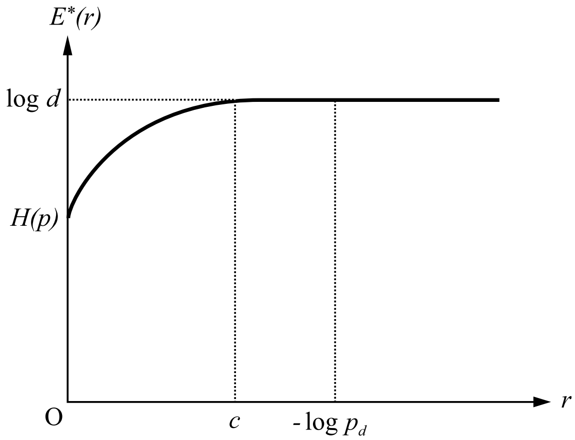

Figure 2: Entanglement yield in entanglement concentration whose success probability exponentially decreases (strong converse). The horizontal axis represents the exponent of the success probability. The vertical axis represents the number of Bell pairs distilled per copy in the asymptotic limit: $E^*(r) = \max_{q:D(q\|p)\leq r} H(q)$. $E^*(r)$ reaches the maximum value, $\log d$, at $r = c \equiv -\log d - (1/d)\sum_i \log p_i$, which is the relative entropy between the uniform distribution $q = (1/d, \cdots, 1/d)$ and $\{p_i\}$.

4 Summary

We have discussed entanglement concentration with exponentially decreasing error probability in the asymptotic limit. Entanglement yield $E(r)$ was derived as a function of an error exponent r. The result links the well-known upper bound of entanglement yield represented by entropy and the lower bound of deterministic concentration. The explicit dependence on the exponent of the success probability was also presented, for the large yield regime, in the form of a strong converse.

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