Mathematics tutorial II - Assessed Coursework 2 (Calculus)

All the solutions should be properly justified and explained. Clarity of the presentation will also be rewarded.

The maximal number of points awarded is 10. The number of points for each exercise is specified between parenthesis. To hand in July 11 at the beginning of the tutorial.

Exercise 1 : (2) Let f be the function defined by:

$$\begin{aligned} f : \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto (\sin(x)\cos(y), xy\exp(yz)). \end{aligned}$$

1. Compute the Jacobian matrix of f.
2. Prove that it is differentiable over $\mathbb{R}^3$.

Exercise 2 : (3) Let f be the function defined by:

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto (x - y)\exp(yx). \end{aligned}$$

1. Prove that this function is of class C^2 over $\mathbb{R}^2$.
2. Compute its Taylor expansion of order 2 at $(\pi, 2)$.
3. Prove that the graph of f admits a tangent plan at $(\pi, 2, f(\pi, 2))$. What is its equation?

Exercise 3 : (2) Let f be the function defined by:

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto \cos(x) + y^2 - 2y + 1. \end{aligned}$$

1. Determine the critical points of f.
2. For each critical point, is it a local minimum, a local maximum, a saddle point, or is it degenerate?

Exercise 4 : (3) We consider the following situation. A small ball of mass m hangs from the ceiling attached to two springs of spring constants k_1 and k_2. For this problem, we consider that the equilibrium length of the strings can be neglected and therefore will be taken to be 0. We consider that the two springs are fixed at two points of distance d from each other. The height of the room is H.

We recall that the potential energy of a spring is $k\delta^2/2$ where k is its spring constant and δ is its elongation from its equilibrium length. We also recall that the gravitational potential energy of an object is mgh where m is its mass, g is the gravitational constant and h is its distance to the floor. To summarize, we consider the following situation:

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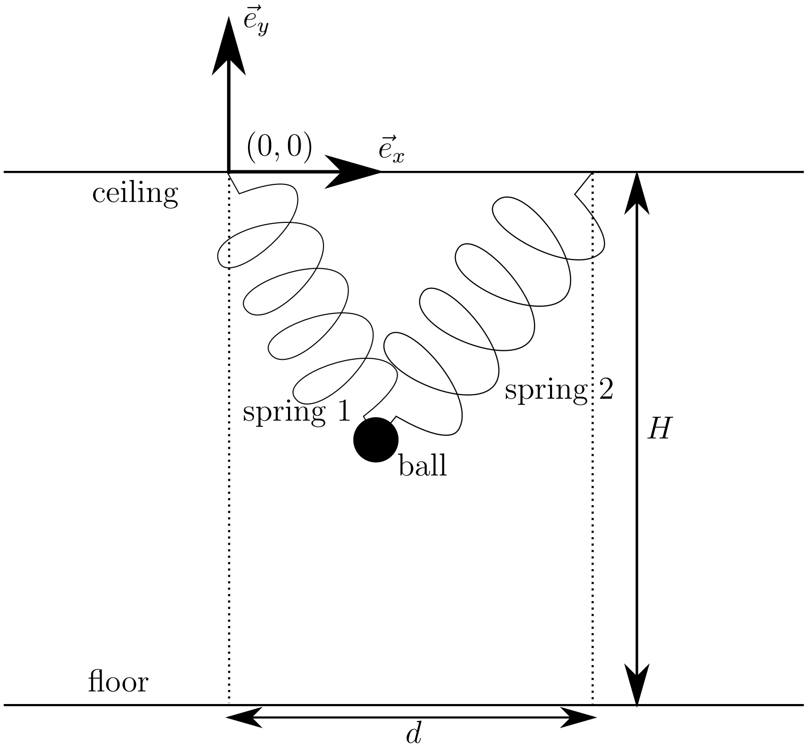

1. Compute the total potential energy $E(x, y)$ of the system in function of the position (x, y) of the ball. [Reminder: it is the sum of the potential energies of the two springs and the gravitational potential energy.]

2. We recall that the system is in equilibrium if its potential energy is minimal. Compute the position of the ball at equilibrium. You can suppose that it is above the floor (H is big enough).