Mathematics tutorial II - Assessed Coursework 1 (Calculus)

All the solutions should be properly justified and explained. Clarity of the presentation will also be rewarded.

The maximal number of points awarded is 10. The number of points for each exercise is specified between parenthesis. To hand in May 16 at the beginning of the tutorial.

Exercise 1 : **(3)** Let $\alpha \in \mathbb{R}$. We consider the function depending on α:

$$\begin{aligned} f_\alpha : \mathbb{R} &\to \mathbb{R}^3 \\ t &\mapsto (\cos(t), \sin(t), \alpha t)\,. \end{aligned}$$

1. Is it injective (= one-one)? [Note: it could depend on α]
2. Is it surjective (= onto)? [Note: it could depend on α]
3. Draw the image of f_0.

Exercise 2 : **(2)** Compute the following limits if they exist. If they do not exist, explain why:

1. $\displaystyle\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x-y^2}{x^2-y^4}$;
2. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y\neq 0}} \frac{\cos x}{\sin y}$.

Exercise 3 : **(2)** We want to study the limit of the following function when (x,y) approaches $(0,0)$:

$$\begin{aligned} f : \mathbb{R}^2 \setminus \{0\} &\to \mathbb{R} \\ (x,y) &\mapsto \frac{xy}{x^2+y^2} \end{aligned}$$

1. For any $\vec{v} = (v_1, v_2) \in \mathbb{R}^2 \setminus \{0\}$, we consider the function

   $$\begin{aligned} g_{\vec{v}} : \mathbb{R} \setminus \{0\} &\to \mathbb{R}^2 \setminus \{0\} \\ t &\mapsto t\vec{v}. \end{aligned}$$

   Does the function $t \mapsto f(g_{\vec{v}}(t))$ converge when t approaches 0? If it is the case, compute the limit.
2. Does the function f converge at $(0,0)$? If it is the case, compute the limit.

Exercise 4 : (3) Let $n \in \mathbb{N}$. We want to study the continuity of the following function:

$$\begin{aligned} g_n : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto \begin{cases} y^n \sin \dfrac{x}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases} \end{aligned}$$

1. Prove that g_n is continuous at any $(x_0, y_0) \in \mathbb{R}^2$ such that $y_0 \neq 0$.
2. Suppose that $n \geqslant 1$. Prove that g_n is continuous at any $(x_0, 0) \in \mathbb{R}^2$.
3. For which real numbers x_0 is the function g_0 continuous at $(x_0, 0)$?