The University of Nagoya
School of Mathematical Sciences
G30 Calculus 2, Spring 2012
Summary of the lecture

1. Preliminaries

1.1. **Multivariable functions.** First of all note that if $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is a multivariable function, it corresponds to the data of m functions

$$f_i : \mathbb{R}^n \to \mathbb{R}$$

called coordinate functions of f and such that:

$$f(x) = f(x_1, \dots, x_n) = (f_1(x), \dots, f_m(x)).$$

When $m = 1$, the function f is called a **scalar-valued function**. (In particular, the coordinate functions of f are scalar-valued functions).

1.2. **Topology in $\mathbb{R}^n$.**

1.1. **Definition.**

(1) Let $a \in \mathbb{R}^n$, and $r \in \mathbb{R}_{\geq 0}$. Define $B(a, r) = \{x \in \mathbb{R}^n | ||x - a|| < r\} \subset \mathbb{R}^n$, called the **open ball (centered at** a **with radius** r). An **open** set of $\mathbb{R}^n$ is a union of open balls. A **neighbourhood** of a is a set containing an open set which in turn contains the point a.

(2) A subset $X \subset \mathbb{R}^n$ is called **closed** if it is the complement in $\mathbb{R}^n$ of an open set and it is said **compact** if it is closed and **bounded**, that is, there exist a non-negative real number M such that for any $x \in X$, $||x|| < M$. We denote the ∂X the set of elements $x \in X$ which do not belong to any neighborhood of X and call it the **boundary** of X. The point of X which are not in ∂X are called **interior points.**

2. Limits and Continuity.

2.1. **Definition.** Let $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function with coordinate functions f_i and $a = (a_1, \dots, a_n) \in X$. We write $\lim_{x \to a} f(x) = L = (L_1, \dots, L_m)$ and say "**the limit of** $f(x)$**, as** x **approaches** a**, equals** L" if we can make the values of f arbitrarily close to L by taking x to be sufficiently close to a but not equal to a. An equivalent definition is to say that:
"For any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\text{If } ||x - a|| < \delta, \text{then } ||f(x) - L|| < \epsilon\text{"}.$$

(Note that $||x-a|| = \sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2}$ is the norm of the vector x-a in $\mathbb{R}^n$ while $||f(x)-L|| = \sqrt{(f_1(x) - L_1)^2 + \cdots + (f_m(x) - L_m)^2}$ is the norm of the vector $f(x) - L$ in $\mathbb{R}^m$.

Note that if the limit exists it is unique. In particular, to prove that a function f has NO limit at some point a, it is common to compute the limit of f by approaching a with different paths and to show that you get different limits.

2.2. **Theorem.** *Let $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function with coordinate functions f_i and $a = (a_1, \dots, a_n) \in X$. The function $f(x)$ has for limit $L = (L_1, \dots, L_m)$ as x approaches a if and only if the coordinate function $f_i(x)$ has for limit L_i as x approaches a for any $i = 1, \dots, m$.*

2.3. **Theorem. Laws for Limits**

(1) *Sum Law for multivariable functions. The limit of a sum is the sum of the limits.*
(2) *Difference Law for multivariable functions. The limit of a difference is the difference of the limits.*
(3) *Constant Multiple Law for multivariable functions. The limit of a constant times a function is the constant times the limit of the function.*
(4) *Product Law for scalar-valued functions. The limit of a product is the product of the limits.*
(5) *Quotient Law for scalar-valued functions. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).*

2.4. **Theorem. The Squeeze Theorem**
If $g(x) \leq f(x) \leq h(x)$ and $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$ then $\lim_{x\to a} f(x) = L$.

2.5. **Definition.** A multivariable function f is **continuous at** a if

$$\lim_{x\to a} f(x) = f(a).$$

For example,

(1) Polynomials in several variables, linear transformations are continuous everywhere.
(2) The sum $F + G$ of two functions $F, G : X \subset \mathbb{R}^n \to \mathbb{R}^m$ that are continuous at $a \in X$ is continuous at a.
(3) For all $k \in \mathbb{R}$, the scalar multiple kF of a function $F : X \subset \mathbb{R}^n \to \mathbb{R}^m$ that is continuous at $a \in X$ is continuous at a.
(4) The product fg and the quotient f/g ($g \neq 0$) of two scalar-valued functions $f, g : X \subset \mathbb{R}^n \to \mathbb{R}$ that are continuous at $a \in X$ are continuous at a.
(5) $F : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in X$ if and only if its coordinate functions $F_i : X \subset \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$ are all continuous at a.
(6) If g is continuous at a and f continuous at $g(a)$ then $f \circ g(x) = f(g(x))$ is continuous at a.

3. Derivatives of multivariable functions

3.1. **Definition.** Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function. The **partial derivative of** $f(x_1, \ldots, x_n)$ **with respect to** x_i is the (ordinary) derivative of the function

$$x \mapsto f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)$$

at x_i. That is,

$$\lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}$$

It is denoted $\frac{\partial f}{\partial x_i}(x)$ or $f_{x_i}(x)$ or $Df_{x_i}(x)$.

3.1. Differentiability of multivariable functions.

3.2. **Definition.**

(1) Let $f : X \subset \mathbb{R}^n \to \mathbb{R}^m$ be a function defined on an open X of $\mathbb{R}^n$, with coordinate functions f_i, $j = 1, \ldots, m$ and let $a \in X$.

We denote $\nabla(f)$ or Df the $m \times n$ matrix with coefficients

$$(\frac{\partial f_i}{\partial x_j})_{i=1,\ldots,m; j=1,\ldots,n}.$$

This matrix is called the **Jacobian matrix** of f. When $m = 1$, we get a row vector and we call it **gradient**. We can also evaluate the matrix at the point a, denoted

$$\nabla(f)(a) = (\frac{\partial f_i}{\partial x_j}(a))_{i=1,\ldots,m; j=1,\ldots,n}.$$

(2) We say that f **is differentiable at** a if all the partial derivatives $f_{j,x_i}(a)$ exist and

$$\lim_{x \to a} \frac{||f(x) - h(x)||}{||x - a||} = 0,$$

with $h : X \subset \mathbb{R}^n \to \mathbb{R}^m$ the function defined by

$$h(x) = f(a) + \nabla(f)(a).(x - a),$$

where we consider $(x - a)$ as a vector column and "." means the product of matrices.

In the case $m = 1$, we can rewrite

$$h(x_1, \ldots, x_n) = f(a) + f_{x_1}(a)(x_1 - a_1) + \cdots + f_{x_n}(a)(x_n - a_n).$$

If f is differentiable at all points of its domain, then we simply say that f is differentiable.

In the case where $n = 2$ and $m = 1$, then $z = f(x, y)$ is the equation of a **surface** in $\mathbb{R}^3$. Moreover, if f is differentiable at (a, b), then the equation $z = h(x, y)$ defines the **tangent plane to the graph of** f **at the point** $(a, b, f(a, b))$.

As with single variable functions, we have the following result:

3.3. **Theorem.** *If a multivariable function is differentiable at some point, then it is continuous at this same point.*

In general, it can be quite difficult to check differentiability using the previous definition but we have the following criterion:

3.4. **Theorem.** *If all partial derivatives of a multivariable function exist and are continuous on a neighborhood of a point, then the function is differentiable at this point.*

3.2. **Properties of derivatives.**

(1) $D(\lambda.f + \mu.g) = \lambda.Df + \mu.Dg$ (f,g, multivariable functions, $\lambda, \mu \in \mathbb{R}$).
(2) $Dfg(a) = g(a)Df(a) + f(a)Dg(a)$ (f, g scalar-valued functions, $a \in \mathbb{R}$).
(3) $D\frac{f}{g}(a) = \frac{g(a)Df(a) - f(a)Dg(a)}{g(a)^2}$ (f, g scalar-valued functions, $a \in \mathbb{R}$ such that $g(a) \neq 0$).

(Note that the two last equalities are equalities of n-uples.)

3.3. **Partial derivatives of higher orders.**

3.5. **Definition.**

(1) If $f : X \subset \mathbb{R}^n \to \mathbb{R}$ is a (scalar-valued) function of n variables, the **kth-order partial derivative with respect to the variables** $x_{i_1}, x_{i_2}, \dots, x_{i_k}$ (in that order),where $i_1, i_2, \dots, i_k$ are integers in the set $\{1, 2, \dots, n\}$ (possibly repeated), is the iterated derivative

$$\frac{\partial^k f}{x_{i_1} \dots x_{i_k}} := \frac{\partial}{x_{i_k}} \dots \frac{\partial}{x_{i_1}} f(x_1, \dots, x_n)$$

Equivalent (and frequently more manageable) notation for this kth-order partial is

$$f_{x_{i_1} x_{i_2} \dots x_{i_k}}(x_1, x_2, \dots, x_n).$$

(2) Assume X is an open subspace of $\mathbb{R}^n$. We say that f is of **class** C^k if the partial derivatives of f up to (and including) order at least k exist and are continuous on X. If f is of class C^k for any k, we say that f is **smooth** or of **class** C^∞.

3.6. **Theorem.** *Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function of class C^k . Then the order in which we calculate any kth-order partial derivative does not matter.*

3.4. **Chain Rule.**

3.7. **Theorem.** *Suppose $X \subset \mathbb{R}^m$ and $T \subset \mathbb{R}^n$ are open and $f : X \to \mathbb{R}^p$ and $g : T \to \mathbb{R}^m$ are multivariable functions such that the range of g is contained in X. If g is differentiable at a and f is differentiable at $g(a) = b$, then the function $f \circ g$ is differentiable at a and we have*

$$Df \circ g(a) = Df(b).Dg(a)$$

3.5. **Taylor Theorem.** Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function of class C^{k+1} at $a \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$. Set $h = (h_1, \dots, h_n) = (x_1 - a_1, \dots, x_n - a_n)$. There exists an element ξ on the line segment joining a and x such that

$$f(x) = f(a) + \sum_{i=1}^{n} f_{x_i}(a)h_i + \frac{1}{2!}\sum_{i,j=1}^{n} f_{x_i x_j}(a)h_i h_j + \cdots + \frac{1}{k!}\sum_{i_1,\dots,i_k=1}^{n} f_{x_{i_1}\dots x_{i_k}}(a)h_{i_1}\dots h_{i_k}$$

$$+\frac{1}{(k+1)!}\sum_{i_1,\dots,i_{k+1}=1}^{n} f_{x_{i_1}\dots x_{i_{k+1}}}(a)h_{i_1}\dots h_{i_{k+1}}.$$

The $n \times n$ matrix $H(f)$ whose (i,j)-th entry is $f_{x_i x_j}(x)$ is called the **Hessian matrix** associated to f. We will denote $Hf(a)$ the real $n \times n$ matrix whose (i,j)-th entry is $f_{x_i x_j}(a)$. We consider $h = (h_1, \dots, h_n)$ as a row vector and write h^T for the transposed of h (this is a column vector). We can write the second Taylor polynomial in n variables as follows:

$$P_2(h) = f(a) + D(f)(a).h^T + \frac{1}{2}Q(h)$$

with $Q(h) = h.H(f)(a).h^T$ (all dots in this formula are matrices products; $Q(h)$ is called a **quadratic form**)

3.6. **Extrema of scalar-valued functions.**

3.8. **Definition.**
(1) A scalar valued function has an **absolute minimum** (resp. **maximum**) **at** c if $f(c) \geq f(x)$ (resp. $f(c) \leq f(x)$) for any x in the domain of f.
(2) the function f has a **local minimum** (resp. **maximum**) **at** c if $f(c) \geq f(x)$ (resp. $f(c) \leq f(x)$) when x belongs to a neighborhood of c. We say that a is a **local extremum** if it is either a local max or min.

3.9. **Theorem.** *Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function differentiable on some neighborhood of $a \in X$. If a is a local extremum for f, then $Df(a)$ is the zero row vector.*

3.10. **Definition.** We call $a \in X$ a **critical point** of f if $Df(a) = 0$. As in the case of single variable functions, a critical point is not necessarily a local extremum. When this happens we say that a is a saddle point. That is, a saddle point is a point a such that $Df(a) = 0$ but a is not a local extremum.

3.11. **Theorem.** *Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function, let a be a critical point and assume f is C^2 on a neighborhood U of a. We set as before $Q(h) = h.H(f)(a).h^T$.*
(1) *If $Q(h) > 0$ for any $h \in U$ (we say that $Q(h)$ is* **definite positive***), then a is a local minimum.*
(2) *If $Q(h) < 0$ for any $h \in U$ (we say that $Q(h)$ is* **definite negative***), then a is a local maximum.*

(3) *If $det(Hf(a)) \neq 0$ and $Q(h)$ is neither definite positive nor definite negative, then a is a saddle point.*

Note that in practise it is not so easy to determine if $Q(h)$ is definite positive /negative. Therefore the importance of the next result:

3.7. **Second derivative test for local extrema.** Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function, let a be a critical point. Assume that f is C^2 on a neighborhood U of a and $det(H(f)(a)) \neq 0$. Let also denote a_{ij} the (i, j)-th entry of the Hessian matrix.

We set d_k the determinant of the submatrix $(a_{ij})_{i,j=1,...k}$ of $Hf(a)$ (such that for example $d_1 = f_{x_1}(a)$ and $d_n = det(Hf(a))$).

(1) If $d_k > 0$ for any k, then f has a local minimum at a.
(2) If $d_k > 0$ for even k and $d_k < 0$ for odd k, then f has a local maximum at a.
(3) Otherwise, a is a saddle point.

3.8. **The Extreme Value Theorem for scalar-valued functions.**

3.12. **Theorem. EVT for scalar-valued functions** *Let $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function such that X is compact and f is continuous on X.*

Then f has both a global minimum and global maximum on X, that is there exist $a_{min}, a_{max} \in X$ such that

$$a_{min} \leq f(x) \leq a_{max}.$$

The points a_{min} and a_{max} can be found among the following points

(1) The set of critical points of f on X.
(2) The global extrema of the restriction of f to the boundary of X

$$f|_{\partial X} : \partial X \to \mathbb{R},$$

$$x \mapsto f(x).$$

(the global extrema of such function (in most cases considered, $f|_{\partial X}$ will be a single variable function) are to be found among the critical points and the endpoints of ∂X: see the summary of Calculus 1:"Algorithm to find Global extrema").

4. **Integration of multivariable functions**

4.1. **Double integrals.**

4.1. **Definition.** Let $R = [a, b] \times [c, d]$ be a rectangle in the plane. A **partition of order n** $\{R_{i,j}\}$ of R is the collection of the following datas:

(1) $\{x_i\} : a = x_0 < x_1 < \cdots < x_n = b$.
(2) $\{y_i\} : c = y_0 < y_1 < \cdots < y_n = d$.

Set $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$ for any $i = 1, \ldots, n$ and $R_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ the rectangle with width Δx_i and height Δy_j.

4.2. **Definition.** Let $f : R \subset \mathbb{R}^2 \to \mathbb{R}$. For any integer n, let $\{R_{i,j}\}$ be a partition of R of order n. For any i, j, let $P_{i,j}$ be a point in the subrectangle $R_{i,j}$. The **double integral of f on R** is defined as

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(P_{i,j})\Delta x_i \Delta y_j$$

provided the limit exists in which case we say that f is **integrable** and we write $\int_a^b \int_c^d f(x,y)dxdy$ or $\int\int_R f(x,y)dxdy$ this limit. The sum $\sum_{i=1}^{n} f(P_{i,j})\Delta x_i \Delta y_j$ is called **Riemann sum** (of order n).

Note that if $f(x,y) \geq 0$ then the double integral of f on R is the volume of the area of $\mathbb{R}^3$ delimited by the surface $z = f(x,y)$ above R.

4.3. **Theorem.** *If f is continuous on R then it is integrable on R. More generally, if f is bounded on R and the set of points where the function is discontinuous has zero area (for example a finite set of points) then f is integrable.*

4.2. **Properties of the double integral.** let $f, g : R \subset \mathbb{R}^2 \to \mathbb{R}$integrable on R and let $\lambda, \mu \in \mathbb{R}$.

(1) $\int\int_R \lambda.f(x,y)+\mu g(x,y)dxdy = \lambda \int\int_R f(x,y)dxdy+\mu \int\int_R g(x,y)dxdy$ (Linearity).
(2) If for any $(x,y) \in R$, $f(x,y) \leq g(x,y)$, then $\int\int_R f(x,y)dxdy \leq \int\int_R g(x,y)dxdy$.
(3) The function $|f|$ is integrable and we have $|\int\int_R f(x,y)dxdy| \leq \int\int_R |f(x,y)|dxdy$.

4.3. **Iterated integral.** Let $f : R \subset \mathbb{R}^2 \to \mathbb{R}$, with R a rectangle $[a,b] \times [c,d]$. We fix y and consider $f(x,y)$ as a single variable function that we assume to be continuous on $[a,b]$. then $F(y) = \int_a^b f(x,y)dx$ is a function of y. Assuming that this function F is continuous on $[c,d]$, then

$$\int_c^d F(y)dy = \int_c^d (\int_a^b f(x,y)dx)dy$$

is well-defined and called an iterated integral. with similar assumptions of continuity as above and replacing x by y, we can also integrate first in y, then in x:

$$\int_a^b (\int_c^d f(x,y)dx)dy.$$

This is also called an iterated integral of f.

4.4. **Theorem.** *(***Fubini Theorem***) Let $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ and let $A \subset R$ be a set zero area and such that A meets everyline parallel to both axis at only finitely many points. Assume that f is continuous on $R \setminus A$. Then f is integrable on R, the double integral can both be computed by the Riemann sum or the iterated integrals and we have*

$$\int\int_R f(x,y)dxdy = \int_a^b (\int_c^d f(x,y)dx)dy = \int_c^d (\int_a^b f(x,y)dx)dy.$$

4.4. **Double integral on elementary domains.**

4.5. **Definition.** We say that D is an **elementary domain** of the plane if it can be described as a subset of $\mathbb{R}^2$ of one of the following types:

(1) $D = \{(x,y) \in \mathbb{R}^2 | x \in [a,b], y \in [\gamma(x), \delta(x)]\}$, where γ, δ are continuous functions on $[a,b]$.
(2) $D = \{(x,y) \in \mathbb{R}^2 | x \in [\alpha(y), \beta(y)], y \in [c,d]\}$, where α, β are continuous functions on $[c,d]$.

(Remark that an elementary domain is a finite portion of the plane and can be simultaneously of both types.)

4.6. **Theorem.** *Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function on an elementary domain D. We define the function $f^{ext} : \mathbb{R}^2 \to \mathbb{R}$ as follows:*

(1) *$f^{ext}(x,y) = f(x,y)$ for any $(x,y) \in D$.*
(2) *$f^{ext}(x,y) = 0$, otherwise.*

Then, if the function f^{ext} is discontinuous at some point, then this point is in ∂D, the boundary of D. The function f^{ext} is integrable on any rectangle R containing D and we set

$$\int\int_D f(x,y)dxdy := \int\int_R f^{ext}(x,y)dxdy.$$

This double integral is independent of the choice of the rectangle R containing D.

4.7. **Theorem.** *Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function on an elementary domain D.*

(1) *Assume D is of type (1), then*

$$\int\int_D f(x,y)dxdy = \int_a^b (\int_{\gamma(x)}^{\delta(x)} f(x,y)dy)dx.$$

(2) *Assume D is of type (2), then*

$$\int\int_D f(x,y)dxdy = \int_c^d (\int_{\alpha(y)}^{\beta(y)} f(x,y)dx)dy.$$

4.5. **Triple integrals.**

4.8. **Definition.** Let $B = [a,b] \times [c,d] \times [p,q]$ be a closed box in $\mathbb{R}^3$. A **partition of order n** $\{B_{i,j,k}\}$ of B is the collection of the following datas:

(1) $\{x_i\} : a = x_0 < x_1 < \cdots < x_n = b$.
(2) $\{y_i\} : c = y_0 < y_1 < \cdots < y_n = d$.
(3) $\{z_i\} : p = z_0 < z_1 < \cdots < z_n = q$.

Set $\Delta x_i = x_i - x_{i-1}$, $\Delta y_i = y_i - y_{i-1}$ and $\Delta z_i = z_i - z_{i-1}$ for any $i = 1, \ldots, n$ and $B_{i,j,k} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ the subboxes of B with volume $\Delta V_{i,j,k} := \Delta x_i . \Delta y_j . \Delta z_k$.

4.9. **Definition.** Let $f : R \subset \mathbb{R}^3 \to \mathbb{R}$. For any integer n, let $\{B_{i,j,k}\}$ be a partition of B of order n. For any i, j, k, let $P_{i,j,k}$ be a point in the subbox $B_{i,j,k}$. The **triple integral of** f **on** B is defined as

$$\int\int\int_B := \lim_{n\to\infty} \sum_{i=1}^{n} f(P_{i,j,k})\Delta V_{i,j,k}$$

provided the limit exists in which case we say that f is **integrable** and we write $\int_a^b \int_c^d \int_p^q f(x,y,z)dxdydz$ or $\int\int\int_B f(x,y,z)dxdydz$ this limit. The sum $\sum_{i=1}^n f(P_{i,j,k})\Delta V_{i,j,k}$ is called **Riemann sum** (of order n).

4.10. **Example.** If f is bounded and discontinuous on a set X with zero volume, then f is integrable. If we assume moreover that X meets every lines parallel to the 3 axes at only finitely many points, then

$$\int\int\int_B f(x,y,z)dxdydz = \int_a^b (\int_c^d (\int_p^q f(x,y,z)dz)dy)dx,$$

or in any other order, where on the right hand side we consider iterated integrals as in semester 1.

4.6. **Triple integral on elementary domains.**

4.11. **Definition.** We say that D is an **elementary domain** of $\mathbb{R}^3$ if it can be described as a subset of $\mathbb{R}^3$ of one of the following types:

(1) $D = \{(x,y,z) \in \mathbb{R}^3 | x \in [a,b], y \in [\gamma(x), \delta(x)], z \in [\varphi(x,y), \Psi(x,y)]\}$ or where $D = \{(x,y,z) \in \mathbb{R}^3 | y \in [c,d], x \in [\alpha(y), \beta(y)], z \in [\varphi(x,y), \Psi(x,y)]\}$ $\gamma, \delta, \alpha, \beta, \varphi, \Psi$ are continuous functions.
(2) $D = \{(x,y,z) \in \mathbb{R}^3 | x \in [\alpha(y,z), \beta(y,z)], y \in [\gamma(z), \delta(z)], z \in [p,q]\}$ or $D = \{(x,y,z) \in \mathbb{R}^3 | x \in [\alpha(y,z), \beta(y,z)], y \in [c,d], z \in [\varphi(y), \Psi(y)]\}$, where $\alpha, \beta, \gamma, \delta, \varphi, \Psi$ are continuous functions.
(3) $D = \{(x,y,z) \in \mathbb{R}^3 | x \in [\alpha(z), \beta(z)], y \in [\gamma(x,z), \delta(x,z)], z \in [p,q]\}$ or $D = \{(x,y,z) \in \mathbb{R}^3 | x \in [a,b], y \in [\gamma(x,z), \delta(x,z)], z \in [\varphi(x), \Psi(x)]\}$, where $\alpha, \beta, \gamma, \delta, \varphi, \Psi$ are continuous functions.

(Remark that an elementary domain is a finite portion of $\mathbb{R}^3$ and can be simultaneously of type 1,2 or 3.)

4.12. **Theorem.** *Let $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ be a continuous function on an elementary domain D. We define the function $f^{ext} : \mathbb{R}^3 \to \mathbb{R}$ as follows:*

(1) *$f^{ext}(x,y,z) = f(x,y,z)$ for any $(x,y,z) \in D$.*
(2) *$f^{ext}(x,y,z) = 0$, otherwise.*

Then, if the function f^{ext} is discontinuous at some point, then this point is in ∂D, the boundary of D. The function f^{ext} is integrable on any box B containing D and we set

$$\int\int\int_D f(x,y)dxdy := \int\int\int_B f^{ext}(x,y)dxdy.$$

This triple integral is independent of the choice of the box B containing D.

4.13. **Theorem.** *Let $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ be a continuous function on an elementary domain D.*

(1) *Assume D is of type (1a), then*
$$\int\int\int_D f(x,y,z)dxdydz = \int_a^b \int_{\gamma(x)}^{\delta(x)} \int_{\varphi(x,y}^{\Psi(x,y)} f(x,y,z)dzdydx.$$

(2) *Assume D is of type (1b), then*
$$\int\int\int_D f(x,y,z)dxdy = \int_c^d \int_{\alpha(y)}^{\beta(y)} \int_{\varphi(x,y)}^{\Psi(x,y)} dzdydz.$$

(3) *Assume D is of type (2a), then*
$$\int\int\int_D f(x,y,z)dxdydz = \int_p^q \int_{\gamma(z)}^{\delta(z)} \int_{\varphi(y,z}^{\Psi(y,z)} f(x,y,z)dxdydz.$$

(4) *Assume D is of type (2b), then*
$$\int\int\int_D f(x,y,z)dxdy = \int_c^d \int_{\alpha(y)}^{\beta(y)} \int_{\varphi(y,z)}^{\Psi(y,z)} f(x,y,z)dxdzdy.$$

(5) *Assume D is of type (3a), then*
$$\int\int\int_D f(x,y,z)dxdydz = \int_p^q \int_{\gamma(z)}^{\delta(z)} \int_{\varphi(x,z}^{\Psi(x,z)} f(x,y,z)dydxdz$$

(6) *Assume D is of type (3b), then*
$$\int\int\int_D f(x,y,z)dxdy = \int_a^b \int_{\alpha(x)}^{\beta(x)} \int_{\varphi(x,z)}^{\Psi(x,z)} f(x,y,z)dydzdx.$$

(Do not learn all these formulas by heart!)

4.7. Change of variables.

4.14. **Theorem.** *Let D be an elementary region of the plane (resp. of $\mathbb{R}^3$) and $f : D \to \mathbb{R}$ be an integrable on D. Let*
$$T : \mathbb{R}^2 \to \mathbb{R}^2; (u,v) \mapsto (x(u,v), y(u,v))$$
(resp.
$$T : \mathbb{R}^3 \to \mathbb{R}^3; (u,v,w) \mapsto (x(u,v,w), y(u,v,w), z(u,v,w)))$$
be a C^1 differentiable function on $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) mapping an elementary region D^* of the plane (resp. $\mathbb{R}^3$) onto D in a one to one fashion (which means that for any $x \in D$, there exists a $x^* \in D^*$ such that $T(x^*) = x$ and this x^* is unique) and denote J_T its Jacobian matrix. Then
$$\int\int_D f(x,y)dxdy = \int\int_{D^*} f(x(u,v), y(u,v)).|det(J_T)|dudv$$
(resp.
$$\int\int\int_D f(x,y,z)dxdydz = \int\int\int_{D^*} f(x(u,v,w), y(u,v,w), z(u,v,w)).|det(J_T)|dudvdw)$$

4.8. **Line integrals.**

4.15. **Definition.** (1) A **path** in $\mathbb{R}^n$ is a continuous function $p : I \subset \mathbb{R} \to \mathbb{R}^n$, where I is an interval of $\mathbb{R}$. The points $p(a)$ and $p(b)$ are called **endpoints** of the path. A **vector field** is a function F from a subset X of $\mathbb{R}^n$ to $\mathbb{R}^n$.

(2) Let $p : [a, b] \to \mathbb{R}^n$ be a C^1 path and $F : X \subset \mathbb{R}^n \to \mathbb{R}^n$ such that $p([a, b]) \subset X$ and F is continuous on $p([a, b])$. The **vector line integral** of F along p , denoted $\int_p F.ds$ is by defined as

$$\int_a^b F(p(t)).p'(t)dt,$$

where the dot represents the dot product of the two n-uples $F(p(t))$ and $p'(t)$.

(3) Let$p : [a, b] \to \mathbb{R}^n$ be a C^1 path and $f : X \subset \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function such that $p([a, b]) \subset X$ and f is continuous on $p([a, b])$. The **scalar line integral** of f along p denoted $\int_p f.ds$ is defined as

$$\int_a^b f(p(t)).||p'(t)||dt.$$

If F is the force field in space, the $\int_p F.ds$ represents the work done by F on a particle as the particle moves along the path p.

4.16. **Definition.** $p : [a, b] \to \mathbb{R}^n$ be a C^1 path. We say that $p' : [c, d] \to \mathbb{R}^n$ is a **reparametrization** of p if there is a one to one and onto function $u : [c, d] \to [a, b]$ of class C^1 with inverse u^{-1} such that $p'(t) = p(u(t))$. If $u(c) = a$ and $u(d) = b$, we say that p' is **orientation-preserving**, if $u(c) = b$ and $u(d) = a$, we say that p' is **orientation-reversing**.

4.17. **Theorem.** (1) *Let $p : [a, b] \to \mathbb{R}^n$ be a C^1 path and $f : X \subset \mathbb{R}^n \to \mathbb{R}$ such that $p([a, b]) \subset X$ and let p' be a reparametrization of p. Then*

$$\int_p f.ds = \int_{p'} f.ds.$$

(2) *Let $p : [a, b] \to \mathbb{R}^n$ be a C^1 path and $f : X \subset \mathbb{R}^n \to \mathbb{R}$ such that $p([a, b]) \subset X$ and let p' be a reparametrization of p. Then*

$$\int_p F.ds = \int_{p'} F.ds,$$

if p' is orientation-preserving and

$$\int_p F.ds = -\int_{p'} F.ds,$$

if p' is orientation-reversing.