The University of Nagoya
School of Mathematical Sciences
G30 Calculus 2, Spring 2012

MIDTERM EXAM

Answer key

Exercise 1. Find the following limits (if it exists):

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$.

(b) $\lim_{(x,y)\to(0,0)} \frac{-x}{\sqrt{x^2+y^2}}$ (Hint: consider different paths of approach)

A:

(a)

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y}) = 0.$$

(b) Consider the limiting value along line $y = kx$ where $k \in \mathbb{R}$.

$$\lim_{(x,y)\to(0,0)} \frac{-x}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{-x}{|x|\sqrt{1+k^2}} = \pm\frac{1}{\sqrt{1+k^2}}.$$

Since the limiting value vary according to k and the direction which x approaches to 0. So the limit does not exist.

Exercise 2.

(1) Show that $|xy(x^2-y^2)| \leq (x^2+y^2)^2$.

(2) Define $f(0,0)$ such that the function

$$f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}$$

is continuous at the origin. (Hint: use the previous inequality and the Sandwich Theorem to compute the limit at $(0,0)$.)

A:

(1) By $(|x|-|y|)^2 \geq 0$, we can obtain

$$|2xy| \leq x^2+y^2.$$

And obviously

$$|x^2-y^2| \leq x^2+y^2 \leq 2(x^2+y^2).$$

Combining the two inequalities,we can obtain

$$|2xy(x^2-y^2)| \leq 2(x^2+y^2)^2.$$

That is

$$|xy(x^2-y^2)| \leq (x^2+y^2)^2.$$

(2) By the previous inequality,we have

$$-(x^2+y^2)^2 \le xy(x^2-y^2) \le (x^2+y^2)^2$$

Then

$$-(x^2+y^2) \le f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2} \le (x^2+y^2).$$

And we have

$$\lim_{(x,y)\to(0,0)} x^2+y^2 = \lim_{(x,y)\to(0,0)} -(x^2-y^2) = 0.$$

By Sandwich Theorem, we obtain

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

So in order to let f(x,y) be continuous at (0,0), we just need to define

$$f(0,0) = 0$$

.

Exercise 3.

(1) Compute the Jacobian matrix of the function

$$F(x,y,z,w) := (3x-7y+z, 5x+2z-8w, y-17z+3w)$$

(2) Is the function F differentiable at any $a \in \mathbb{R}^4$? Justify your answer.

A:

(1) Let $f_1 := 3x-7y+z$,$f_2 := 5x+2z-8w$,$f_3 := y-17z+3w$. Then $F = (f_1, f_2, f_3)$.

$$\mathbf{D}F = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial w} \end{pmatrix} = \begin{pmatrix} 3 & -7 & 1 & 0 \\ 5 & 0 & 2 & -8 \\ 0 & 1 & -17 & 3 \end{pmatrix}.$$

(2) By (1), we can see for $i = 1,2,3$, $\partial f_i/x$, $\partial f_i/y$, $\partial f_i/z$ and $\partial f_i/w$ exist and continuous at any $a \in \mathbb{R}^4$. So F is differentiable at any $a \in \mathbb{R}^4$.

Exercise 4. Find the equation of the plane tangent to the surface $z = x^2 - xy - y^2$ at $(1,1,-1)$.

A: Set $f(x,y) = x^2 - xy - y^2 = z$, then

$$\frac{\partial f}{\partial x} = 2x - y, \frac{\partial f}{\partial y} = -x - 2y.$$

$$f_x(1,1) = 1, f_y(1,1) = -3, f(1,1) = -1.$$

Thus the equation is

$$z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) = -1 + (x-1) - 3(y-1).$$

That is

$$x - 3y - z + 1 = 0.$$

Exercise 5. Let $z = 4e^x ln(y), x = ln(ucos(v)), y = usin(v)$.

Express $\partial z/\partial u$ and $\partial z/\partial v$ as functions of u and v both by using the Chain Rule and by expressing z directly in terms of u and v before differentiation.

A:

(1) By Chain Rule:

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= 4e^x lny \cdot \frac{\cos(v)}{u\cos(v)} + 4e^x \frac{1}{y} \cdot \sin(v) \\
&= 4(u\cos(v))ln(u\sin(v)) \cdot \frac{1}{u} + 4(u\cos(v))\frac{sin(v)}{u\sin(v)} \\
&= 4cos(v)(ln(u\sin(v)) + 1).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= 4e^x lny \cdot \frac{-u\sin(v)}{u\cos(v)} + 4e^x \frac{1}{y} \cdot u\cos(v) \\
&= 4u(-\sin(v)ln(u\sin(v)) + \frac{\cos^2(v)}{\sin(v)}).
\end{aligned}$$

(2) By expressing z directly in terms of u and v before differentiation:

$$z = 4e^x ln(y) = 4ucos(v)ln(usin(v)).$$

Then

$$\begin{aligned}
\frac{\partial z}{\partial u} &= 4cos(v)(1 \cdot \ln(u\sin(v)) + \frac{sin(v)}{u\sin(v)}) \\
&= 4cos(v)(ln(u\sin(v)) + 1).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial v} &= 4u(-sin(v) \cdot \ln(u\sin(v)) + \cos(v)\frac{u\cos(v)}{u\sin(v)}) \\
&= 4u(-\sin(v)ln(u\sin(v)) + \frac{\cos^2(v)}{\sin(v)}).
\end{aligned}$$