The University of Nagoya
School of Mathematical Sciences
G30 Tutorials 2, Spring 2012

ASSESSED COURSEWORK 3

Deadline: July 12th, 14:45

Exercise 1. Consider the function

$$f(x_1, \ldots, x_n) = e^{x_1 + 2x_2 + \cdots + nx_n}.$$

(a) Calculate $Df(0, \ldots, 0)$ and $Hf(0, \ldots, 0)$.
(b) Determine the first and second-order Taylor polynomials of f at 0.
(c) Write the Taylor polynomial of f using part (a).

Proof. (1) Case (a): Because

$$\frac{\partial f}{\partial x_i} = if,$$

we have

$$\frac{\partial f}{\partial x_i}(0, \ldots, 0) = i.$$

Then,

$$Df(0, \ldots, 0) = (1, 2, \ldots, n).$$

And,

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = i \frac{\partial f}{\partial x_j} = ijf.$$

So

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(0, \ldots, 0) = ij.$$

Then,

$$Hf(0, \ldots, 0) = \begin{pmatrix} 1 & \ldots & n \\ \ldots & ij & \ldots \\ n & \ldots & n^2 \end{pmatrix}_{n \times n}.$$

(2) Case (b): Let $x := (x_1, \ldots, x_n)$. The Taylor expansion for a n variables function f at the point $a = (a_1, \ldots, a_n)$ is

$$\begin{aligned} f(x_1, \ldots, x_n) &= f(a) + \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x_i} \right|_{x=a} (x_i - a_i) + \ldots \\ &+ \frac{1}{n!} \sum_{(i_1, \ldots, i_n) \in (1, \ldots, n)} \left. \frac{\partial^n f}{\partial x_{i_1} \ldots x_{i_n}} \right|_{x=a} (x_{i_1} - a_{i_1}) \ldots (x_{i_n} - a_{i_n}) + \ldots. \end{aligned}$$

Using the case (a), we can get that, the first order Taylor polynomials of f at 0 is

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\bigg|_{x=0} x_i = \sum_{i=1}^{n} i x_i,$$

the second order Taylor polynomials of f at 0 is

$$\frac{1}{2!} \sum_{\{i,j\}\in\{1,\dots,n\}} \frac{\partial^2 f}{\partial x_i \partial x_j}\bigg|_{x=0} x_i x_j = \frac{1}{2!} \sum_{\{i,j\}\in\{1,\dots,n\}} ij x_i x_j.$$

(3) Case (c): Using (a), (b), we have the Taylor polynomial of f at the point 0 as follows,

$$\begin{aligned} f(x) &= f(0) + Df(0)x^T + xHf(0)x^T + \dots \\ &= 1 + (1, \dots, n)x^T + x \begin{pmatrix} 1 & \dots & n \\ \dots & ij & \dots \\ n & \dots & n^2 \end{pmatrix}_{n\times n} x^T + \dots. \end{aligned}$$

□

Exercise 2. Concerning the function

$$f(x, y) = 4x + 6y - 12^2 - y^2 :$$

(a) There is a unique critical point. Find it.
(b) Determine whether this critical point is a maximum, a minimum, or a saddle point.

Proof. (1) Case (a):
Since f is a polynomial, it is differentiable everywhere, any extremum must occur where $\partial f/\partial x$, $\partial f/\partial y$ vanish simultaneously. Thus, we solve

$$\begin{cases} \frac{\partial f}{\partial x} = 4 - 24x = 0 \\ \frac{\partial f}{\partial y} = 6 - 2y = 0. \end{cases}$$

and find that the only solution is $x = 1/6$, $y = 3$. Consequently, $(1/6, 3)$ is the only critical point of this function.

(2) Case (b): To determine whether $(1/6, 3)$ is is a maximum or minimum or saddle point, we use the second derivative test for f at the critical point $(1/6, 3)$, we calculate:

$$\begin{aligned} d_1 &= f_{xx}(\frac{1}{6}, 3) - 24 < 0, \\ d_2 &= Hf(\frac{1}{6}, 3) = \begin{pmatrix} -24 & 0 \\ 0 & -2 \end{pmatrix} = 48 > 0. \end{aligned}$$

So f has a local maximum at $(1/6, 3)$.

□

Exercise 3.

(a) Evaluate the integral

$$\int_{-1}^{2}\int_{y^2-2y}^{2-y}(x+y)\mathrm{d}x\mathrm{d}y$$

(b) Sketch the region of the plane determined by the limits of integration of the previous integral.

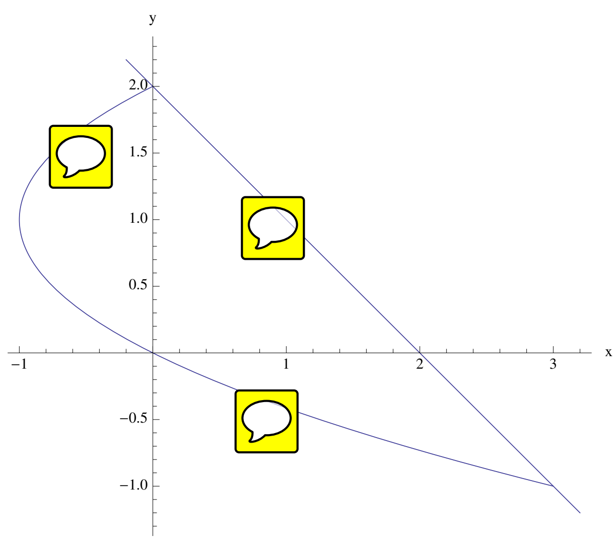
Solution (a)

$$\begin{aligned}\int_{-1}^{2}\int_{y^2-2y}^{2-y}(x+y)\mathrm{d}x\mathrm{d}y &= \int_{-1}^{2}\frac{1}{2}x^2+xy\Big|_{y^2-2y}^{2-y}\mathrm{d}y\\ &= \int_{-1}^{2}(-\frac{1}{2}y^4+y^3-\frac{1}{2}y^2+2)\mathrm{d}y\\ &= -\frac{1}{10}y^5+\frac{1}{4}y^4-\frac{1}{6}y^3+2y\Big|_{-1}^{2}\\ &= \frac{99}{20}.\end{aligned}$$

Solution (b) The plane determined by the limits of integration is between the lines $x = 2-y$ and $x = y^2-2y$ on the interval $y \in [-1, 2]$, i.e.,

$$D = \{(x, y)\big|y^2-2y \le x \le 2-y, -1 \le y \le 2\}.$$

It is shown in the following figure.



Exercise 4. Consider the integral

$$\int_{0}^{2}\int_{x^2}^{2x}(2x+1)\mathrm{d}y\mathrm{d}x.$$

(a) Evaluate this integral.
(b) Sketch the region of integration.

(c) Write an equivalent iterated integral with the order of integration reversed. Evaluate this new integral and check that your answer agrees with part (a).

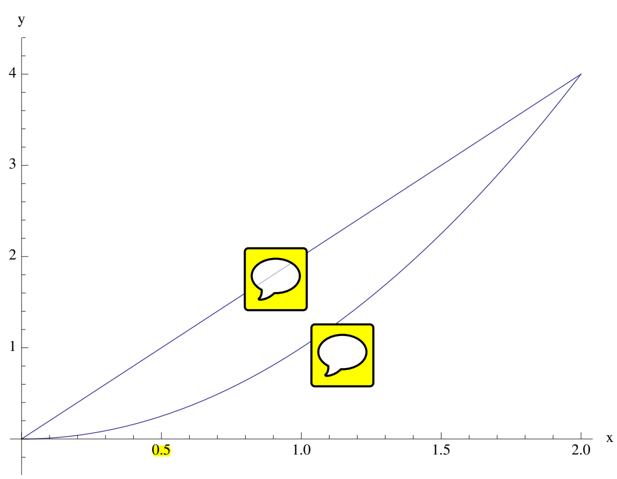
Solution (a)

$$\begin{aligned}\int_0^2 \int_{x^2}^{2x} (2x+1)\mathrm{d}y\mathrm{d}x &= \int_0^2 (2x+1)y\Big|_{x^2}^{2x}\mathrm{d}x \\ &= \int_0^2 (-2x^3+3x^2+2x)\mathrm{d}x \\ &= -\frac{1}{2}+x^3+x^2\Big|_0^2 \\ &= 4.\end{aligned}$$

Solution (b) The region of integration D is bewteen the lines $y=2x$ and $y=x^2$ on the interval $x \in [0,2]$, i.e.,

$$D = \{(x,y)\big| x^2 \leq y \leq 2x, 0 \leq x \leq 2\}.$$

It is shown in the following figure.



Solution (c) We can solve x from the limits of integration.

$$x = \frac{y}{2}, x = \sqrt{y}.$$

It means that D can also be described as

$$D = \{(x,y)\big| \frac{y}{2} \leq x \leq \sqrt{y}, 0 \leq y \leq 4\}.$$

Then we change the order of integration.

$$\begin{aligned}
\int_0^2 \int_{x^2}^{2x} (2x+1)\mathrm{d}y\mathrm{d}x &= \int_0^4 \int_{y/2}^{\sqrt{y}} (2x+1)\mathrm{d}x\mathrm{d}y \\
&= \int_0^4 x^2 + x\Big|_{y/2}^{\sqrt{y}} \mathrm{d}y \\
&= \int_0^4 (\sqrt{y} - \frac{y^2}{4} + \frac{y}{2})\mathrm{d}y \\
&= \frac{2}{3}y^{3/2} - \frac{1}{12}y^3 + \frac{1}{4}y^2\Big|_0^4 \\
&= 4.
\end{aligned}$$

We see that this new integral agrees with the original one in part (a).

Exercise 5. Evaluate the integral

$$\int_{-1}^{2}\int_{1}^{z^2}\int_{0}^{y+z} 3yz^2 dxdydz.$$

Solution

$$\begin{aligned}
\int_{-1}^{2}\int_{1}^{z^2}\int_{0}^{y+z} 3yz^2\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_{-1}^{2}\int_{1}^{z^2} 3xyz^2\Big|_0^{y+z} \mathrm{d}y\mathrm{d}z \\
&= \int_{-1}^{2}\int_{1}^{z^2} (3y^2z^2 + 3yz^3)\mathrm{d}y\mathrm{d}z \\
&= \int_{-1}^{2} y^3z^2 + \frac{3}{2}y^2z^3)\Big|_1^{z^2} \mathrm{d}z \\
&= \int_{-1}^{2} (z^8 - z^2 + \frac{3}{2}z^7 - \frac{3}{2}z^3)\mathrm{d}z \\
&= \frac{1}{9}z^9 - \frac{1}{4}z^4 + \frac{3}{16}z^8 - \frac{3}{8}z^4\Big|_{-1}^{2} \\
&= \frac{1539}{16}.
\end{aligned}$$