The University of Nagoya
School of Mathematical Sciences
G30 Tutorials 2, Spring 2012
ASSESSED COURSEWORK 2
Deadline: June 14th, 14:45

Exercise 1. Evaluate or explain why the limit fails to exist:

(a) $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{(x+y)^2}{x^2+y^2}$.

(b) $\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3}$.

Solution (a) As (x, y) approaches $\mathbf{0}$ along the x-axis (i.e. $y = 0$), the value of f is

$$\lim_{x\to 0, y=0} f(x,y) = \lim_{x\to 0, y=0} \frac{(x+y)^2}{x^2+y^2} = \lim_{x\to 0} \frac{(x+0)^2}{x^2+0^2} = 1.$$

However, as the straight line $y = -x$, the value of f is

$$\lim_{x\to 0, y=-x} f(x,y) = \lim_{x\to 0, y=-x} \frac{(x+y)^2}{x^2+y^2} = \lim_{x\to 0} \frac{(x-x)^2}{x^2+(-x)^2} = 0.$$

Since the limit of f depends on which direction it approaches the origin, it fails to exist.

We can also use polar coordinates here. Let $x = r\cos\theta, y = r\sin\theta$. For $r \neq 0$, f can be rewritten as

$$\begin{aligned} f(x,y) &= \frac{(r\cos\theta + r\sin\theta)^2}{(r\cos\theta)^2 + (r\sin\theta)^2} \\ &= \frac{r^2(\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta)}{r^2(\cos^2\theta + \sin^2\theta)} \\ &= 1 + \sin 2\theta. \end{aligned}$$

Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} 1 + \sin 2\theta = 1 + \sin 2\theta.$$

There is no restriction for θ. Since the limit depends on θ, it fails to exist.

Solution(b) As in the question (a), if we consider (x, y) approaches the origin along a straight line $y = kx$, the value of f is

$$\begin{aligned} \lim_{x\to 0, y=kx} f(x,y) &= \lim_{x\to 0, y=kx} \frac{x^4y^4}{(x^2+y^4)^3} \\ &= \lim_{x\to 0} \frac{x^4(kx)^4}{(x^2+(kx)^4)^3} = \lim_{x\to 0} \frac{k^4x^2}{(1+k^4x^2)^3} = 0. \end{aligned}$$

It means the value of f tends to the constant 0 as (x, y) approaches the origin along any straight line.

However, if we consider the value of f along the parabola $x = y^2$

$$\lim_{x\to 0, x=y^2} f(x,y) = \lim_{x\to 0, x=y^2} \frac{x^4y^4}{(x^2+y^4)^3} = \lim_{x\to 0} \frac{x^4x^2}{(x^2+x^2)^3} = \lim_{x\to 0} \frac{x^6}{8x^6} = \frac{1}{8}.$$

It is different from the limit value of f along straight lines, so the limit of f fails to exist.

Exercise 2. Evaluate the partial derivatives $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, and $\frac{\partial F}{\partial z}$ for the given functions F.

(a) $F(x,y,z) = \frac{x+y+z}{(1+x^2+y^2+z^2)^{3/2}}$.

(b) $F(x,y,z) = \sin(x^2y^3z^5)$.

Solution (a)

$$\begin{aligned}
\frac{\partial F}{\partial x}(x,y,z) &= \frac{\partial}{\partial x}\frac{x+y+z}{(1+x^2+y^2+z^2)^{3/2}} \\
&= \frac{\partial}{\partial x}(x+y+z)\frac{1}{(1+x^2+y^2+z^2)^{3/2}} \\
&+ (x+y+z)\frac{\partial}{\partial x}\frac{1}{(1+x^2+y^2+z^2)^{3/2}} \\
&= \frac{1}{(1+x^2+y^2+z^2)^{3/2}} \\
&+ (-\frac{3}{2})\frac{x+y+z}{(1+x^2+y^2+z^2)^{5/2}}\frac{\partial}{\partial x}(1+x^2+y^2+z^2) \\
&= \frac{1}{(1+x^2+y^2+z^2)^{3/2}} + \frac{-3x(x+y+z)}{(1+x^2+y^2+z^2)^{5/2}} \\
&= \frac{1+x^2+y^2+z^2-3x(x+y+z)}{(1+x^2+y^2+z^2)^{5/2}}.
\end{aligned}$$

We can evaluate $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ in the same way. In fact, since x, y and z are symmetric to each other, we can get

$$\frac{\partial F}{\partial y} = \frac{1+x^2+y^2+z^2-3y(x+y+z)}{(1+x^2+y^2+z^2)^{5/2}},$$

$$\frac{\partial F}{\partial z} = \frac{1+x^2+y^2+z^2-3z(x+y+z)}{(1+x^2+y^2+z^2)^{5/2}}.$$

Solution (b)

$$\begin{aligned}
\frac{\partial F}{\partial x}(x,y,z) &= \frac{\partial}{\partial x}\sin(x^2y^3z^5) \\
&= \cos(x^2y^3z^5)\frac{\partial}{\partial x}x^2y^3z^5 \\
&= 2xy^3z^5\cos(x^2y^3z^5).
\end{aligned}$$

$$\begin{aligned}\frac{\partial F}{\partial y}(x,y,z) &= \frac{\partial}{\partial y}\sin(x^2y^3z^5)\\ &= \cos(x^2y^3z^5)\frac{\partial}{\partial y}x^2y^3z^5\\ &= 3x^2y^2z^5\cos(x^2y^3z^5).\end{aligned}$$

$$\begin{aligned}\frac{\partial F}{\partial z}(x,y,z) &= \frac{\partial}{\partial z}\sin(x^2y^3z^5)\\ &= \cos(x^2y^3z^5)\frac{\partial}{\partial z}x^2y^3z^5\\ &= 5x^2y^3z^4\cos(x^2y^3z^5).\end{aligned}$$

Exercise 3. Compute the gradient of the following functions

(a) $f(x,y) = \frac{x-y}{x^2+y^2+1}$ at $(2,-1)$
(b) $f(x,y,z) = xy + y\cos(z) - x\sin(yz)$ at $(2,-1,\pi)$.

Solution (a) Here we also need to evaluate the partial derivatives.

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= \frac{\partial}{\partial x}\frac{x-y}{x^2+y^2+1}\\ &= \frac{1}{x^2+y^2+1}\frac{\partial}{\partial x}(x-y) + (x-y)\frac{\partial}{\partial x}\frac{1}{x^2+y^2+1}\\ &= \frac{1}{x^2+y^2+1} + (x-y)(-1)\frac{2x}{(x^2+y^2+1)^2}\\ &= \frac{x^2+y^2+1-2x(x-y)}{(x^2+y^2+1)^2}\\ &= \frac{-x^2+2xy+y^2+1}{(x^2+y^2+1)^2}.\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y}(x,y) &= \frac{\partial}{\partial y}\frac{x-y}{x^2+y^2+1}\\ &= \frac{1}{x^2+y^2+1}\frac{\partial}{\partial y}(x-y) + (x-y)\frac{\partial}{\partial y}\frac{1}{x^2+y^2+1}\\ &= \frac{-1}{x^2+y^2+1} + (x-y)(-1)\frac{2y}{(x^2+y^2+1)^2}\\ &= -\frac{x^2+y^2+1+2y(x-y)}{(x^2+y^2+1)^2}\\ &= -\frac{x^2+2xy-y^2+1}{(x^2+y^2+1)^2}.\end{aligned}$$

Therefore the gradient of $f(x, y)$ at $(2, -1)$ is

$$(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})\big|_{(2,-1)} = (-\frac{1}{6}, 0).$$

Solution (b) We evaluate the partial derivatives as following

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y, z) &= \frac{\partial}{\partial x} xy + y\cos(z) - x\sin(yz) \\ &= y - \sin(yz).\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y}(x, y, z) &= \frac{\partial}{\partial y} xy + y\cos(z) - x\sin(yz) \\ &= x + \cos(z) - xz\cos(yz).\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial z}(x, y, z) &= \frac{\partial}{\partial z} xy + y\cos(z) - x\sin(yz) \\ &= -y\sin(z) - xy\cos(yz).\end{aligned}$$

So the gradient of $f(x, y, z)$ at $(2, -1, \pi)$ is

$$(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})\big|_{(2,-1,\pi)} = (-1, 1 + 2\pi, -2).$$

Exercise 4. Let f be the function defined as follows:

- $f(x, y) = \frac{xy^2 - x^2y + 3x^3 - y^3}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$
- $f(0, 0) = 0$.

(a) Calculate $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ for $(x, y) \neq (0, 0)$.
(b) Find $f_x(0, 0)$, $f_y(0, 0)$.

Solution (a) For $(x, y) \neq (0, 0)$ we have

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y) &= \frac{\partial}{\partial x} \frac{xy^2 - x^2y + 3x^3 - y^3}{x^2 + y^2} \\ &= \frac{1}{x^2 + y^2} \frac{\partial}{\partial x}(xy^2 - x^2y + 3x^3 - y^3) \\ &+ (xy^2 - x^2y + 3x^3 - y^3) \frac{\partial}{\partial x} \frac{1}{x^2 + y^2} \\ &= \frac{y^2 - 2xy + 9x^2}{x^2 + y^2} - \frac{(xy^2 - x^2y + 3x^3 - y^3)(2x)}{(x^2 + y^2)^2} \\ &= \frac{3x^4 + 8x^2y^2 + y^4}{(x^2 + y^2)^2}.\end{aligned}$$

$$
\begin{aligned}
\frac{\partial f}{\partial y}(x,y) &= \frac{\partial}{\partial y}\frac{xy^2 - x^2y + 3x^3 - y^3}{x^2+y^2} \\
&= \frac{1}{x^2+y^2}\frac{\partial}{\partial y}(xy^2 - x^2y + 3x^3 - y^3) \\
&+ (xy^2 - x^2y + 3x^3 - y^3)\frac{\partial}{\partial y}\frac{1}{x^2+y^2} \\
&= \frac{2xy - x^2 - 3y^2}{x^2+y^2} - \frac{(xy^2 - x^2y + 3x^3 - y^3)(2y)}{(x^2+y^2)^2} \\
&= -\frac{x^4 + y^4 + 4x^3y + 2x^2y^2}{(x^2+y^2)^2}.
\end{aligned}
$$

Solution (b) Due to the definition of partial derivatives,

$$
f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{3h^3}{h^2} - 0}{h} = 3.
$$

$$
f_y(0,0) = \lim_{h\to 0}\frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0}\frac{\frac{-h^3}{h^2} - 0}{h} = -1.
$$

Exercise 5. Let $z = g(x,y)$ be a class C^2 function and let $x = e^r\cos\Theta$, $y = e^r\sin\Theta$.

(a) Use the Chain Rule to find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \Theta}$. Write $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ in terms of $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \Theta}$.

(b) Use Part (a) and the Chain Rule to prove that

$$
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^{-2r}(\frac{\partial^2 z}{\partial r^2} + \frac{\partial^2 z}{\partial \Theta^2})
$$

Solution (a) According to the Chain Rule we have

$$
\begin{aligned}
(1)\qquad \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}e^r\cos\Theta + \frac{\partial z}{\partial y}e^r\sin\Theta, \\
(2)\qquad \frac{\partial z}{\partial \Theta} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial \Theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \Theta} = -\frac{\partial z}{\partial x}e^r\sin\Theta + \frac{\partial z}{\partial y}e^r\cos\Theta.
\end{aligned}
$$

The above two equations (1) and (2) form linear equations with $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial x}$ as variables. We can solve the equations and get

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial r}e^{-r}\cos\Theta - \frac{\partial z}{\partial \Theta}e^{-r}\sin\Theta, \\
\frac{\partial z}{\partial y} &= \frac{\partial z}{\partial r}e^{-r}\sin\Theta + \frac{\partial z}{\partial \Theta}e^{-r}\cos\Theta.
\end{aligned}
$$

Solution (b) Use the results of Part (a) and Chain Rule we have

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r}\frac{\partial z}{\partial r} \\
&= \frac{\partial}{\partial x}\Big(\frac{\partial z}{\partial r}\Big)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\Big(\frac{\partial z}{\partial r}\Big)\frac{\partial y}{\partial r} \\
&= \frac{\partial^2 z}{\partial x^2}e^{2r}\cos^2\Theta + 2\frac{\partial^2 z}{\partial x \partial y}e^{2r}\cos\Theta\sin\Theta + \frac{\partial^2 z}{\partial y^2}e^{2r}\sin^2\Theta
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial \Theta^2} &= \frac{\partial}{\partial \Theta}\frac{\partial z}{\partial \Theta} \\
&= \frac{\partial}{\partial x}\Big(\frac{\partial z}{\partial \Theta}\Big)\frac{\partial x}{\partial \Theta} + \frac{\partial}{\partial y}\Big(\frac{\partial z}{\partial \Theta}\Big)\frac{\partial y}{\partial \Theta} \\
&= \frac{\partial^2 z}{\partial x^2}e^{2r}\sin^2\Theta - 2\frac{\partial^2 z}{\partial x \partial y}e^{2r}\cos\Theta\sin\Theta + \frac{\partial^2 z}{\partial y^2}e^{2r}\cos^2\Theta
\end{aligned}$$

Therefore,

$$\frac{\partial^2 z}{\partial r^2} + \frac{\partial^2 z}{\partial \Theta^2} = \Big(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\Big)e^{2r}.$$

If we put e^{2r} to the left hand side, we get the form in Question (b).

We can also evaluate the second-order partial derivatives $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ from $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, which have been gotten in Part (a). The evaluation is similar.